Massive Integrability: From Fishnet Theories to Feynman Graphs and Back

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based on

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with *J. Miczajka, D. Müller, H. Münkler*
and JHEP12(2020) with *J. Miczajka*

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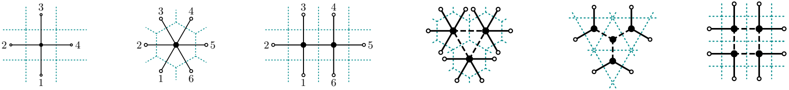
Motivation

- QFT computations are hard
- Toy model: Planar $\mathcal{N} = 4$ SYM theory
- Highly constraining symmetries (Yangian)
- What does this teach us about generic building blocks of QFT?



Integrability in $\mathcal{N} = 4$ SYM Theory: Yangian

- The Yangian is an infinite dimensional extension of a Lie algebra $\mathfrak{g}$.
- It underlies rational quantum integrable models (e.g. $\mathrm{AdS}_5/\mathrm{CFT}_4$).

Yangian algebra $Y[\mathfrak{g}]$ (first realization): [Drinfel'd 1985]

$$\text{Level 0}: \quad \mathrm{J}^a = \sum_{k=1}^{n} \mathrm{J}_k^a \in \mathfrak{g}, \qquad [\mathrm{J}^a, \mathrm{J}^b] = f^{ab}{}_c \mathrm{J}^c$$

$$\text{Level 1}: \quad \widehat{\mathrm{J}}^a = f^a{}_{bc} \sum_{j<k=1}^{n} \mathrm{J}_j^c \, \mathrm{J}_k^b + \sum_{j=1}^{n} s_j \mathrm{J}_j^a, \quad [\mathrm{J}^a, \widehat{\mathrm{J}}^b] = f^{ab}{}_c \widehat{\mathrm{J}}^c$$

In $\mathcal{N} = 4$ SYM theory we have

- Lie algebra $\mathfrak{g} = \mathfrak{psu}(2,2|4)$
- Yangian $\simeq$ dual conformal + conformal symmetry

From $\mathcal{N} = 4$ SYM to Feynman Graphs

Double-scaling limit:

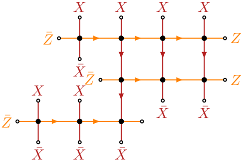
$$\begin{tikzcd}
\underline{\mathcal{N} = 4 \text{ SYM}} & \underline{\gamma\text{-Deformation}} & \underline{\text{Fishnets}} \\
\mathcal{L}_{\mathcal{N}=4} \arrow[r, "X\,Y \to e^{i\gamma_j(\dots)}XY"] & \mathcal{L}^{\gamma}_{\mathcal{N}=4} \arrow[r, "g\to 0,\ \gamma_3 \to i\infty", "\xi = g e^{-i\gamma_3/2}\text{ fix}"'] & \mathcal{L}_{\text{bi}}
\end{tikzcd}$$

Resulting non-unitary bi-scalar fishnet theory: [Gürdoğan Kazakov 2015]

$$\mathcal{L}_{\text{bi}} = N_{\text{c}}\, \mathrm{tr}\Big(-\partial_\mu \bar{X} \partial^\mu X - \partial_\mu \bar{Z} \partial^\mu Z + \xi^2 \bar{X} \bar{Z} X Z\Big)$$

- Correlators given by single fishnet Feynman graphs.
- Fishnet integrals inherit a conformal Yangian symmetry $Y[\mathfrak{so}(1,5)]$:

$\widehat{\mathrm{J}}^a$



$= 0.$

[Chicherin, Kazakov FL, Müller Zhong 2017]

Can we introduce masses?

Massive Generalization?

