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Non-Hermiticity: a new paradigm for model building in particle physics

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Some thanks

To **the organisers**, to **you**, and to my collaborators:

- **Jean Alexandre** (King's College London)
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- **John Ellis** (King's College London)
- **Dries Seynaeve** (former PhD student at King's College London)

Model building strategies

To go **beyond the Standard Model of particle physics**, we can:

- **Add new degrees of freedom**:
  gauge singlets, extra Higgs doublets, right-handed neutrinos, SUSY partners, hidden sectors, …
- **Relax assumptions**:
  number of spatial dimensions, Lorentz invariance, locality, CPT invariance, ...

Why not non-Hermiticity?

All Hermitian matrices have real eigenvalues …

… but **matrices with real eigenvalues need not be Hermitian.**

Enter PT-symmetric QM

- Relax Hermiticity of the Hamiltonian in favour of another **antilinear symmetry**, e.g., combined action of parity (P) and time-reversal (T).
  [Bender & Boettcher '98]
- PT symmetry is sufficient to give **real eigenvalues** and **unitary evolution**.
  [Bender & Boettcher '98, Bender, Brody & Jones '02]

For a recent review, see Bender, PT Symmetry in Quantum and Classical Physics, World Scientific '2019.

But we want to do **non-Hermitian quantum field theory** …

Scalar model

A simple **scalar model** with ***c*-number Lagrangian** involving **real parameters**:

[Alexandre, PM & Seynaeve '17]

$$\mathcal{L} = \partial_\nu \phi_i^* \partial^\nu \phi_i - m_i^2 \phi_i^* \phi_i - \mu^2 (\phi_1^* \phi_2 - \phi_2^* \phi_1) \neq \mathcal{L}^* \qquad i = 1,2$$

"Naïve" $\mathcal{PT}$ symmetry if we have a complex scalar and a complex pseudoscalar, i.e.,

$$\mathcal{P}\colon\ \phi_1 \longrightarrow +\phi_1, \qquad \phi_2 \longrightarrow -\phi_2$$
$$\mathcal{T}\colon\ \phi_1 \longrightarrow +\phi_1^*, \qquad \phi_2 \longrightarrow +\phi_2^*$$

Matrix model: eigenvalues

Non-Hermitian squared mass matrix: $M^2 = \begin{pmatrix} m_1^2 & \mu^2 \\ -\mu^2 & m_2^2 \end{pmatrix}$

With the $\mathcal{PT}$ symmetry of $\mathcal{L}$ translating into the **pseudo-Hermiticity** of M^2:

$$[P \cdot M^2 \cdot P]_{ij} = [M^2]_{ji} \qquad P = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$$

Mass spectrum: $M_\pm^2 = \frac{m_1^2+m_2^2}{2} \pm \left[\left(\frac{m_1^2-m_2^2}{2}\right)^2 - \mu^4\right]^{1/2} \in \mathbb{R}$ if $\eta \equiv \frac{2|\mu^2|}{|m_1^2-m_2^2|} \leq 1$

M^2 is defective at the **exceptional point** at $\eta = 1$.

Matrix model: eigenvectors

Eigenvectors $(m_1^2 - m_2^2, \mu^2 > 0)$:

$$\boldsymbol{e}_+ = N\begin{pmatrix} \eta \\ -1 + \sqrt{1-\eta^2} \end{pmatrix} \qquad \boldsymbol{e}_- = N\begin{pmatrix} -1 + \sqrt{1-\eta^2} \\ \eta \end{pmatrix} \qquad \eta = \frac{2\mu^2}{m_1^2 - m_2^2}$$

- **Not** orthogonal with respect to the Dirac inner product: $\boldsymbol{e}_\pm^* \cdot \boldsymbol{e}_\mp \neq 0$.
- **Not** orthonormal with respect to the $\mathcal{PT}$ inner product: $\boldsymbol{e}_\pm^* \cdot P \cdot \boldsymbol{e}_\pm \ngtr 0$.
- Orthonormal with respect to the $\mathcal{C}'\mathcal{PT}$ inner product: $\boldsymbol{e}_\pm^* \cdot C' \cdot P \cdot \boldsymbol{e}_\pm = 1$.

  [Bender, Brody & Jones ’02; see also Alexandre, Ellis & PM ’20b; cf. Mannheim ‘18]

$$C' = R \cdot P \cdot R^{-1} = \frac{1}{\sqrt{1-\eta^2}}\begin{pmatrix} 1 & -\eta \\ \eta & -1 \end{pmatrix} \qquad R \cdot M^2 \cdot R^{-1} = \widehat{M}^2 = \begin{pmatrix} M_+^2 & 0 \\ 0 & M_-^2 \end{pmatrix}$$

Variational procedure

The action is not Hermitian, so

$$\frac{\partial \mathcal{L}}{\partial \phi_i^*} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi_i^*} = 0 \;\not\Leftrightarrow\; \frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi_i} = 0$$

Prescription: choose one of these pairs of **Euler-Lagrange equations** to fix the dynamics.

