A Loop Summit - new perturbative results and methods in precision physics

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Function spaces in QFT at higher loops and their algorithmic exploration

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Der Wissenschaftsfonds.

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon, N) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+N+1} \underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^N}_{f(\varepsilon, N, x, y, z)}$$

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The integrand is

- hypergeometric in N:

$$\frac{f(\varepsilon, N+1, x, y, z)}{f(\varepsilon, N, x, y, z)} \in \mathbb{Q}(\varepsilon, N, x, y, z)$$

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$$\frac{D_x f(\varepsilon, N, x, y, z)}{f(\varepsilon, N, x, y, z)} \in \mathbb{Q}(\varepsilon, N, x, y, z)$$

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$$\Big\downarrow \quad \text{Ablinger's MultiIntegrate.m (9 hours)}$$

$$a_0(\varepsilon,N)F(\varepsilon,N)+a_1(\varepsilon,N)F(\varepsilon,N+1)+\cdots+a_5(\varepsilon,N)F(\varepsilon,N+5)=0$$

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$$a_0(\varepsilon, N) F(\varepsilon, N) + a_1(\varepsilon, N) F(\varepsilon, N+1) + \cdots + a_5(\varepsilon, N) F(\varepsilon, N+5) = 0$$

recurrence solver $\big\downarrow$

$F(\varepsilon, N) =$ expression in terms of special functions

Recurrence solving

A recurrence solver (Sigma.m)

GIVEN a recurrence

$a_0(N), \ldots, a_\delta(N)$: polynomials in N
$h(N)$: expression in indefinite nested sums defined over hypergeometric products.

$$a_0(N)F(N) + \cdots + a_\delta(N)F(N+\delta) = h(N);$$

together with initial values $F(0), \ldots, F(\delta-1) \in \mathbb{K}$

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DECIDE constructively if $F(N)$ can be expressed in terms indefinite nested sums defined over hypergeometric products.

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DECIDE constructively if $F(N)$ can be expressed in terms indefinite nested sums defined over hypergeometric products.

Special cases of indefinite nested sums over hypergeometric products:

$$S_{2,1}(N)=\sum_{i=1}^{n}\frac{1}{i^2}\sum_{j=1}^{i}\frac{1}{j}\qquad\text{(harmonic sums)}$$

J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018 [arXiv:hep-ph/9810241];
J.A.M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037 [arXiv:hep-ph/9806280].

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Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{k=1}^{n} \frac{2^k}{k} \sum_{i=1}^{k} \frac{2^{-i}}{i} \sum_{j=1}^{i} \frac{S_1(j)}{j}$$ (generalized harmonic sums)

S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083];
J. Ablinger, J. Blümlein and CS, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378].

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Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{k=1}^{N} \frac{1}{(1+2k)^2} \sum_{j=1}^{k} \frac{1}{j^2} \sum_{i=1}^{j} \frac{1}{1+2i}$$ (cyclotomic harmonic sums)

J. Ablinger, J. Blümlein and CS, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063].

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Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{j=1}^{N} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} \qquad \text{(binomial sums)}$$

J. Ablinger, J. Blümlein, C. G. Raab and CS, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

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DECIDE constructively if $F(N)$ can be expressed in terms indefinite nested sums defined over hypergeometric products.

Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{h=1}^{N} 2^{-2h}(1-\eta)^h \binom{2h}{h} \sum_{k=1}^{h} \frac{2^{2k}}{k^2 \binom{2k}{k}}$$ (generalized binomial sums)

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, CS, K. Schönwald. Nucl.Phys.B 932. 2018. [arXiv:1804.02226].
J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. Nucl.Phys.B 955. 2020. [arXiv:2004.08916]

Sigma.m is based on difference ring/field theory

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$$a_0(\varepsilon, N) F(\varepsilon, N) + a_1(\varepsilon, N) F(\varepsilon, N+1) + \dots + a_5(\varepsilon, N) F(\varepsilon, N+5) = 0$$

Sigma $\downarrow$

$$F(\varepsilon, N) = F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + \dots$$

A refined recurrence solver (Sigma.m)

GIVEN a recurrence

$a_0(\varepsilon, N), \ldots, a_\delta(\varepsilon, N)$: polynomials in ε, N
$h_l(N), h_{l+1}(N), \ldots, h_\lambda(N)$:
expressions in indefinite nested sums
defined over hypergeometric products.

$$a_0(\varepsilon, N)F(\varepsilon, N) + \cdots + a_\delta(\varepsilon, N)F(\varepsilon, N+\delta)$$
$$= h_l(N)\varepsilon^l + h_{l+1}(N)\varepsilon^{l+1} + \ldots h_\lambda(N)\varepsilon^r + O(\varepsilon^{r+1});$$

together with ε-expansions of $F(0), \ldots, F(\delta-1)$ up to a certain order.

A refined recurrence solver (Sigma.m)

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$h_l(N), h_{l+1}(N), \dots, h_\lambda(N)$:
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$$a_0(\varepsilon,N)F(\varepsilon,N)+\dots+a_\delta(\varepsilon,N)F(\varepsilon,N+\delta)$$
$$=h_l(N)\varepsilon^l+h_{l+1}(N)\varepsilon^{l+1}+\dots h_\lambda(N)\varepsilon^r+O(\varepsilon^{r+1});$$

together with ε-expansions of $F(0), \dots, F(\delta-1)$ up to a certain order.

DECIDE constructively if the coefficients $F_i(N)$ of

$$F(N)=F_l(N)\,\varepsilon^l+F_{l+1}(N)\,\varepsilon^{l+1}+\dots+F_\lambda(N)\,\varepsilon^r+O(\varepsilon^{r+1})$$

can be given in terms of indefinite nested sums defined over hypergeometric products.

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656[cs.SC]
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon, N)\Big[F(N)\Big] \\
+&a_1(\varepsilon, N)\Big[F(N+1)\Big] \\
+& \\
&\vdots \\
+&a_\delta(\varepsilon, N)\Big[F(N+\delta)\Big]
\end{aligned}
$$

$$= h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots$$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]$$
$$+a_1(\varepsilon,N)\Big[F(N+1)\Big]$$
$$+$$
$$\vdots$$
$$+a_\delta(\varepsilon,N)\Big[F(N+\delta)\Big]$$
$$=h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots$$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$
\begin{aligned}
& a_0(\varepsilon, N)\Big[F_0(N) + F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots\Big] \\
+ & a_1(\varepsilon, N)\Big[F_0(N+1) + F_1(N+1)\varepsilon + F_2(N+1)\varepsilon^2 + \dots\Big] \\
+ & \\
\vdots & \\
+ & a_\delta(\varepsilon, N)\Big[F(N+\delta)\Big]
\end{aligned}
$$

$$= h_0(N) + h_1(N)\varepsilon + h_1(N)\varepsilon^2 + \dots$$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$
\begin{aligned}
& a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
+& a_1(\varepsilon,N)\Big[F_0(N+1)+F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
+& \\
& \vdots\\
+& a_\delta(\varepsilon,N)\Big[F_0(N+\delta)+F_1(N+\delta)\varepsilon+F_2(N+\delta)\varepsilon^2+\dots\Big]\\
& \qquad\qquad = h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

given (in terms of indefinite nested sums and products)

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[F_0(N+1)+F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_\delta(\varepsilon,N)\Big[F_0(N+\delta)+F_1(N+\delta)\varepsilon+F_2(N+\delta)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad=h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,N)F_0(N)+a_1(0,N)F_0(N+1)+\dots+a_\delta(0,N)F_0(N+\delta)=h_0(N)}
$$

Ansatz (for power series)

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[F_0(N+1)+F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_\delta(\varepsilon,N)\Big[F_0(N+\delta)+F_1(N+\delta)\varepsilon+F_2(N+\delta)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad= h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,N)F_0(N)+a_1(0,N)F_0(N+1)+\dots+a_\delta(0,N)F_0(N+\delta)=h_0(N)}
$$

REC solver: Given the initial values $F_0(1),F_0(2),\dots,F_0(\delta)$, **decide** if $F_0(N)$ can be written in terms of indefinite nested sums and products.

