Karlsruhe Institute of Technology

Institute for Theoretical Particle Physics

Third order corrections to $b \to c\ell\bar{\nu}$

A Loop Summit — Cadenabbia, July 25-30, 2021

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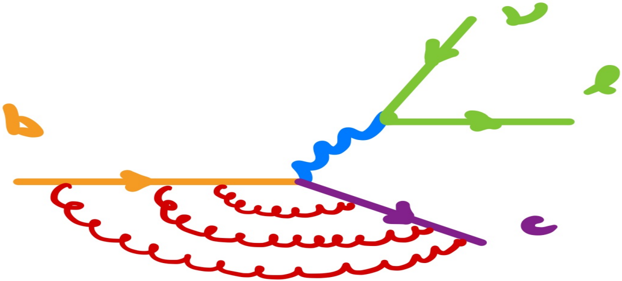
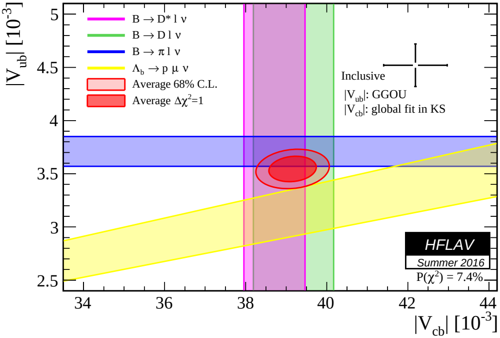
TTP KIT



Motivation



- tension between inclusive and exclusive determinations
- current uncertainty on $|V_{cb}|$: $\approx 2\%$ ➪ 1% (?) important for
  $B_s \to \mu^+\mu^-$
  $K \to \pi\nu\bar{\nu}$
  ϵ_K

- $|V_{cb}|_{\text{incl.}} = (42.19 \pm 0.78) \times 10^{-3}$ $\leftarrow 0.78 = \sqrt{(0.50_{\Gamma_{\text{sl}}})^2 + \ldots}$
  $|V_{ub}|_{\text{incl.}} = (\ 4.32 \pm 0.12_{\text{exp}} \pm 0.13_{\text{th}}) \times 10^{-3}$
  theory uncertainties dominate

$\Gamma(B \to X_c \ell \bar{\nu})$

$$\Gamma = \Gamma_0 + \Gamma_{\mu_\pi} \frac{\mu_\pi^2}{m_b^2} + \Gamma_{\mu_G} \frac{\mu_G^2}{m_b^2} + \Gamma_{\rho_D} \frac{\rho_D^3}{m_b^3} + \Gamma_{\rho_{LS}} \frac{\rho_{LS}^3}{m_b^3} + \ldots$$

- Γ_0
  up to $\mathcal{O}(\alpha_s^2)$: [Jezabek,Kühn'89; Nir'89 ...; Gambino et al.'05; Melnikov'08; Biswas,Melnikov'08; Pak,Czarnecki'08; Dowling,Piclum,Czarnecki'08]
  NEW: $\mathcal{O}(\alpha_s^3)$ [Fael,Schönwald,Steinhauser'20] partial cross checks: [Czakon,Czarnecki,Dowling'21]
- $\Gamma_{\mu_\pi}, \Gamma_{\mu_G}$
  up to $\mathcal{O}(\alpha_s)$: [Becher,Boos,Lunghi'07; Alberti,Gambino,Nandi'14; Mannel,Pivovarov,Rosenthal'15]
- Γ_{ρ_D}
  up to $\mathcal{O}(\alpha_s)$: [Mannel,Pivovarov'19]
- $1/m_b^4, 1/m_b^5$: [Dassinger,Mannel,Turczyk'07; Mannel,Turczyk,Uraltsev'10; Mannel,Vos'18; Fael,Mannel,Vos'19]

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lepton energy moments and hadronic invariant mass moments

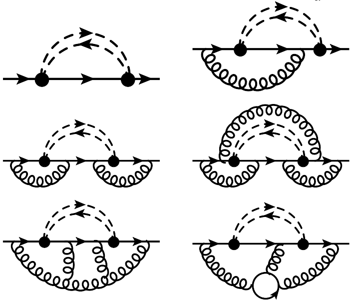
➪ fit: compare theory to experiment (Belle,Babar,CDF,CLEO,DELPHI)