In the massless case, $\mathcal{N}=4$ SYM theory was the starting point:

$\mathcal{N}=4$ SYM → γ-deformation → fishnet theory → Feynman graphs

Introduce masses into $\mathcal{N}=4$ SYM by giving VEV to one of the scalars:

$$\hat{\Phi} = \langle \Phi \rangle + \Phi$$

Leads to massive propagators with difference mass ($x^\mu_{jk} = x^\mu_j - x^\mu_k$):

$$1/x^2_{jk} \quad \to \quad 1/\hat{x}^2_{jk} = 1/(x^2_{jk} + (m_j - m_k)^2)$$

Well known massive dual conformal symmetry generated by: [Alday, Henn 2010 Plefka, Schuster]

$$\begin{aligned}
&\mathrm{P}^{\hat{\mu}}_j = -i\partial^{\hat{\mu}}_{x_j}, && \mathrm{L}^{\hat{\mu}\hat{\nu}}_j = ix^{\hat{\mu}}_j \partial^{\hat{\nu}}_{x_j} - ix^{\hat{\nu}}_j \partial^{\hat{\mu}}_{x_j}, \\
&\mathrm{D}_j = -i\big(x_{j\mu}\partial^\mu_{x_j} + m_j\partial_{m_j} + \Delta_j\big), \\
&\mathrm{K}^{\hat{\mu}}_j = -2ix^{\hat{\mu}}_j\big(x_{j\nu}\partial^\nu_{x_j} + m_j\partial_{m_j} + \Delta_j\big) + i(x^2_j + m^2_j)\partial^{\hat{\mu}}_{x_j}.
\end{aligned}$$

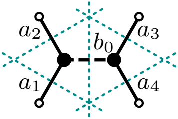
Mass interpreted as *fifth dimension* (radial direction in AdS): $x^{\hat{\mu}=5} = m_j$.

. . . but no massive Yangian symmetry (integrability) known.

Massive Feynman Integrals

Consider massive Feynman integrals directly, e.g.

$p_j^\mu = x_j^\mu - x_{j+1}^\mu$



$$= \int \frac{\mathrm{d}^D x_0 \mathrm{d}^D x_{\bar{0}}}{\hat{x}_{01}^{2a_1} \hat{x}_{02}^{2a_2} x_{0\bar{0}}^{2b_0} \hat{x}_{\bar{0}3}^{2a_3} \hat{x}_{\bar{0}4}^{2a_4}},$$

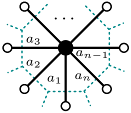
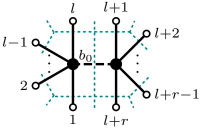
Use massive dual conformal (level-zero) generators J^a to build Yangian level-one generators:

$$\widehat{\mathrm{J}}^a = \tfrac{1}{2} f^a{}_{bc} \sum_{j<k=1}^{n} \mathrm{J}_j^c \mathrm{J}_k^b + \sum_{j=1}^{n} s_j \mathrm{J}_j^a,$$

with e.g. the level-one momentum generator

$$\widehat{\mathrm{P}}^{\hat{\mu}} = \tfrac{i}{2} \sum_{j,k=1}^{n} \mathsf{sign}(k-j) \big(\mathrm{P}_j^{\hat{\mu}} \mathrm{D}_k + \mathrm{P}_{j\nu} \mathrm{L}_k^{\nu\hat{\mu}} \big) + \sum_{j=1}^{n} s_j \mathrm{P}_j^{\hat{\mu}}$$

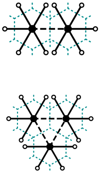
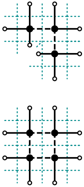
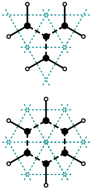
Massive Yangian at 1 and 2 Loops

	1 loop	2 loops
	$I_n =$ 	$I_{lr} =$ 
Level 0	$\mathrm{J}^a I_n = 0$ if conformal condition (cc)	$\mathrm{J}^a I_{lr} = 0$ if conformal condition
Level 1	$\widehat{\mathrm{J}}^a I_n = 0$ always	$\widehat{\mathrm{J}}^a I_{lr} = 0$ if cc, else $\widehat{\mathrm{P}}^\mu I_{lr} = 0$

here the (dual) conformal condition is $\sum_{k\in\text{vertex}} a_k = D$

Higher Loops

Conjecture:
All planar Feynman graphs, which are cut along a closed contour from one of the three regular tilings of the plane, have massive Yangian symmetry if all internal propagators are massless. External propagators can be massive or massless.

Triangle	Square	Hexagon
		

- Agrees with findings on regular tilings in massless limit [Chicherin, Kazakov FL, Müller Zhong 2017]
- Supported by numerical evidence.