Ansatz (for power series)

$$
\begin{aligned}
& a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
& +a_1(\varepsilon,N)\Big[F_0(N+1)+F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
& +\\
& \vdots\\
& +a_\delta(\varepsilon,N)\Big[F_0(N+\delta)+F_1(N+\delta)\varepsilon+F_2(N+\delta)\varepsilon^2+\dots\Big]\\
& \qquad\qquad = h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,N)F_0(N)+a_1(0,N)F_0(N+1)+\dots+a_\delta(0,N)F_0(N+\delta)=h_0(N)}
$$

Ansatz (for power series)

$$
\begin{aligned}
& a_0(\varepsilon,N)\Big[F_0(N)+F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
+& a_1(\varepsilon,N)\Big[F_0(N+1)+F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
+& \\
& \vdots \\
+& a_\delta(\varepsilon,N)\Big[F_0(N+\delta)+F_1(N+\delta)\varepsilon+F_2(N+\delta)\varepsilon^2+\dots\Big]\\
&\qquad = h_0(N)+h_1(N)\varepsilon+h_1(N)\varepsilon^2+\dots
\end{aligned}
$$

$\Downarrow$ constant terms must agree

$$
\boxed{a_0(0,N)F_0(N)+a_1(0,N)F_0(N+1)+\dots+a_\delta(0,N)F_0(N+\delta)=h_0(N)}
$$

$$
\begin{aligned}
&\quad a_0(\varepsilon,N)\Big[F_1(N)\varepsilon + F_2(N)\varepsilon^2 + \dots\Big]\\
&+a_1(\varepsilon,N)\Big[F_1(N+1)\varepsilon + F_2(N+1)\varepsilon^2 + \dots\Big]\\
&+\\
&\ \vdots\\
&+a_\delta(\varepsilon,N)\Big[F_1(N+\delta)\varepsilon + F_2(N+\delta)\varepsilon^2 + \dots\Big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad = h_0'(N) + h_1'(N)\varepsilon + h_2'(N)\varepsilon^2 + \dots
\end{aligned}
$$

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_1(N)\varepsilon+F_2(N)\varepsilon^2+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[F_1(N+1)\varepsilon+F_2(N+1)\varepsilon^2+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_\delta(\varepsilon,N)\Big[F_1(N+\delta)\varepsilon+F_2(N+\delta)\varepsilon^2+\dots\Big]\\
&\qquad\qquad\qquad\qquad=\underbrace{h_0'(N)}_{\mathbf{=0}}+h_1'(N)\varepsilon+h_2'(N)\varepsilon^2+\dots
\end{aligned}
$$

Devide by ε

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[F_1(N)+F_2(N)\varepsilon+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[F_1(N+1)+F_2(N+1)\varepsilon+\dots\Big]\\
+&\\
\vdots&\\
+&a_\delta(\varepsilon,N)\Big[F_1(N+\delta)+F_2(N+\delta)\varepsilon+\dots\Big]=h_1'(N)+h_2'(N)\varepsilon+\dots
\end{aligned}
$$

Repeat to get $F_1(N), F_2(N), \dots$

Remark: Works the same for Laurent series.

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656[cs.SC]
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

$$F(N)=\int_0^1 dx\int_0^1 dy\int_0^1 dz x^{\varepsilon/2}y^{\varepsilon/2}(1-z)^{-\frac{3\varepsilon}{2}-2}z^{\frac{\varepsilon}{2}+N+1}(1-xz)^{\varepsilon/2}\times(1-yz)^{\varepsilon/2}(x+y-1)^N$$

$\downarrow$ (package `MultiIntegrate.m`)

$$a_0(\varepsilon,N)F(N)+a_1(\varepsilon,N)F(N+1)+\cdots+a_5(\varepsilon,N)F(N+5)=0$$

$$F(2)=\frac{20}{27\varepsilon^3}-\frac{40}{27\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{1393}{486}+\frac{5\zeta_2}{18}\right)+\dots$$

$$\vdots$$

$$F(6)=\frac{22}{147\varepsilon^3}-\frac{535}{2058\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{630043}{1234800}+\frac{11\zeta_2}{196}\right)+\dots$$

$$F(N) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+N+1} (1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^N$$

↓ (package `MultiIntegrate.m`)

$$a_0(\varepsilon, N)F(N) + a_1(\varepsilon, N)F(N+1) + \cdots + a_5(\varepsilon, N)F(N+5) = 0$$

$$F(2) = \frac{20}{27\varepsilon^3} - \frac{40}{27\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{1393}{486} + \frac{5\zeta_2}{18}\right) + \dots$$

$$\vdots$$

$$F(6) = \frac{22}{147\varepsilon^3} - \frac{535}{2058\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{630043}{1234800} + \frac{11\zeta_2}{196}\right) + \dots$$

↓ (summation package `Sigma.m`)

$$\boxed{F(N) = F_{-3}(N)\,\varepsilon^{-3} + F_{-2}(N)\,\varepsilon^{-2} + F_{-1}(N)\,\varepsilon^{-1} + \dots}$$

We get

$$F_{-3}(N) = \frac{8(-1)^N}{3(N+1)(N+2)} + \frac{8(2N+3)}{3(N+1)^2(N+2)}$$

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$$F_{-3}(N) = \frac{8(-1)^N}{3(N+1)(N+2)} + \frac{8(2N+3)}{3(N+1)^2(N+2)}$$

$$F_{-2}(N) = -\frac{4(-1)^N\left(3N^3+18N^2+31N+18\right)}{3(N+1)^3(N+2)^2} - \frac{4\left(6N^3+32N^2+51N+26\right)}{3(N+1)^3(N+2)^2}$$

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$$F_{-2}(N) = -\frac{4(-1)^N\left(3N^3+18N^2+31N+18\right)}{3(N+1)^3(N+2)^2} - \frac{4\left(6N^3+32N^2+51N+26\right)}{3(N+1)^3(N+2)^2}$$

$$\begin{aligned}F_{-1}(N) = (-1)^N \Bigg(&\frac{2\left(9N^5+81N^4+295N^3+533N^2+500N+204\right)}{3(N+1)^4(N+2)^3} + \frac{\zeta_2}{(N+1)(N+2)} \\ &+ \frac{2\left(18N^5+150N^4+490N^3+755N^2+536N+132\right)}{3(N+1)^4(N+2)^3} + \frac{(2N+3)\zeta_2}{(N+1)^2(N+2)} \\ &+ \left(-\frac{4}{(N+1)^2(N+2)} + \frac{4(-1)^N}{(N+1)(N+2)}\right) S_2 \\ &+ \left(\frac{4(-1)^N}{3(N+1)(N+2)} - \frac{4(N+9)}{3(N+1)^2(N+2)}\right) S_{-2}\end{aligned}$$

Find a recurrence and solve it for the integral/sum

$$D_\varepsilon(n) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \ldots, x_7) dx_1 dx_2 \ldots dx_7$$
$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \ldots$$

ε-recurrence solver

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$a_0(\varepsilon, n) D_\varepsilon(n) + \ldots + a_d(\varepsilon, n) D_\varepsilon(n+d) = h(\varepsilon, n)$$

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ε-recurrence solver

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \ldots, i_7)$$

MultiSum Package
(K. Wegschaider)

$$a_0(\varepsilon, n) D_\varepsilon(n) + \ldots + a_d(\varepsilon, n) D_\varepsilon(n+d) = h(\varepsilon, n)$$

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$$D_\varepsilon(n) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \ldots, x_7) dx_1 dx_2 \ldots dx_7$$
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ε-recurrence solver

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \ldots, i_7)$$

MultiSum Package
(K. Wegschaider)

Holonomic/difference ring approach

$$a_0(\varepsilon, n) D_\varepsilon(n) + \ldots + a_d(\varepsilon, n) D_\varepsilon(n+d) = h(\varepsilon, n)$$

Solving coupled systems

[coming, e.g., from IBP methods]

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x), \ldots, \hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x), \ldots, \hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \ldots \\ \hat{I}_\lambda(x) \end{pmatrix} = A(x) \begin{pmatrix} \hat{I}_1(x) \\ \ldots \\ \hat{I}_\lambda(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \ldots \\ \hat{R}_\lambda(x) \end{pmatrix}$$

given

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$\downarrow$ uncoupling algorithms
(Zürcher,Abramov/Zima,Gauss,...)