[Gambino,Schwanda'14; Alberti,Gambino,Healey,Nandi'15; Gambino,Healey,Turczyk'16; Bordone,Capdevila,Gambino'21]

➪ $|V_{cb}|$ and μ_π^2, μ_G^2, ρ_D^3, ρ_{LS}^3, m_b, m_c

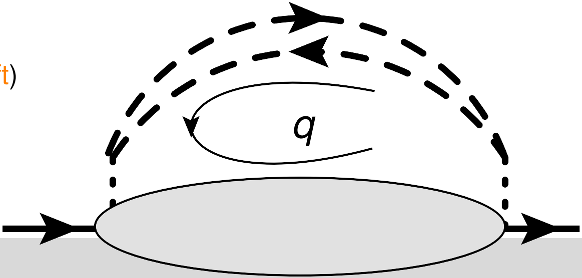
important: proper definition of bottom and charm quark masses

Method – key ideas

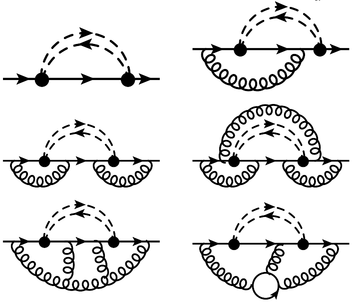


- optical theorem
- integrate out $(\ell\bar{\nu})$ loop
- loop momentum through $(\ell\bar{\nu})$ loop: q
  1-loop integration over q possible
  remaining 0, 1, 2, 3 loops
- asymptotic expansion [Beneke,Smirnov'97]
  $m_b \approx m_c$: $\delta = 1 - m_c/m_b$

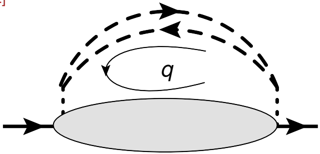
  [Dowling,Piclum,Czarnecki'08]
- $|k^\mu| \sim m_b$ (hard)
  $|k^\mu| \sim \delta \cdot m_b$ (ultra-soft)
- expansion up to δ^{12}
- analytic calculation



Method – key ideas

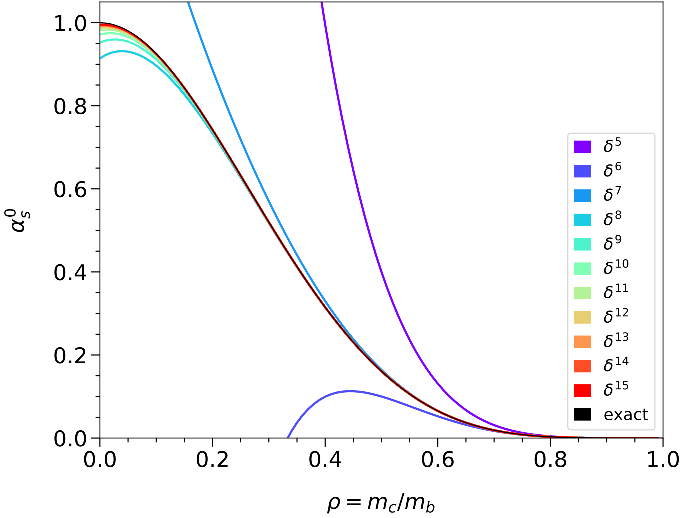


- 1450 5-loop diagrams
- asymptotic expansion cross checked against `asy` [Pak,Smirnov'10]
- automated partial fraction decomposition: `LIMIT` [Herren'20]
- number of 3-loop integrals: $\approx 25\,000\,000$
- reduction to MIs: `FIRE` [Smirnov,Chuharev'19] and `LiteRed` [Lee'12]
- scalar integrals with powers up to ± 12
  ➪ interm. expr. $\approx$ 100 GB
- number of MIs:
  2 loops: 3+3; 3-loops: 20+19 [Lee,Smirnov'10; Fael,Schönwald,Steinhauser'20]