Momentum Space Symmetry

Note: $\widehat{\mathrm{P}}$ is symmetry even if dual conformal symmetry is broken.

Translate level-one momentum generators in x-space into dual momentum space:

$$\widehat{\mathrm{P}}^\mu \to \bar{\mathrm{K}}^\mu$$

Here the momenta relate to the dual x-variables

$$p_j^\mu = x_j^\mu - x_{j+1}^\mu$$

New massive generalization of momentum space conformal symmetry:

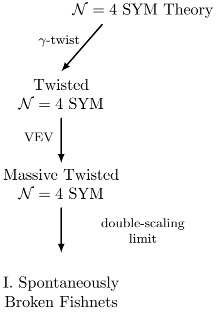
$$\begin{aligned}
\bar{\mathrm{P}}_j^\mu &= p_j^\mu\,, \qquad\qquad \bar{\mathrm{L}}_j^{\mu\nu} = p_j^\mu \partial_{p_j}^\nu - p_j^\nu \partial_{p_j}^\mu\,,\\
\bar{\mathrm{D}}_j &= p_{j\nu}\partial_{p_j}^\nu + \tfrac{m_j\partial_{m_j} + m_{j+1}\partial_{m_{j+1}}}{2} + \bar{\Delta}_j\,,\\
\bar{\mathrm{K}}_j^\mu &= p_j^\mu \partial_{p_j}^2 - 2\Big[p_{j\nu}\partial_{p_j}^\nu + \tfrac{m_j\partial_{m_j} + m_{j+1}\partial_{m_{j+1}}}{2} + \bar{\Delta}_j\Big]\partial_{p_j}^\mu\,.
\end{aligned}$$

... as opposed to known massive *dual* conformal symmetry [Alday, Henn 2010
Plefka, Schuster]

Where Does Massive Integrability Come From?

? → massive Feynman graphs

Can we define a massive version of the fishnet theory?



$\mathcal{N}=4$ SYM Theory

γ-twist

Twisted $\mathcal{N}=4$ SYM

VEV

Massive Twisted $\mathcal{N}=4$ SYM

double-scaling limit

I. Spontaneously Broken Fishnets

<u>First idea:</u> Spontaneous Symmetry Breaking in massless Fishnet Theory [Karananas, Kazakov Shaposhnikov 2019] [FL, Miczajka 2020]

- product-mass propagators $p^2 + m_j m_k$
- not the Feynman graphs with Yangian symmetry

Where Does Massive Integrability Come From?

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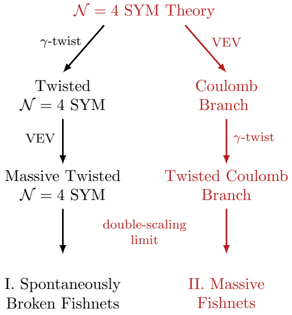

$\mathcal{N} = 4$ SYM Theory

γ-twist

VEV

Twisted $\mathcal{N} = 4$ SYM

Coulomb Branch

VEV

γ-twist

Massive Twisted $\mathcal{N} = 4$ SYM

Twisted Coulomb Branch

double-scaling limit

I. Spontaneously Broken Fishnets

II. Massive Fishnets

Second idea: Double-scaling limit of Coulomb branch $\mathcal{N} = 4$ SYM theory [FL, Miczajka 2020]

- similar route as in massless case
- yields the desired difference mass propagators $p^2 + (m_j - m_k)^2$

Double-Scaling Limit of Coulomb-Branch $\mathcal{N}=4$ SYM

$$\mathcal{L}_{\text{Coul}} \quad \xrightarrow{\mathcal{P}_\gamma ?} \quad \mathcal{L}^\gamma_{\text{Coul}} \to \text{ double-scaling limit} \to \text{massive Feynman graphs}$$

Critical step: γ-deformation $\mathcal{P}_\gamma$ only well-defined on R-symmetry singlets

- Ad hoc solution: average over ways to break up traces in Lagrangian:

$$Q : \operatorname{tr}(\Phi_1\Phi_2\dots\Phi_n) \mapsto \tfrac{1}{n}\big(\Phi_1\Phi_2\dots\Phi_n + \Phi_2\Phi_3\dots\Phi_1 + \dots\big)$$

- and define deformation as $\mathcal{L}^\gamma_{\text{Coul}} = Q^{-1}\mathcal{P}_\gamma Q\mathcal{L}_{\text{Coul}}$.