1. $\hat{I}_1(x)$ is a solution of

$$d_0(x)\hat{I}_1(x) + d_1(x)D_x\hat{I}_1(x) + \dots + d_\lambda(x)D_x^\lambda\hat{I}_1(x) = \hat{r}(x)$$

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
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2. For $i = 2, \dots, r$ we get

$$\hat{I}_i(x) = \text{LinCom}(\hat{I}_1(x), \dots, D_x^{\lambda-1}\hat{I}_1(x)) + \text{LinCom}(..., D^i\hat{R}_i(x), \dots)$$

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x), \dots, \hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} = A(x) \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \dots \\ \hat{R}_\lambda(x) \end{pmatrix}$$

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DE-solver
(see, e.g., [arXiv:1810.12261])

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DE-solver
(see, e.g., [arXiv:1810.12261])

REC-solver

The DE-REC approach

DE system

$$D_x \hat{I}(x) = A\, \hat{I}(x) + \hat{R}(x)$$

The DE-REC approach

DE system	OreSys package (S. Gerhold) → uncoupling algorithm	uncoupled DE system
$D_x\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$		$\sum_i d_i(x) D_x^i \hat{I}_1(x) = \hat{r}(x)$ $\hat{I}_k(x) = \text{expr}_k(D_x^i \hat{I}_1(x)), k > 1$

The DE-REC approach

DE system	OreSys package (S. Gerhold) → uncoupling algorithm	uncoupled DE system
$D_x \hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$		$\sum_i d_i(x) D_x^i \hat{I}_1(x) = \hat{r}(x)$ $\hat{I}_k(x) = \mathsf{expr}_k(D_x^i \hat{I}_1(x)), k > 1$

$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

The DE-REC approach

DE system

$$D_x \hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

→ OreSys package (S. Gerhold) — uncoupling algorithm →

uncoupled DE system

$$\sum_i d_i(x) D_x^i \hat{I}_1(x) = \hat{r}(x)$$

$$\hat{I}_k(x) = \mathrm{expr}_k(D_x^i \hat{I}_1(x)), k > 1$$

$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

↓ coeff. comparison w.r.t. x^N

linear recurrence

$$\sum_i a_i(N) I_1(N+i) = r(N)$$

The DE-REC approach

DE system

$$D_x \hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

→ OreSys package (S. Gerhold) / uncoupling algorithm →

uncoupled DE system

$$\sum_i d_i(x) D_x^i \hat{I}_1(x) = \hat{r}(x)$$

$$\hat{I}_k(x) = \mathsf{expr}_k(D_x^i \hat{I}_1(x)),\, k > 1$$

↓ coeff. comparison w.r.t. x^N

$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

linear recurrence

$$\sum_i a_i(N) I_1(N+i) = r(N)$$

← `Sigma`'s REC-solver ←

closed form solutions of $I_1(N)$ in the class of nested sums and products – if this is possible

The DE-REC approach (SolveCoupledSystem package)

DE system
$$D_x \hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

OreSys package (S. Gerhold)

uncoupling algorithm

uncoupled DE system
$$\sum_i d_i(x) D_x^i \hat{I}_1(x) = \hat{r}(x)$$
$$\hat{\mathbf{I}}_{\mathbf{k}}(\mathbf{x}) = \text{expr}_{\mathbf{k}}(\mathbf{D}_{\mathbf{x}}^{\mathbf{i}} \hat{\mathbf{I}}_{\mathbf{1}}(\mathbf{x})), \mathbf{k} > \mathbf{1}$$

$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

extract coefficient

SumProduction package

coeff. comparison w.r.t. x^N

linear recurrence
$$\sum_i a_i(N) I_1(N+i) = r(N)$$

Sigma's REC-solver

closed form solutions of
$I_1(N), \mathbf{I_2}(\mathbf{N}), \ldots, \mathbf{I_n}(\mathbf{N})$
in the class of nested sums and products
– if this is possible

J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. Comput. Phys. Comm. 202, pp. 33-112.

General strategy:

$$\Big\downarrow \text{IBP methods}$$

- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- $\hat{D}(x) = \mathsf{LinComb}(\hat{I}_1(x), \ldots, \hat{I}_u(x))$

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$$I_i(N) = \varepsilon^{-3} F_{-3}(N) + \varepsilon^{-2} F_{-2}(N) + \varepsilon^{-1} F_{-1}(N) + \ldots$$

General strategy:

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$$I_i(N) = \varepsilon^{-3} F_{-3}(N) + \varepsilon^{-2} F_{-2}(N) + \varepsilon^{-1} F_{-1}(N) + \ldots$$

$\downarrow$ plug into $\hat{D}(x) = \sum_{N=0}^{\infty} D(N) x^N$

$$D(N) = \varepsilon^{-3} D_{-3}(N) + \varepsilon^{-2} D_{-2}(N) + \varepsilon^{-1} D_{-1}(N) + \varepsilon^{0} D_0(N) + \ldots$$

Concrete calculations:

- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The 3-Loop Non-Singlet Heavy Flavor Contributions and Anomalous Dimensions for the Structure Function $F_2(x, Q^2)$ and Transversity. Nuclear Physics B 886, 2014. arXiv:1406.4654 [hep-ph].
- J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS. The $O(\alpha_s^3 T_F^2)$ Contributions to the Gluonic Operator Matrix Element. Nuclear Physics B 885, 2014. arXiv:1405.4259 [hep-ph].
- J. Ablinger, J. Blümlein, A. De Freitas A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The Transition Matrix Element $A_{gq}(N)$ of the Variable Flavor Number Scheme at $O(\alpha_s^3)$. Nuclear Physics B 882, 2014. arXiv:1402.0359 [hep-ph].
- A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, CS. The $O(\alpha_s^3)$ Heavy Flavor Contributions to the Charged Current Structure Function $xF_3(x, Q^2)$ at Large Momentum Transfer. Physical Review D 92(114005), 2015. arXiv:1508.01449 [hep-ph].
- A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Non-Singlet Heavy Flavor Contributions to the Structure Function $g_1(x, Q^2)$ at Large Momentum Transfer. Nucl. Phys. B 897, 2015. arXiv:1504.08217 [hep-ph].
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Pure Singlet Heavy Flavor Contributions to the Structure Function $F_2(x, Q^2)$ and the Anomalous Dimension. Nuclear Physics B 890, 2015. arXiv:1409.1135.
- A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, A. von Manteuffel, CS. The Asymptotic 3-Loop Heavy Flavor Corrections to the Charged Current Structure Functions $F_L^{W^+-W^-}(x, Q^2)$ and $F_2^{W^+-W^-}(x, Q^2)$. Physical Review D 94(11), 2016. arXiv:1609.06255 [hep-ph].
- J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. Calculating Three Loop Ladder and V-Topologies for Massive Operator Matrix Elements by Computer Algebra. Comput. Phys. Comm. 202, 2016. arXiv:1509.08324 [hep-ph].
- J. Ablinger, J. Blümlein, P. Marquard, N. Rana, CS. Heavy Quark Form Factors at Three Loops in the Planar Limit, 2018. arXiv:1804.07313 [hep-ph].
- J. Ablinger, A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, P. Marquard, N. Rana, CS. The Heavy Quark Form Factors at Two Loops. Physical Review D, 2018. arXiv:1712.09889.

General strategy:

$$\Big\downarrow \text{IBP methods}$$

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- $\hat{D}(x) = \text{LinComb}(\hat{I}_1(x), \dots, \hat{I}_u(x))$

$$\Big\downarrow \text{solver for } \hat{I}_i(x) = \sum_{N=0}^{\infty} I_i(N) x^N$$

$$I_i(N) = \varepsilon^{-3} F_{-3}(N) + \varepsilon^{-2} F_{-2}(N) + \varepsilon^{-1} F_{-1}(N) \underbrace{+ \varepsilon^{0} F_0(N) + \dots + \varepsilon^{6} F_6(N)}_{\text{expensive calculation}}$$

$$\Big\downarrow \text{plug into } \hat{D}(x) = \sum_{N=0}^{\infty} D(N) x^N$$

$$D(N) = \varepsilon^{-3} D_{-3}(N) + \varepsilon^{-2} D_{-2}(N) + \varepsilon^{-1} D_{-1}(N) + \varepsilon^{0} D_0(N) + \dots$$

General strategy:

$\downarrow$ IBP methods

- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
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$\downarrow$ solver for $\hat{I}_i(x) = \sum_{N=0}^{\infty} I_i(N) x^N$

$$I_i(N) = \varepsilon^{-3} F_{-3}(N) + \varepsilon^{-2} F_{-2}(N) + \varepsilon^{-1} F_{-1}(N) \underbrace{+\varepsilon^{0} F_0(N) + \cdots + \varepsilon^{6} F_6(N)}_{\text{often unknown functions}}$$

$\downarrow$ plug into $\hat{D}(x) = \sum_{N=0}^{\infty} D(N) x^N$

$$D(N) = \varepsilon^{-3} D_{-3}(N) + \varepsilon^{-2} D_{-2}(N) + \varepsilon^{-1} D_{-1}(N) + \varepsilon^{0} D_0(N) + \ldots$$