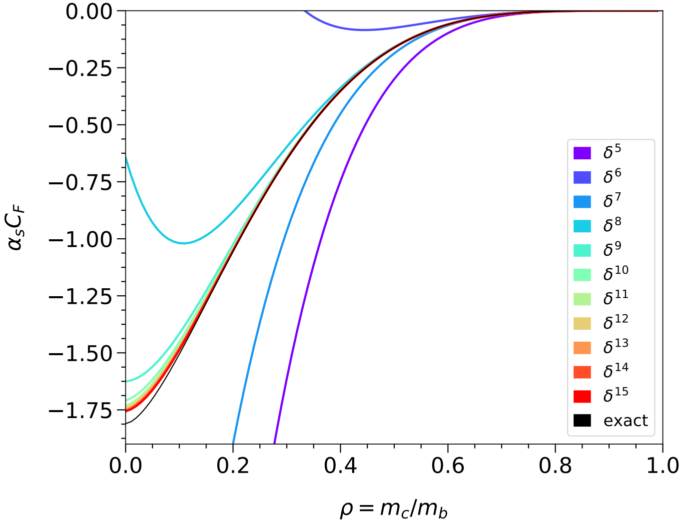


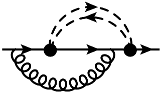
LO

$$X_0 = 1 - 8\rho^2 - 12\rho^4 \log(\rho^2) + 8\rho^6 - \rho^8$$



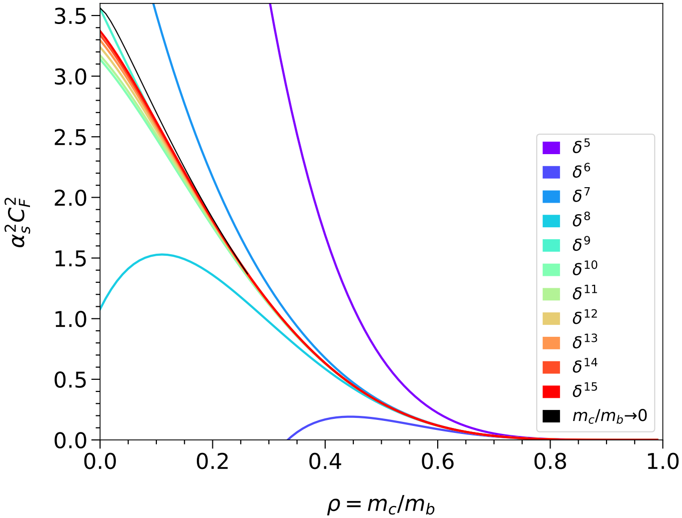
NLO

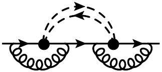




exact: [Nir'89,. . .]

NNLO

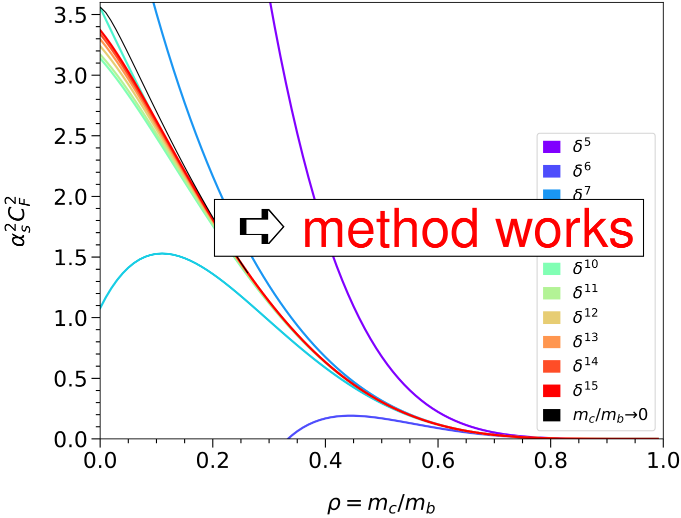




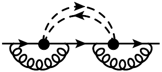
$\rho \to 0$:

[Pak,Czarnecki'08]

NNLO



⇨ method works



$\rho \to 0$:

[Pak,Czarnecki'08]

N³LO: $\Gamma(B \to X_c \ell \bar{\nu}) = \Gamma_0 \left[X_0 + C_F \sum_{n\geq 1} \left(\frac{\alpha_s}{\pi}\right)^n X_n \right] + \ldots$