Most restrictive limit results in massive fishnet theory: [FL, Miczajka 2020]

$$\boxed{\begin{aligned}\mathcal{L}_{\text{MF}} =& N_{\mathsf{c}}\operatorname{tr}\big(-\partial_\mu\bar{X}\partial^\mu X - \partial_\mu\bar{Z}\partial^\mu Z + \xi^2\bar{X}\bar{Z}XZ\big)\\ &- N_{\mathsf{c}}(m_a-m_b)^2 X^a{}_b\bar{X}^b_a - N_{\mathsf{c}}(m_a-m_b)^2 Z^a{}_b\bar{Z}^b_a.\end{aligned}}$$

Planar amplitudes in 1-to-1 correspondence with Yangian-invariant massive Feynman integrals (integrability).

Integrability on the Coulomb Branch?

Integrable

$$\begin{tikzcd}
\text{Planar } \mathcal{N} = 4 \text{ SYM} \arrow[r, "\text{VEV}"] \arrow[d, "\gamma\text{-deformation} + \text{double scaling limit}"'] & \text{Coulomb Branch } \mathcal{N} = 4 \text{ SYM} \arrow[d, "\gamma\text{-deformation} +\text{double scaling limit}"] \\
\text{Fishnet Theory} & \text{Massive Fishnet Theory}
\end{tikzcd}$$

Integrable [Gürdoğan Kazakov 2015]

[FL, Miczajka 2020] Integrable

Summary and Directions

- Surprising integrability of massive Feynman integrals.
- Massive generalization of momentum space conformal symmetry.
- Leads to massive fishnet theory.
- Yangian allows to bootstrap Feynman integrals (not shown here).
- Is $\mathcal{N}=4$ SYM theory on the Coulomb branch integrable?

Summary and Directions

- Surprising integrability of massive Feynman integrals.
- Massive generalization of momentum space conformal symmetry.
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- Is $\mathcal{N}=4$ SYM theory on the Coulomb branch integrable?

Thank you!

Yangian Bootstrap

Level 0: Dependence on conformal variables $u_{ij} = \frac{x_{ij}^2+(m_i-m_j)^2}{-4m_i m_j}$

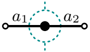
Level 1: Yangian PDEs for conformal function $\phi(u_{ij})$

Example: One-loop integrals with massive propagators for $D = \sum_j a_j$

$n = 2$: fixed by symmetries

Gauß' hypergeometric function

$$I_2 =$$

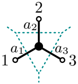


$$= \frac{\pi^{D/2}\Gamma_{D/2}}{\Gamma_D m_1^{a_1} m_2^{a_2}}\, {}_2F_1\left[{a_1, a_2 \atop (D+1)/2}; u\right].$$

$n = 3$: fixed by symmetries

Srivastava's hypergeometric function

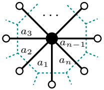
$$I_3 =$$



$$= \frac{\pi^{D/2}\Gamma_{D/2}}{\Gamma_D m_1^{a_1} m_2^{a_2} m_3^{a_3}} H_C(u, v, w).$$

$n =$generic: 'experiments' yield natural conjecture [FL, Miczajka Müller,Münkler 2020]

$$I_n =$$



$$= \frac{\pi^{D/2}\Gamma_{D/2}}{\Gamma_D \prod_{j=1}^n m_j^{a_j}} \sum_{k_{ij}=0}^{\infty} \frac{\prod_{j=1}^n (a_j)_{\sum_{\alpha \in B_{n|j}} k_\alpha}}{(\frac{D+1}{2})_{\sum_{\alpha\in B_n} k_\alpha}} \prod_{\alpha \in B_n} \frac{u_\alpha^{k_\alpha}}{k_\alpha!}.$$

Conjecture was confirmed in [Ananthanarayan, Banik Friot, Ghosh 2020]