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$\downarrow$ plug into $\hat{D}(x) = \sum_{N=0}^{\infty} D(N) x^N$

$$D(N) = \underbrace{\varepsilon^{-3} D_{-3}(N) + \varepsilon^{-2} D_{-2}(N) + \varepsilon^{-1} D_{-1}(N)}_{\text{often nice}} + \underbrace{\varepsilon^0 D_0(N)}_{\text{partially nice}} + \ldots$$

Computing large moments and guessing recurrences

The DE-REC approach (SolveCoupledSystem package)

DE system
$$D_x \hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

OreSys package (S. Gerhold)
uncoupling algorithm →

uncoupled DE system
$$\sum_i d_i(x) D_x^i \hat{I}_1(x) = \hat{r}(x)$$
$$\mathbf{\hat{I}_k}(\mathbf{x}) = \text{expr}_\mathbf{k}(\mathbf{D_x^i}\mathbf{\hat{I}_1}(\mathbf{x})), \mathbf{k} > \mathbf{1}$$

$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

extract coefficient
SumProduction package

coeff. comparison w.r.t. x^N ↓

linear recurrence
$$\sum_i a_i(N) I_1(N+i) = r(N)$$

← `Sigma's` REC-solver

closed form solutions of $I_1(N), \mathbf{I_2}(\mathbf{N}), \ldots, \mathbf{I_n}(\mathbf{N})$ in the class of nested sums and products – if this is possible

J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. Comput. Phys. Comm. 202, pp. 33-112.

The method of large moments (SolveCoupledSystem)

DE system
$$D_x\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$$

OreSys package (S. Gerhold) → uncoupling algorithm

uncoupled DE system
$$\sum_i d_i(x)D_x^i\hat{I}_1(x) = \hat{r}(x)$$
$$\hat{I}_k(x) = \text{expr}_k(D_x^i\hat{I}_1(x)), k>1$$

$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N)x^N$$

extract coefficient
SumProduction package

coeff. comparison w.r.t. x^N

linear recurrence
$$\sum_i a_i(N)I_1(N+i) = r(N)$$

Sigma's REC-solver ←

closed form solutions of $I_1(N)$
in the class of nested sums and products
– if th

Given $r(0), r(1), \ldots, r(8000)$,
compute $I_1(0), I_1(1), \ldots, I_1(8000)$

J. Blümlein, CS, Physics Letters B 771, pp. 31-36., 2017; arXiv:1701.04614 [hep-ph].

General strategy:

$\Big\downarrow$ IBP methods

- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- $\hat{D}(x) = \text{LinComb}(\hat{I}_1(x), \dots, \hat{I}_u(x))$

$\Big\downarrow$ solver for $\hat{I}_i(x) = \sum_{N=0}^{\infty} I_i(N) x^N$

$$I_i(N) = \varepsilon^{-3} F_{-3}(N) + \varepsilon^{-2} F_{-2}(N) + \varepsilon^{-1} F_{-1}(N) \underbrace{+\varepsilon^0 F_0(N) + \cdots + \varepsilon^6 F_6(N)}_{\text{only numbers}}$$

$N = 0, 1, \dots, 8000$

$\Big\downarrow$ plug into $\hat{D}(x) = \sum_{N=0}^{\infty} D(N) x^N$

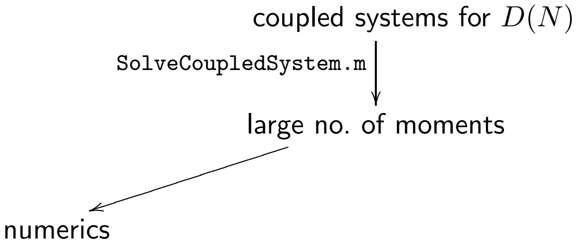
$$D(N) = \underbrace{\varepsilon^{-3} D_{-3}(N) + \varepsilon^{-2} D_{-2}(N) + \varepsilon^{-1} D_{-1}(N)}_{\text{numbers}} + \underbrace{\varepsilon^0 D_0(N)}_{\text{numbers}} + \dots$$

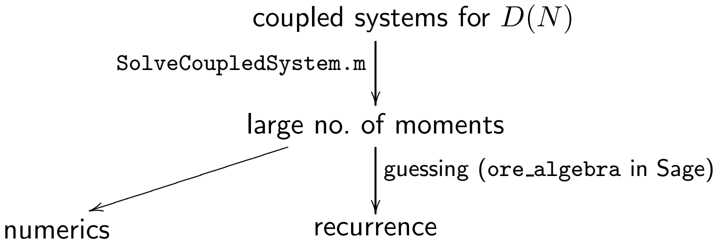
$N = 0, 1, \dots, 8000$

coupled systems for $D(N)$

`SolveCoupledSystem.m` ↓

large no. of moments


coupled systems for $D(N)$
SolveCoupledSystem.m
large no. of moments
numerics


coupled systems for $D(N)$
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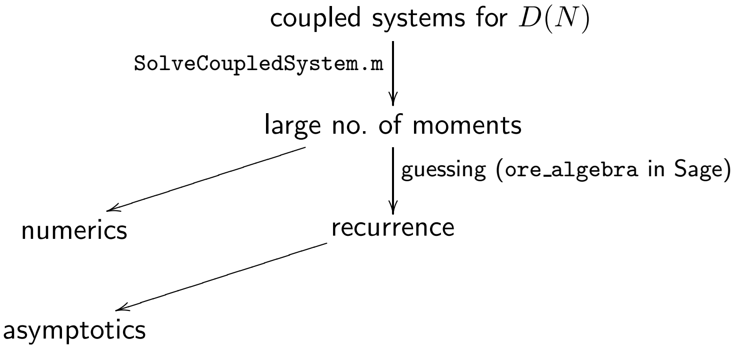
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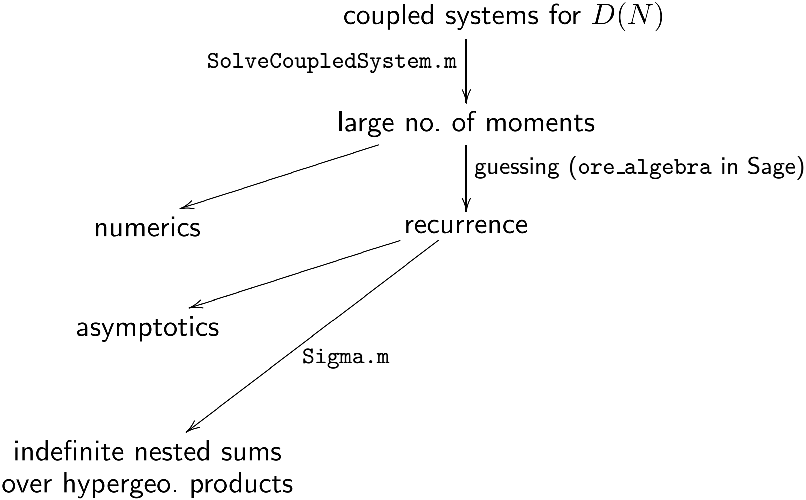
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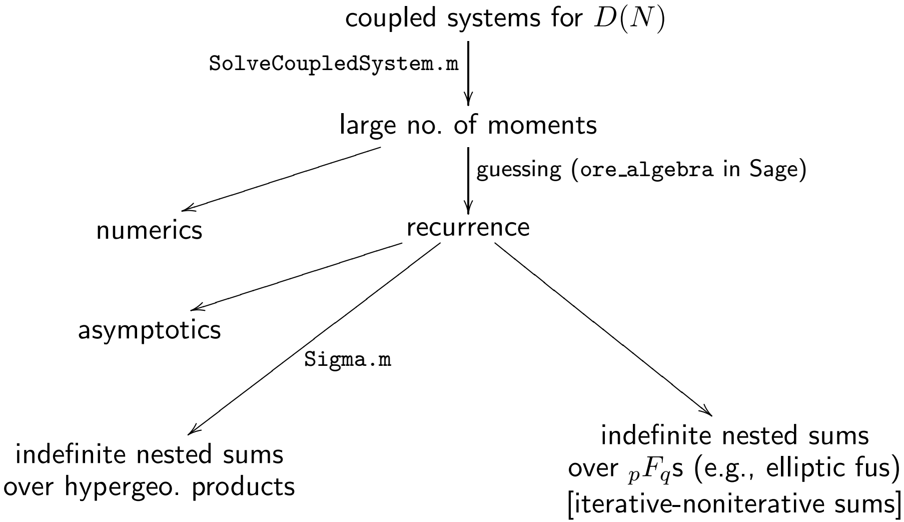
Sigma.m
indefinite nested sums
over hypergeo. products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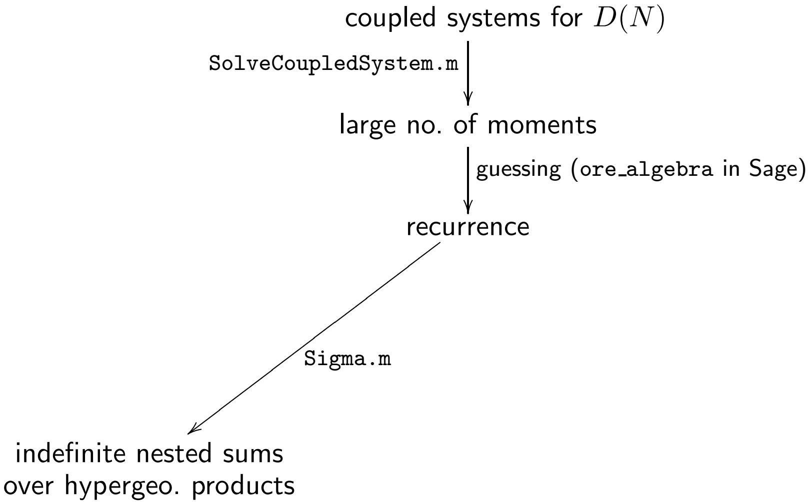
indefinite nested sums
over ${}_pF_q$s (e.g., elliptic fus)
[iterative-noniterative sums]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SolveCoupledSystem.m
large no. of moments
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General strategy:

$$\Big\downarrow \text{IBP methods}$$

- Recursively defined coupled DE systems for unknown MIs $\hat{I}_i(x)$
- $\hat{D}(x) = \text{LinComb}(\hat{I}_1(x), \ldots, \hat{I}_u(x))$

$$\Big\downarrow \text{solver for } \hat{I}_i(x) = \sum_{N=0}^{\infty} I_i(N) x^N$$

$$I_i(N) = \varepsilon^{-3} F_{-3}(N) + \varepsilon^{-2} F_{-2}(N) + \varepsilon^{-1} F_{-1}(N) \underbrace{+\varepsilon^0 F_0(N) + \cdots + \varepsilon^6 F_6(N)}_{\text{only numbers}}$$

$N = 0, 1, \ldots, 8000$

$$\Big\downarrow \text{plug into } \hat{D}(x) = \sum_{N=0}^{\infty} D(N) x^N$$

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$$D(N) = \underbrace{\varepsilon^{-3} D_{-3}(N) + \varepsilon^{-2} D_{-2}(N) + \varepsilon^{-1} D_{-1}(N)}_{\text{nice}} + \underbrace{\varepsilon^0 D_0(N)}_{\text{partially nice}} + \ldots$$

all N solution

Concrete calculations of large moments:

- The three-loop splitting functions $P_{qg}^{(2)}$ and $P_{gg}^{(2,\mathrm{N_F})}$ [Nucl.Phys.B./2017]
  1. computed ~ 2400 moments
  2. guessed and solved all recurrences

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Mathematica is loaded...

In[4]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[5]:= **initial** $=<<$ **iFile16**

In[4]:= **<< Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[5]:= **initial =<< iFile16**

Out[5]= {37, 34577/1296, 7598833/151875, 13675395569/230496000,
475840076183/7501410000, 1432950323678333/21965628762000,
21648380901382517/328583783127600,
52869784323778576751/80221899453696000,
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14061543374120479886110159898869387/226643167590350326435656036000000,
715586522666491903324905785178619936571168370307700222807811495895030000000,
16286729046359273892841271257418854056836413/269396588055480390401343344736943104000000,
1428729642632302467951426905844691837805299/23940759575034122827861315961573673600000,
498938690219595294505102809199154550783080767/8468883667852979813171262304054002720000000,
555296381919643885816767597997123847103620469689/9546343089354642655682952088937477747472000000};

In[6]:= **rec =<< rFile16**

Out[6]=

$$(n+1)^4(n+2)^2(2n+3)(2n+5)(2n+7)(2n+9)(2n+11)\Big(309237645312n^{32} + 38256884318208n^{31} +$$

$$2282100271087616n^{30} + 87428170197762048n^{29} + 2417273990256001024n^{28} + 51388547929265405952n^{27} +$$

$$873862324676687036416n^{26} + 12209268055143308328960n^{25} + 142860861222820240162816n^{24} +$$

$$1419883954103469621510144n^{23} + 12115561235109256405319680n^{22} + 89479384946084038000803840n^{21} +$$

$$575561340618928527623274496n^{20} + 3239547818363227419971647488n^{19} + 16009805333085271423330779136n^{18} +$$

$$69631814641718655426881659392n^{17} + 266892117418348771052573667328n^{16} +$$

$$901901113782416884441719270144n^{15} + 2685821385767154471801366647296n^{14} +$$

$$7038702625583766161604414471744n^{13} + 16195069575749412648646633248128n^{12} +$$

$$32602540883321212533013752639288n^{11} + 57154680141624618025310553466704n^{10} +$$

$$86710462147941775492301231896818n^{9} + 112917328975807075881545543668548n^{8} +$$

$$124873767581470867343743078943772n^{7} + 115624836314544572769501784072647n^{6} +$$

$$87938536330971046886456627610048n^{5} + 53481897815980319933589323279298n^{4} +$$

$$25000430622737750756669804052204n^{3} + 8430930497463933665464836129855n^{2} +$$

$$1825177817831282261293155379650n + 190428196025667395685609855000\Big)(2n+1)^4 f[n]$$

$$
\begin{aligned}
&-(n+2)^3(2n+3)^3(2n+7)(2n+9)(2n+11)\Big(12369505812480n^{38} + 161315106167193 6n^{37} + \\
&101748284195864576n^{36} + 4135139115563745280n^{35} + 121713599527855849472n^{34} + \\
&2765050919624810430464n^{33} + 50453046277771391664128n^{32} + 759760507477065230974976n^{31} + \\
&9628262076527899425374208n^{30} + 104191253579306374131613696n^{29} + 973595596739520084325171200n^{28} + \\
&7924537790312611436520013824n^{27} + 56571687381518195331462463488n^{26} + \\
&356133102136059681954436399104n^{25} + 1985507231916669869451824553984n^{24} + \\
&9836060321685410187563260035072n^{23} + 43406506634905372676489415905280n^{22} + \\
&170945808151999530921656848106496n^{21} + 601507760131008511164113355409920n^{20} + \\
&1892149418896523531194676203153920n^{19} + 5321173806292333448534132495165440n^{18} + \\
&13370912745727662541153592039812160n^{17} + 29987002021632029091547005084057760n^{16} + \\
&59921270253255984811455083696758912n^{15} + 106434458966741189159011567116493072n^{14} + \\
&167533688453539238956436945725341004n^{13} + 232781742346547554435545097479210510n^{12} + \\
&284125621128876904663642986868770746n^{11} + 302806836393712159148051277734975424n^{10} + \\
&279679164311116651162116055961513301n^{9} + 221781415386984655607595031093415136n^{8} + \\
&149214365004640710156345950062395186n^{7} + 83882523964213110328265187672574356n^{6} + \\
&38609679702395410742361774562392789n^{5} + 14149471988638475521561721269939086n^{4} + \\
&3963748138857399502678254252169734n^{3} + 795659668131014454843348852372480n^{2} + \\
&101701393436276172443717692853400n + 6204709909986751913151675960000\Big)\, f[n+1]
\end{aligned}
$$