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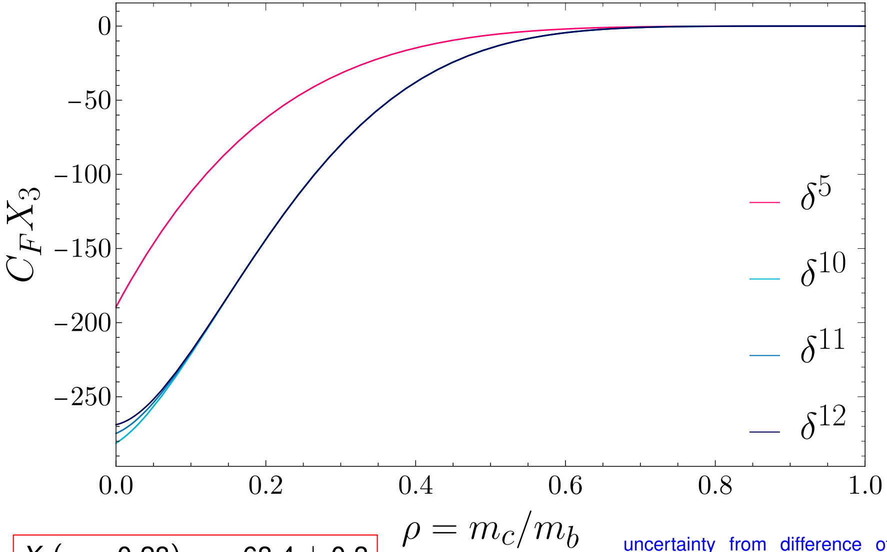

$X_3(\rho = 0.28) = -68.4 \pm 0.3$

uncertainty from difference of δ^{11} and δ^{12} expansion $\times 5$

Renormalization schemes

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pole masses: bad convergence behaviour
$\overline{\mathrm{MS}}$ scheme (m_b): better but still not good
kinetic scheme: optimal for B decays

- [Bigi,Shifman,Uraltsev,Vainshtein'96] ➪ $\Gamma_{\rm sl} \simeq \frac{G_F^2|V_{cb}|^2}{192\pi^3}\left(M_B - \overline{\Lambda}\right)^5$
- $m_b^{\rm kin}(\mu) = m_b^{\rm OS} - [\overline{\Lambda}(\mu)]_{\rm pert} - \frac{[\mu_\pi^2(\mu)]_{\rm pert}}{2m_b^{\rm kin}(\mu)} - \ldots$

$\overline{\Lambda}$: binding energy of B meson

μ_π^2: kinetic energy of b quark inside B meson

[Bigi, Shifman,Uraltsev,Vainshtein'97; 2 loops: Czarnecki,Melnikov,Uraltsev'98; 3 loops: Fael,Schönwald,Steinhauser'20]

$$m_b^{\rm kin}(1\text{ GeV}) = 4163 + 259 + 78 + 26\text{ MeV} = 4526\text{ MeV}$$

Starting point: $m_b^{\rm OS}, m_c^{\rm OS}$
➪ m_b: transform to $m_b^{\rm kin}$
➪ m_c: transform to $m_c^{\rm kin}$ or $\overline{m}_c(\mu_c)$ $\qquad \mu_c = 2\text{ GeV}, 3\text{ GeV}, \ldots$

1S [Hoang,Ligeti,Manohar'99], PS [Beneke'98]

Numerical results

$$\Gamma(B \to X_c \ell \bar{\nu}) = \Gamma_0 X_0 \left[1 + \sum_{n\geq 1} \left(\frac{\alpha_s}{\pi}\right)^n Y_n\right] + \mathcal{O}\left(\frac{\Lambda^2_{\rm QCD}}{m_b^2}\right)$$

$$\alpha_s \equiv \alpha_s^{(4)}$$

	Y_1	$Y_2^{\rm rem}$	$\beta_0 Y_2^{\beta_0}$	$Y_3^{\rm rem}$	$\beta_0^2 Y_3^{\beta_0^2}$
$m_b^{\rm OS}, m_c^{\rm OS}$	-1.72	3.08	-16.17	48.8	-212.1
$m_b^{\rm kin}, m_c^{\rm kin}$	-0.94	0.33	-4.08	-5.4	-15.4
$m_b^{\rm kin}, \overline{m}_c(3\text{ GeV})$	-1.67	-3.39	-3.85	-97.7	69.1
$m_b^{\rm kin}, \overline{m}_c(2\text{ GeV})$	-1.25	-1.21	-2.43	-68.8	67.9
$\overline{m}_b(\overline{m}_b), \overline{m}_c(3\text{ GeV})$	3.07	-21.81	35.17	-56.7	119.4
$m_b^{\rm PS}, \overline{m}_c(2\text{ GeV})$	-0.47	-6.10	-2.31	-93.1	-7.19
$m_b^{\rm 1S}, \overline{m}_c(m_b^{\rm 1S})$	-3.59	-0.98	-19.39	-39.83	-80.22
$m_b^{\rm 1S}$, m_c via HQET	-1.38	0.73	-7.05	5.04	-38.09

HQET: $m_b^{\rm OS} - m_c^{\rm OS} = M_B - M_D + (1/M_B - 1/M_D)\lambda_1/2 + \ldots$

Numerical results

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$$\Gamma(B \to X_c \ell \bar{\nu}) = \Gamma_0 X_0 \left[1 + \sum_{n\geq 1} \left(\frac{\alpha_s}{\pi}\right)^n Y_n\right] + \mathcal{O}\left(\frac{\Lambda^2_{\rm QCD}}{m_b^2}\right)$$

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Numerical results

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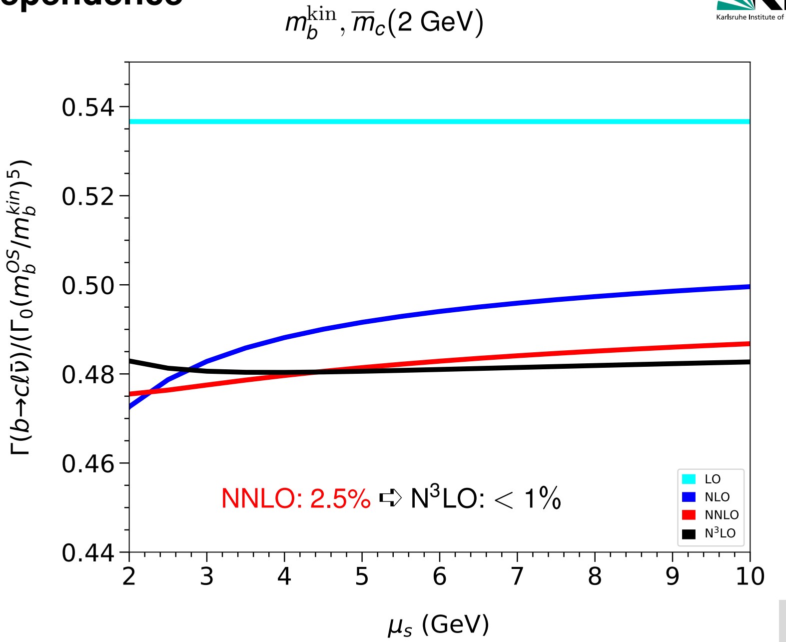
$$\Gamma(B \to X_c \ell \bar{\nu}) = \Gamma_0 X_0 \left[1 + \sum_{n \geq 1} \left(\frac{\alpha_s}{\pi}\right)^n Y_n\right] + \mathcal{O}\left(\frac{\Lambda^2_{\mathrm{QCD}}}{m_b^2}\right)$$

$\alpha_s \equiv \alpha_s^{(4)}$

	Y_1	Y_2	Y_3
$m_b^{\mathrm{OS}}, m_c^{\mathrm{OS}}$	-1.72	-13.09	-163.3
$m_b^{\mathrm{kin}}, m_c^{\mathrm{kin}}$	-0.94	-3.75	-20.8
$m_b^{\mathrm{kin}}, \overline{m}_c(3\text{ GeV})$	-1.67	-7.24	-28.6
$m_b^{\mathrm{kin}}, \overline{m}_c(2\text{ GeV})$	-1.25	-3.64	-0.9
$\overline{m}_b(\overline{m}_b), \overline{m}_c(3\text{ GeV})$	3.07	-13.36	62.7
$m_b^{\mathrm{PS}}, \overline{m}_c(2\text{ GeV})$	-0.47	-8.41	-100.3
$m_b^{\mathrm{1S}}, \overline{m}_c(m_b^{\mathrm{1S}})$	-3.59	-20.37	-120.1
m_b^{1S}, m_c via HQET	-1.38	-7.78	-33.05