$$
\begin{aligned}
&+2(n+3)^2(2n+5)^3(2n+9)(2n+11)\big(24739011624960n^{40} + 3317836466356224n^{39} + 215508170284466176n^{38} + 9032884062187945984n^{37} + \\
&274636134389959884800n^{36} + 6455501959255126179840n^{35} + 122094572934385260036096n^{34} + 1909387225793663151898624n^{33} + \\
&25180108291969215434326016n^{32} + 284171960705270647479074816n^{31} + 2775794400720227034854326272n^{30} + \\
&23677622163992853854566219776n^{29} + 177624312783583749157935120384n^{28} + 1178515602115604757944201871360n^{27} + \\
&6947091965313419323781358354432n^{26} + 36515023100308314818702129258496n^{25} + 171621148571344894953594594017280n^{24} + \\
&722837793013976317556258102507520n^{23} + 2732534027077907914497042720534528n^{22} + 9281028665970648470895368668485120n^{21} + \\
&28337819215557708948254385336117248n^{20} + 77786125749274632150536464583130752n^{19} + 191877161455672780973502244537632256n^{18} + \\
&424953221702140663089937921965135648n^{17} + 843818276409975584824720931649555264n^{16} + \\
&1499359936674956711935311062995422344n^{15} + 2378007025570977662661938772843220240n^{14} + \\
&3355671771434535852147325502571953770n^{13} + 4196375762867184563407432891655585484n^{12} + \\
&4627675779563752366067861596232781096n^{11} + 4473175960511956000526499430851993603n^{10} + \\
&3761696365025837909581516781307249585n^{9} + 2726553473467254373993685951699145492n^{8} + \\
&1683383212304999468664293798012773485n^{7} + 871926653651504419744271839781064837n^{6} + \\
&371307437598003570058538796122994147n^{5} + 126427972742886389602285855482966072n^{4} + 33048762330145623969058704448697313n^{3} + \\
&6217924746857741077419160100404560n^{2} + 748298077423337427195946099994100n + 43181089548034246077698611794000\big)\, f[n+2]
\end{aligned}
$$

$$
\begin{aligned}
&-2(n+4)^2(2n+5)(2n+7)^3(2n+11)\Big(24739011624960n^{40}+3322784268681216n^{39}+216160919414112256n^{38}+9074528155284275200n^{37}+\\
&276348048819456311296n^{36}+6506479077331107315712n^{35}+123266585640616142569472n^{34}+1931040885785102661976064n^{33}+\\
&25510503383281445462081536n^{32}+288418124175428279391485952n^{31}+2822442799033603081019326464n^{30}+\\
&24120717233320712351821332480n^{29}+181295944719289040999116701696n^{28}+1205246297785423925076555694080n^{27}+\\
&7119049557560114436136213413888n^{26}+37496933571993839665392189775872n^{25}+176616172467048982234270428880896n^{24}+\\
&745539218875020737621728364206080n^{23}+2824909633156578132652259733712896n^{22}+9618101958268071244680677589035520n^{21}+\\
&29441860528446423517613263360742912n^{20}+81033563306363873505877563416477312n^{19}+200454769103641040142838133702338304n^{18}+\\
&445286624972461749049425309485328992n^{17}+887028447418790661018847407251573152n^{16}+\\
&1581538101499869694224895701784875304n^{15}+2517550244392724509968791166585362672n^{14}+\\
&3566593026520465155504695877897282630n^{13}+4479066125207404898722179511912639638n^{12}+\\
&4962006990874351800791769650243464872n^{11}+4819992643914265990647887896664485209n^{10}+\\
&4074895386694182240941538222230233221n^{9}+2970477229398746689186622534784613554n^{8}+\\
&1845274131994015990683957902602775337n^{7}+962091291302144537393228847830431614n^{6}+\\
&412595107814836563208757757032740146n^{5}+141540723940232563767779647013785485n^{4}+37292931812630561528276365992452010n^{3}+\\
&7074865777225416725452872895397100n^{2}+858794112392644074221312049837000n+49997386738260112603615104780000\Big)f[n+3]
\end{aligned}
$$

$$
\begin{aligned}
&+(n+5)^3(2n+5)(2n+7)(2n+9)^4\Big(12369505812480n^{38} + 1546355730284544n^{37} + 93441851805138944n^{36} + \\
&3636063211393908736n^{35} + 102413434086873890816n^{34} + 2225107112182077718528n^{33} + \\
&38808234188348931964928n^{32} + 558299807912629375074304n^{31} + 6755648626273815474733056n^{30} + \\
&69769132238801205785001984n^{29} + 621900006220029229458259968n^{28} + 4826558182244413850688946176n^{27} + \\
&32840774268722977511855751168n^{26} + 196981883700048989849717882880n^{25} + \\
&1046061529031136798450810839040n^{24} + 4934888224954929426023144030208n^{23} + \\
&20735286278224836075286873214976n^{22} + 77745549200390911029444008457216n^{21} + \\
&260448286122609254214904458392064n^{20} + 780087654447729149285799146869248n^{19} + \\
&2089276462852113795051294249728512n^{18} + 5001455921015163002705347586646080n^{17} + \\
&10691068512696184477385875851523744n^{16} + 20374769440121072185247660725156544n^{15} + \\
&34542976501702600883669655947085712n^{14} + 51947527795197316142253213880200764n^{13} + \\
&69039779136078090572935768218052854n^{12} + 80712286124402599779679594199103258n^{11} + \\
&82519759833385882007812859351392458n^{10} + 73248127158607338722648198918322201n^{9} + \\
&55935262205790259307904762197107653n^{8} + 36322355479155199114489624391144238n^{7} + \\
&19756597118002557191991191826327042n^{6} + 8822212911433711339358062994077203n^{5} + \\
&3145597282374650512689680780380605n^{4} + 859907105684964990690798899478888n^{3} + \\
&168963309995629650025632011492580n^{2} + 21205680751316222158938757272000n + \\
&1274120732351744651125603886400\Big)\, f[n+4]
\end{aligned}
$$

$$
\begin{aligned}
&-(n+5)^2(n+6)^4(2n+5)(2n+7)(2n+9)^3(2n+11)^4\Big(309237645312n^{32}+28361279668224n^{31}+\\
&1249518729297920n^{30}+35220794552352768n^{29}+713726163159089152n^{28}+11076866026783113216n^{27}+\\
&136959486138712588288n^{26}+1385658801437173350400n^{25}+11691772665924577918976n^{24}+\\
&83438339505976242995200n^{23}+508989054278115477684224n^{22}+2675508113418826174332928n^{21}+\\
&12193213796145039633072128n^{20}+48399020537651722726242304n^{19}+167881257973769248139515904n^{18}+\\
&510012482113388176546187776n^{17}+1358662126092561923541267968n^{16}+3174925021159974655053814528n^{15}+\\
&6504205668151125355938798848n^{14}+11663792381020901870157176128n^{13}+\\
&18263581057905911985340656960n^{12}+24881010123632244515458585528n^{11}+\\
&29346856353503020415409305704n^{10}+29775859546803351930591002266n^{9}+25770328899499991754425455738n^{8}+\\
&18817114309842270306167785140n^{7}+11424980760825630752861027739n^{6}+5656051955667821083952617134n^{5}+\\
&2221448212382554437709999491n^{4}+664859653803075491350122060n^{3}+142190920852333874895041748n^{2}+\\
&19313175036907229252501700n+1248723341516324359641600\Big)\, f[n+5]==0
\end{aligned}
$$

In[7]:= $\mathbf{recSol = SolveRecurrence[rec, f[n]]}$

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Out[7]=
$$\Big\{\Big\{0, \frac{(3+2n)(3+4n)}{(1+n)^2(1+2n)^2}\Big\}$$
$$\Big\{0, -\frac{(3+2n)\big(-8-9n+2n^2\big)}{(1+n)^2(1+2n)^2}\Big\}$$
$$\Big\{0, -\frac{(3+2n)\big(-5+8n^2\big)}{2(1+n)^2(1+2n)^2} + \frac{(3+2n)\sum_{i=1}^{n}\frac{1}{i}}{(1+n)(1+2n)} + \frac{2(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big\}$$
$$\Big\{0, \frac{(3+2n)\big(-513-2184n-2416n^2+768n^4\big)}{2(1+n)^3(1+2n)^3} + \frac{14(3+2n)\sum_{i=1}^{n}\frac{1}{i^2}}{(1+n)(1+2n)} + \Big(-\frac{2(3+2n)\big(3+44n+48n^2\big)}{(1+n)^2(1+2n)^2} + \frac{48(3+2n)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\sum_{i=1}^{n}\frac{1}{i} +$$
$$\frac{12(3+2n)\big(\sum_{i=1}^{n}\frac{1}{i}\big)^2}{(1+n)(1+2n)} + \frac{56(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)} -$$
$$\frac{4(3+2n)\big(3+44n+48n^2\big)\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)^2(1+2n)^2} + \frac{48(3+2n)\big(\sum_{i=1}^{n}\frac{1}{-1+2i}\big)^2}{(1+n)(1+2n)}\Big\}$$