μ dependence



Numerical results (2)

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$$\Gamma(B \to X_c \ell \bar{\nu}) = \Gamma_0 X_0 \left[1 + \sum_{n\geq 1} \left(\frac{\alpha_s}{\pi}\right)^n Y_n\right] + \mathcal{O}\left(\frac{\Lambda^2_{\mathrm{QCD}}}{m_b^2}\right)$$

$$\alpha_s \equiv \alpha_s^{(4)}$$

$$\Gamma(B \to X_c \ell \bar{\nu})/\Gamma_0 =$$

m_b^{kin}, m_c^{kin} :	$0.633\,(1 - 0.066 - 0.018 - 0.007)$	≈ 0.575
m_b^{kin}, $\overline{m}_c(3\text{ GeV})$:	$0.700\,(1 - 0.116 - 0.035 - 0.010)$	≈ 0.587
m_b^{kin}, $\overline{m}_c(2\text{ GeV})$:	$0.648\,(1 - 0.087 - 0.018 - 0.0003)$	≈ 0.580

Updated fit [Bordone,Capdevila,Gambino'21]

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- experimental moments from 2014 [Belle,Babar,CDF,CLEO,DELPHI]
- $\mathcal{O}(\alpha_s^3)$ corrections to $\Gamma(B \to X_c \ell \bar{\nu})$ [Fael,Schönwald,Steinhauser'20]
- $\mathcal{O}(\alpha_s^3)$ corrections to $\overline{m}_b - m_b^{\rm kin}$ relation [Fael,Schönwald,Steinhauser'20]
- [FLAG'19]

  $\overline{m}_c(3\text{ GeV}) = 0.988(7)$ GeV

  $\overline{m}_b(\overline{m}_b) = 4.198(12)$ GeV $\longrightarrow m_b^{\rm kin} = 4.565(19)$ GeV

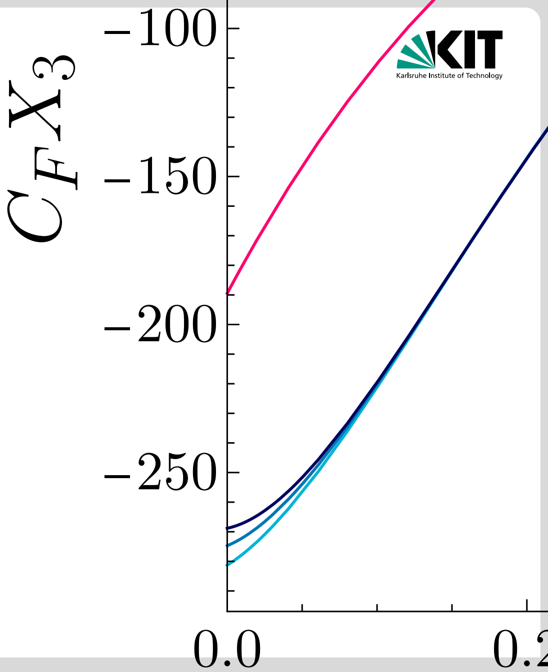
$$\boxed{|V_{cb}| = 42.16(30)_{\rm th}(32)_{\rm exp}(25)_\Gamma \times 10^{-3}}$$

- $\Gamma(B \to X_c \ell \bar{\nu})_{\mathcal{O}(\alpha_s^3)}$:

  shift $|V_{cb}|$ by $+0.6\%$

  reduce uncertainty: $(50)_\Gamma$ ➪ $(25)_\Gamma$
- 1.2% uncertainty
- $(32)_{\rm exp}$ ➪ improvements from Belle II