$$
\begin{aligned}
&\{0, \frac{1}{16(1+n)^4(1+2n)^4}\big(72519+572343n+1814716n^2+2918100n^3+2442240n^4+912896n^5+24576n^6-\\
&49152n^7\big)+\frac{16(3+2n)\sum_{i=1}^n\frac{1}{i^3}}{3(1+n)(1+2n)}+\Big(-\frac{(3+2n)\big(29+307n+322n^2\big)}{4(1+n)^2(1+2n)^2}+\frac{44(3+2n)\sum_{i=1}^n\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\sum_{i=1}^n\frac{1}{i^2}+\\
&\Big(\frac{(3+2n)\big(91+259n+974n^2+1784n^3+1024n^4\big)}{4(1+n)^3(1+2n)^3}+\frac{22(3+2n)\sum_{i=1}^n\frac{1}{i^2}}{(1+n)(1+2n)}+\frac{24(3+2n)\sum_{i=1}^n\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}-\\
&\frac{4(3+2n)\big(-13-4n+16n^2\big)\sum_{i=1}^n\frac{1}{-1+2i}}{(1+n)^2(1+2n)^2}+\frac{16(3+2n)\big(\sum_{i=1}^n\frac{1}{-1+2i}\big)^2}{(1+n)(1+2n)}\Big)\sum_{i=1}^n\frac{1}{i}+\Big(-\\
&\frac{(3+2n)\big(19+92n+80n^2\big)}{(1+n)^2(1+2n)^2}+\frac{40(3+2n)\sum_{i=1}^n\frac{1}{-1+2i}}{(1+n)(1+2n)}\Big)\Big(\sum_{i=1}^n\frac{1}{i}\Big)^2+\frac{20(3+2n)\big(\sum_{i=1}^n\frac{1}{i}\big)^3}{3(1+n)(1+2n)}+\\
&\frac{64(3+2n)\sum_{i=1}^n\frac{1}{(-1+2i)^3}}{3(1+n)(1+2n)}-\frac{3(3+2n)\big(63+209n+150n^2\big)\sum_{i=1}^n\frac{1}{(-1+2i)^2}}{(1+n)^2(1+2n)^2}+\\
&\Big(\frac{(3+2n)\big(347+1795n+4302n^2+4856n^3+2048n^4\big)}{2(1+n)^3(1+2n)^3}+\frac{48(3+2n)\sum_{i=1}^n\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}\Big)\sum_{i=1}^n\frac{1}{-1+2i}-\\
&\frac{4(3+2n)\big(19+92n+80n^2\big)\big(\sum_{i=1}^n\frac{1}{-1+2i}\big)^2}{(1+n)^2(1+2n)^2}+\frac{32(3+2n)\big(\sum_{i=1}^n\frac{1}{-1+2i}\big)^3}{3(1+n)(1+2n)}-\\
&\frac{8(3+2n)\sum_{i=1}^n\frac{\left(\sum_{j=1}^i\frac{1}{j}\right)^2}{i}}{(1+n)(1+2n)}-\frac{16(3+2n)\sum_{i=1}^n\frac{\left(\sum_{j=1}^i\frac{1}{j}\right)^2}{-1+2i}}{(1+n)(1+2n)}\\
&-\frac{32(3+2n)\sum_{i=1}^n\frac{\left(\sum_{j=1}^i\frac{1}{j}\right)\sum_{j=1}^i\frac{1}{-1+2j}}{i}}{(1+n)(1+2n)}+\frac{64(3+2n)\sum_{i=1}^n\frac{\left(\sum_{j=1}^i\frac{1}{j}\right)\sum_{j=1}^i\frac{1}{-1+2j}}{-1+2i}}{(1+n)(1+2n)}+\\
&\frac{32(3+2n)\sum_{i=1}^n\frac{\left(\sum_{j=1}^i\frac{1}{-1+2j}\right)^2}{i}}{(1+n)(1+2n)}+\frac{64(3+2n)\sum_{i=1}^n\frac{\left(\sum_{j=1}^i\frac{1}{-1+2j}\right)^2}{-1+2i}}{(1+n)(1+2n)}\},\{1,0\}\}
\end{aligned}
$$

In[8]:= $\mathbf{sol = FindLinearCombination[recSol, \{0, initial\}, n, 7, MinInitialValue \to 1]}$

In[8]:= $\mathbf{sol = FindLinearCombination[recSol, \{0, initial\}, n, 7, MinInitialValue \to 1]}$

Out[8]=

$$\frac{1}{3(1+n)^4(1+2n)^4}(111+1920n+11765n^2+32545n^3+46476n^4+35376n^5+13440n^6+1968n^7)+\frac{32(3+2n)\sum_{i=1}^{n}\frac{1}{i^3}}{9(1+n)(1+2n)}-$$
$$\frac{(3+2n)(-3+101n+126n^2)\sum_{i=1}^{n}\frac{1}{i^2}}{3(1+n)^2(1+2n)^2}-\frac{(3+2n)(115+921n+1967n^2+1524n^3+340n^4)\sum_{i=1}^{n}\frac{1}{i}}{3(1+n)^3(1+2n)^3}+$$
$$\frac{44(3+2n)(\sum_{i=1}^{n}\frac{1}{i^2})\sum_{i=1}^{n}\frac{1}{i}}{3(1+n)(1+2n)}-\frac{(3+2n)(23+139n+130n^2)(\sum_{i=1}^{n}\frac{1}{i})^2}{3(1+n)^2(1+2n)^2}+\frac{40(3+2n)(\sum_{i=1}^{n}\frac{1}{i})^3}{9(1+n)(1+2n)}+$$
$$\frac{128(3+2n)\sum_{i=1}^{n}\frac{1}{(-1+2i)^3}}{9(1+n)(1+2n)}-\frac{4(3+2n)(77+261n+190n^2)\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{3(1+n)^2(1+2n)^2}+\frac{16(3+2n)(\sum_{i=1}^{n}\frac{1}{i})\sum_{i=1}^{n}\frac{1}{(-1+2i)^2}}{(1+n)(1+2n)}+$$
$$\frac{2(3+2n)(13-153n-303n^2+12n^3+172n^4)\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)^3(1+2n)^3}+\frac{88(3+2n)(\sum_{i=1}^{n}\frac{1}{i^2})\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)(1+2n)}-$$
$$\frac{4(3+2n)(-41-53n+2n^2)(\sum_{i=1}^{n}\frac{1}{i})\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)^2(1+2n)^2}+\frac{80(3+2n)(\sum_{i=1}^{n}\frac{1}{i})^2\sum_{i=1}^{n}\frac{1}{-1+2i}}{3(1+n)(1+2n)}+$$
$$\frac{32(3+2n)(\sum_{i=1}^{n}\frac{1}{(-1+2i)^2})\sum_{i=1}^{n}\frac{1}{-1+2i}}{(1+n)(1+2n)}-\frac{4(3+2n)(23+139n+130n^2)(\sum_{i=1}^{n}\frac{1}{-1+2i})^2}{3(1+n)^2(1+2n)^2}+$$
$$\frac{32(3+2n)(\sum_{i=1}^{n}\frac{1}{i})(\sum_{i=1}^{n}\frac{1}{-1+2i})^2}{3(1+n)(1+2n)}+\frac{64(3+2n)(\sum_{i=1}^{n}\frac{1}{-1+2i})^3}{9(1+n)(1+2n)}-\frac{16(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})^2}{i}}{3(1+n)(1+2n)}-$$
$$\frac{32(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})^2}{-1+2i}}{3(1+n)(1+2n)}-\frac{64(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})\sum_{j=1}^{i}\frac{1}{-1+2j}}{i}}{3(1+n)(1+2n)}+$$
$$\frac{128(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{j})\sum_{j=1}^{i}\frac{1}{-1+2j}}{-1+2i}}{3(1+n)(1+2n)}+\frac{64(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{-1+2j})^2}{i}}{3(1+n)(1+2n)}+$$
$$\frac{128(3+2n)\sum_{i=1}^{n}\frac{(\sum_{j=1}^{i}\frac{1}{-1+2j})^2}{-1+2i}}{3(1+n)(1+2n)}$$

In[9]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[10]:= **sol = TransformToSSums[sol];**

In[11]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;**

In[9]:= $<<$ **HarmonicSums.m**

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In[10]:= **sol = TransformToSSums[sol];**

In[11]:= **sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;**

$$\mathrm{Out[11]}= \frac{1}{3(1+n)^4(1+2n)^4}\big(111+1920n+11765n^2+32545n^3+46476n^4+35376n^5+13440n^6+1968n^7\big)+\frac{64(3+2n)^2S[1,n]}{3(1+n)(1+2n)^2}+\frac{64(3+2n)(2+3n)S[1,n]^2}{3(1+n)(1+2n)^2}+\Big(-\frac{2(3+2n)\big(147+985n+1871n^2+1268n^3+212n^4\big)}{3(1+n)^3(1+2n)^3}+\frac{224(3+2n)S[2,2n]}{3(1+n)(1+2n)}+\frac{128(3+2n)S[-2,2n]}{3(1+n)(1+2n)}\Big)S[1,2n]-\frac{4(3+2n)\big(23+123n+114n^2\big)S[1,2n]^2}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)S[1,2n]^3}{3(1+n)(1+2n)}+\frac{64(3+2n)S[2,n]}{3(1+n)(1+2n)}-\frac{4(3+2n)\big(53+229n+190n^2\big)S[2,2n]}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)S[3,2n]}{3(1+n)(1+2n)}+\Big(-\frac{64(3+2n)^2}{3(1+n)(1+2n)^2}-\frac{128(3+2n)(2+3n)S[1,2n]}{3(1+n)(1+2n)^2}\Big)S[-1,2n]-\frac{64(3+2n)(2+3n)S[-1,2n]^2}{3(1+n)(1+2n)^2}-\frac{32(3+2n)\big(1+8n+8n^2\big)S[-2,2n]}{3(1+n)^2(1+2n)^2}+\frac{64(3+2n)S[-3,2n]}{3(1+n)(1+2n)}-\frac{128(3+2n)S[-2,1,2n]}{3(1+n)(1+2n)}$$

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HarmonicSums by Jakob Ablinger © RISC-Linz

In[10]:= $\mathbf{sol = TransformToSSums[sol];}$

In[11]:= $\mathbf{sol = ReduceToBasis[MultipleSumLimit[sol, n, 2]//ToStandardForm, n]//CollectProdSum;}$

In[12]:= $\mathbf{SExpansion[sol, n, 2]}$

Out[12]=
$$\ln 2^2\left(\frac{64\mathrm{LG}[n]}{n}+\frac{160}{3n^2}-\frac{44}{n}\right)+\ln 2\left(\left(\frac{320}{3n^2}-\frac{88}{n}\right)\mathrm{LG}(n)+\frac{64\mathrm{LG}[n]^2}{n}-\frac{430}{3n^2}+\frac{160\zeta_2}{3n}-\frac{14}{n}\right)+\zeta_2\left(\frac{160\mathrm{LG}[n]}{3n}+\frac{40}{n^2}-\frac{84}{n}\right)+\left(\frac{160}{3n^2}-\frac{44}{n}\right)\mathrm{LG}[n]^2+\left(-\frac{430}{3n^2}-\frac{14}{n}\right)\mathrm{LG}[n]+\frac{64\mathrm{LG}[n]^3}{3n}+\frac{64\ln 2^3}{3n}+\frac{145}{2n^2}+\frac{32\zeta_3}{n}+\frac{41}{n}$$

Special function algorithms

Harmonic sums (Borwein, Hoffman, Broadhurst, Vermaseren, Remmiddi, Blümlein,...)

$$S_{2,1}(n) = \boxed{\sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}}$$

Harmonic sums (Borwein, Hoffman, Broadhurst, Vermaseren, Remmiddi, Blümlein,. . .)

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Integral representation:

$$= \int_0^1 \frac{x^n - 1}{1 - x} \left(\int_0^x \frac{\int_0^y \frac{1}{1-z} dz}{y} dy - \zeta(2) \right) dx, \qquad \zeta(z) := \sum_{i=1}^{\infty} 1/i^z$$

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Asymptotic expansion:

$$= \left(\frac{1}{30n^5} - \frac{1}{6n^3} + \frac{1}{2n^2} - \frac{1}{n} \right) \ln(n)$$

$$- \frac{1}{100n^5} - \frac{1}{6n^4} + \frac{13}{36n^3} - \frac{1}{4n^2} - \frac{1}{n} + 2\zeta(3) + O\left(\frac{\ln(n)}{n^6}\right).$$

limit computations **numerical evaluation**

- Generalized algorithms for generalized harmonic sums

$$\sum_{k=1}^{n}\frac{2^k\sum_{i=1}^{k}\frac{2^{-i}\sum_{j=1}^{i}\frac{S_1(j)}{j}}{i}}{k}=-\frac{21\zeta(2)^2}{20n}+\frac{1}{8n^2}+\frac{295}{216n^3}-\frac{1115}{96n^4}+O(n^{-5})$$
$$+\Big(\frac{1}{2n}-\frac{3}{4n^2}+\frac{19}{12n^3}-\frac{5}{n^4}+O(n^{-5})\Big)\zeta(2)$$
$$+2^n\Big(\frac{3}{2n}+\frac{3}{2n^2}+\frac{9}{2n^3}+\frac{39}{2n^4}+O(n^{-5})\Big)\zeta(3)$$
$$+\Big(\frac{1}{n}+\frac{3}{4n^2}-\frac{157}{36n^3}+\frac{19}{n^4}+O(n^{-5})\Big)(\log(n)+\gamma)$$
$$+\Big(\frac{1}{2n}-\frac{3}{4n^2}+\frac{19}{12n^3}-\frac{5}{n^4}+O(n^{-5})\Big)(\log(n)+\gamma)^2)$$

[Ablinger, Blümlein, CS, J. Math. Phys. 54, 2013, arXiv:1302.0378 [math-ph]]

- Generalized algorithms for cyclotomic harmonic sums

$$
\begin{aligned}
\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{1+2i}}{j^2}}{(1+2k)^2} =&\Big(-3+\frac{35\zeta(3)}{16}\Big)\zeta(2)-\frac{31\zeta(5)}{8}\\
&+\frac{1}{n}-\frac{33}{32n^2}+\frac{17}{16n^3}-\frac{4795}{4608n^4}+O(n^{-5})\\
&+\log(2)\Big(6\zeta(2)-\frac{1}{n}+\frac{9}{8n^2}-\frac{7}{6n^3}+\frac{209}{192n^4}+O(n^{-5})\Big)\\
&+\Big(-\frac{7}{4}-\frac{7}{16n}+\frac{7}{16n^2}-\frac{77}{192n^3}+\frac{21}{64n^4}+O(n^{-5})\Big)\zeta(3)\\
&+\Big(\frac{1}{16n^2}-\frac{1}{8n^3}+\frac{65}{384n^4}+O(n^{-5})\Big)(\log(n)+\gamma)
\end{aligned}
$$

[Ablinger, Blümlein, CS, J. Math. Phys. 52, 2011, arXiv:1302.0378 [math-ph]]

- Generalized algorithms for nested binomial sums

$$\sum_{j=1}^{n}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2} = 7\zeta(3)+\sqrt{\pi}\sqrt{n}\Bigg\{\Bigg[-\frac{2}{n}+\frac{5}{12n^2}-\frac{21}{320n^3}-\frac{223}{10752n^4}+\frac{671}{49152n^5}$$
$$+\frac{11635}{1441792n^6}-\frac{1196757}{136314880n^7}-\frac{376193}{50331648n^8}+\frac{201980317}{18253611008n^9}$$
$$+O(n^{-10})\Bigg]\ln(\bar{n})-\frac{4}{n}+\frac{5}{18n^2}-\frac{263}{2400n^3}+\frac{579}{12544n^4}+\frac{10123}{1105920n^5}$$
$$-\frac{1705445}{71368704n^6}-\frac{27135463}{11164188672n^7}+\frac{197432563}{7927234560n^8}+\frac{405757489}{775778467840n^9}$$
$$+O(n^{-10})\Bigg\}$$

Ablinger, Blümlein, CS, ACAT 2013, arXiv:1310.5645 [math-ph]
Ablinger, Blümlein, Raab, CS, J. Math. Phys. 55, 2014. arXiv:1407.1822 [hep-th]

Conclusion

Our calculations rely on

1. symbolic summation and integration methods to derive recurrences
2. flexible recurrence solver (in the setting of difference rings)
3. coupled systems solver
4. the large moment method

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    - to extract properties from the result

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    - to extract properties from the result
6. stable and efficient software packages

Main CA-packages

In[1]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= << **MultiIntegrate.m**

MultiIntegrate by Jakob Ablinger © RISC-Linz

In[3]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[4]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[5]:= << **SumProduction.m**

SumProduction by Carsten Schneider © RISC-Linz

In[6]:= << **OreSys.m**

OreSys by Stefan Gerhold (optimized by Carsten Schneider) © RISC-Linz

In[7]:= << **SolveCoupledSystem.m**

SolveCoupledSystem by Carsten Schneider © RISC-Linz

Conclusion

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Within the RISC-DESY cooperation

J. Ablinger, A. Behring (KIT), J. Blümlein, A. De Freitas, P. Marquard, M. Saragnese, CS, K. Schönwald (KIT)

I expect that we discover and explore many

new algorithms in CA and results in QFT!