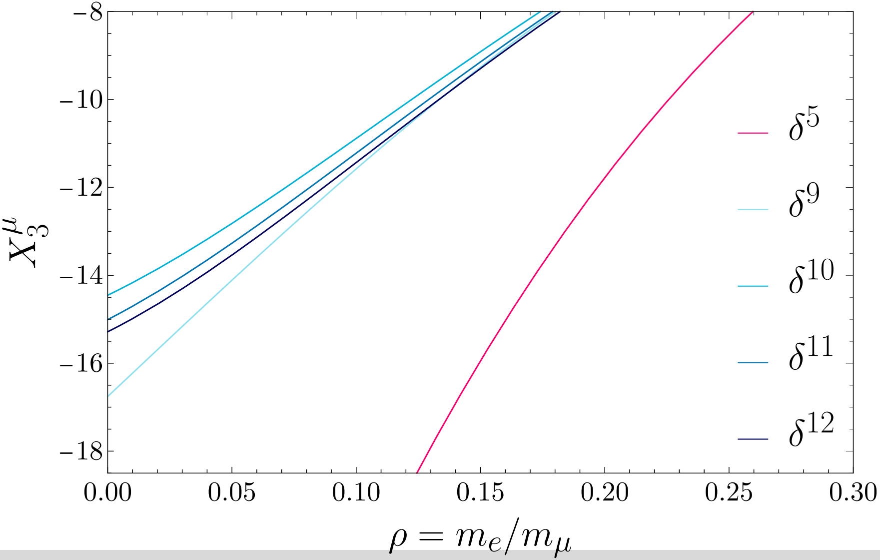
$b \to u\ell\bar{\nu}$



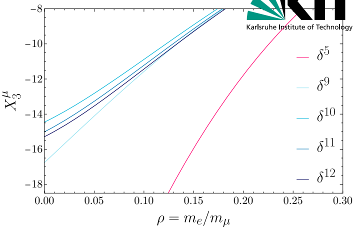
$X_3^u \approx -202 \pm 20$

(uncertainty estimate from behaviour of $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha_s^2)$ terms)

$\mu^- \to e^- \nu_\mu \bar{\nu}_e$



$\mu^- \to e^- \nu_\mu \bar{\nu}_e$



$$\frac{1}{\tau_\mu} \equiv \Gamma(\mu^- \to e^- \nu_\mu \bar{\nu}_e)$$
$$= \frac{G_F^2 m_\mu^5}{192\pi^3} (1 + \Delta q)$$

$$\Delta q^{(1)} \approx \frac{\alpha(m_\mu)}{\pi} \left(\frac{25}{8} - 3\zeta_2\right)$$ [Kinoshita,Sirlin'59; Berman'58]

$\Delta q^{(2)}$: [van Ritbergen,Stuart'98; Seidensticker,Steinhauser'99]

$$\Delta q^{(3)} \approx \left(\frac{\alpha(m_\mu)}{\pi}\right)^3 (-15.3 \pm 2.3)$$ [Fael,Schönwald,Steinhauser'20]

$$\tau_\mu^{\rm exp} = (2.1969811 \pm 2.2 \times 10^{-6})\ \mu\text{s}$$
$$\delta\tau_\mu\Big|_{\alpha^2} = 41 \times 10^{-6}\ \mu\text{s}$$
$$\delta\tau_\mu\Big|_{\alpha^3} = (0.09 \pm 0.01) \times 10^{-6}\ \mu\text{s}$$

Conclusions

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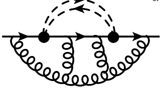

- $\Gamma(b \to c\ell\bar{\nu})$ to $\mathcal{O}(\alpha_s^3)$
- expansion around $m_c \approx m_b$
- good convergence in physical point
  $m_c/m_b \approx 0.3$
- use $m_b^{\rm kin}$, $m_b^{\rm 1S}$, $m_b^{\rm PS}$ and $m_c^{\rm kin}$ or $\overline{m}_c(\mu_c)$
  α_s^3 corrections $\lesssim 1\%$
- $\delta|V_{cb}| = +0.6\%$; reduction of uncertainty
- $m_c \to 0$ ⇨ α_s^3 corrections to $\Gamma(b \to u\ell\bar{\nu})$
- $3^{\rm rd}$ order corrections to $\Gamma(\mu^- \to e^-\nu_\mu\bar{\nu}_e)$
- outlook: apply similar approach to moments
  $(E_\ell, q^2, q_0, M_X, \dots)$