The choices are **physically equivalent**; they are equivalent up to a field redefinition …

[Alexandre, PM & Seynaeve '17]

… but this isn't really satisfactory …

Noether's theorem

If the **Euler-Lagrange equations** are not mutually consistent, conserved currents **do not** correspond to symmetries of the Lagrangian.

$$\delta\mathcal{L} = \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\nu\frac{\partial\mathcal{L}}{\partial\partial_\nu\phi_i}\right)}_{\neq 0}\delta\phi_i + \delta\phi_i^*\underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi_i^*} - \partial_\nu\frac{\partial\mathcal{L}}{\partial\partial_\nu\phi_i^*}\right)}_{= 0} + \partial_\nu j_\delta^\nu$$

The current is conserved if

[Alexandre, PM & Seynaeve '17]

$$\delta\mathcal{L} = \left(\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\nu\frac{\partial\mathcal{L}}{\partial\partial_\nu\phi_i}\right)\delta\phi_i$$

Scalar model: conserved current

For our scalar model, the **conserved current** is

[Alexandre, PM & Seynaeve '17]

$$j^{\nu} = i[\phi_1^* \partial^{\nu} \phi_1 - (\partial^{\nu} \phi_1^*)\phi_1] - i\,[\phi_2^* \partial^{\nu} \phi_2 - (\partial^{\nu} \phi_2^*)\phi_2]$$

corresponding to ($U(1) \times U(1)$)

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \longrightarrow e^{i\alpha P} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} e^{+i\alpha} \phi_1 \\ e^{-i\alpha} \phi_2 \end{pmatrix}$$

with

$$\mathcal{L} \longrightarrow \partial_{\nu} \phi_i^* \partial^{\nu} \phi_i - m_i^2 \phi_i^* \phi_i - \mu^2 \big(e^{-2i\alpha} \phi_1^* \phi_2 - e^{+2i\alpha} \phi_2^* \phi_1 \big)$$

$$\delta\mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial \phi_j} - \partial_{\nu} \frac{\partial \mathcal{L}}{\partial \partial_{\nu} \phi_j} \right) \delta\phi_j = 2i\alpha\mu^2 (\phi_1^* \phi_2 - \phi_2^* \phi_1)$$

The Goldstone theorem

The existence of the conserved current is sufficient to ensure **the Goldstone theorem** continues to hold in the case of a **spontaneously broken global symmetry**:

[Alexandre, Ellis, PM & Seynaeve '18]

$$\mathcal{L} = \partial_\nu \phi_i^* \partial^\nu \phi_i + m_1^2 |\phi_1|^2 - m_2^2 |\phi_2|^2 - \mu^2 (\phi_1^* \phi_2 - \phi_2^* \phi_1) - \frac{g}{4} |\phi_1|^4, \qquad m_1^2, m_2^2 > 0$$

$$\left.\begin{aligned} \left.\frac{\partial U}{\partial \phi_1^*}\right|_{\phi_a = v_a} &= \frac{g}{2} |v_1|^2 v_1 - m_1^2 v_1 + \mu^2 v_2 = 0 \\ \left.\frac{\partial U}{\partial \phi_2^*}\right|_{\phi_a = v_a} &= m_2^2 v_2 - \mu^2 v_1 = 0 \end{aligned}\right\} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \sqrt{2 \frac{m_1^2 m_2^2 - \mu^4}{g m_2^2}} \begin{pmatrix} 1 \\ \frac{\mu^2}{m_2^2} \end{pmatrix} e^{i\alpha}$$

Away from the exceptional point, we have a **single massless, Goldstone mode**.

The Englert-Brout-Higgs mechanism

Gauging the global $U(1)$ symmetry (which is itself subtle):

$$\mathcal{L} = -\frac{1}{4}F_{\nu\rho}F^{\nu\rho} + D_\nu^*\phi_i^* D^\nu \phi_i + m_1^2|\phi_1|^2 - m_2^2|\phi_2|^2 - \mu^2(\phi_1^*\phi_2 - \phi_2^*\phi_1) - \frac{g}{4}|\phi_1|^4$$

$$D_\nu = \partial_\nu - iqA_\nu$$

The **Englert-Brout-Higgs mechanism** is still borne out …

… all the way to the exceptional point:

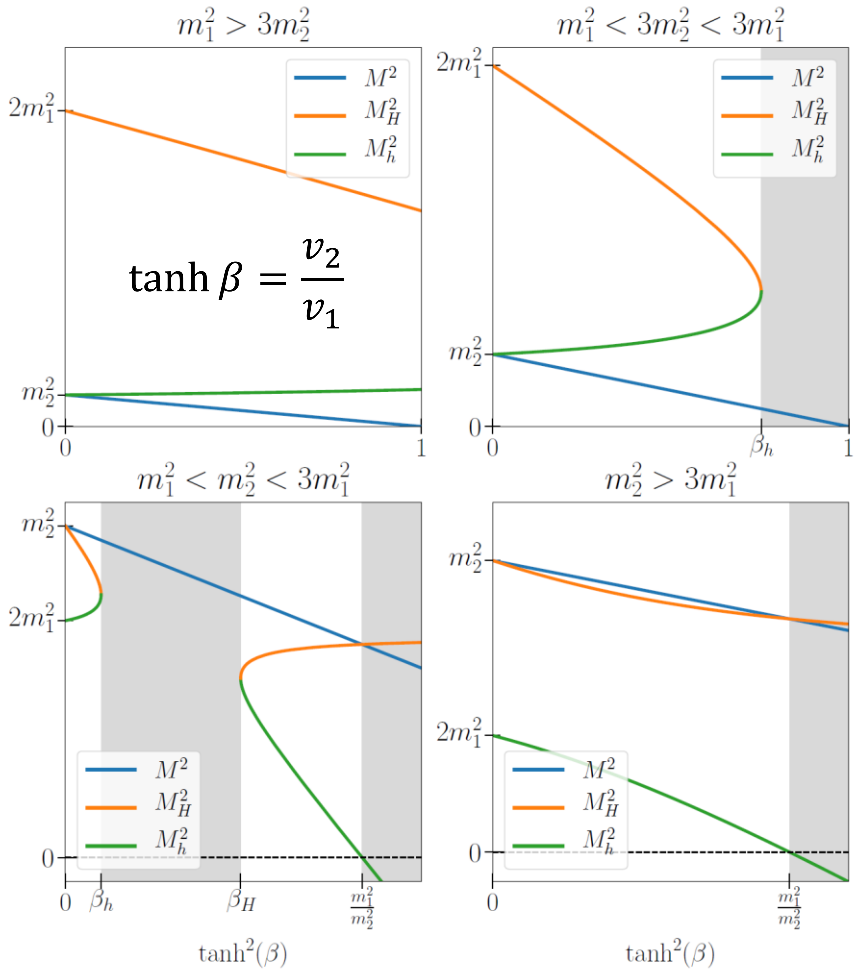
[Alexandre, Ellis, PM & Seynaeve ’19 & ‘20]

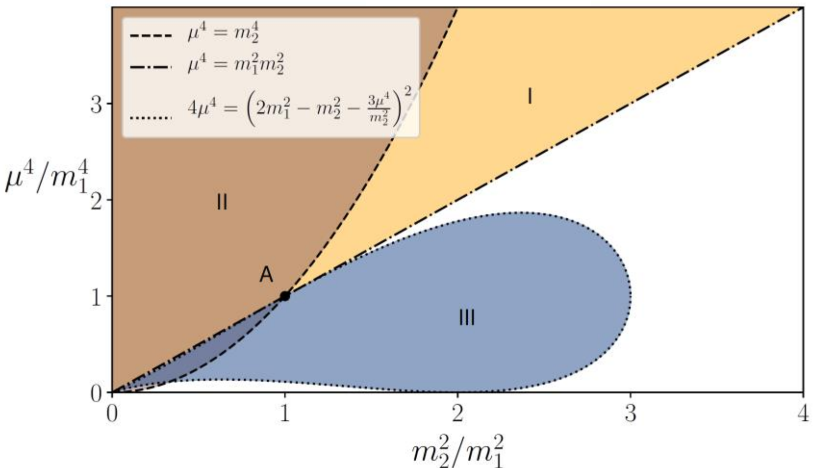
$$M_A^2 = 2q^2\left(|v_1|^2 + |v_2|^2\right) \longrightarrow 2q^2\left(|v_1|^2 + |v_1|^2\right) = \frac{4}{g}\left(m_1^2 - m_2^2\right) \text{ at } \mu^2 = \pm m_2^2$$

The $\mathcal{C}'\mathcal{PT}$ norm of **the Goldstone mode** is ill defined, but its Hermitian norm is defined.

[cf. Mannheim ’19, Fring & Taira ’20a, ‘20b & ’20c, based on a similarity transformation of this model, for which $|v_2|^2 \longrightarrow -|v_2|^2$.]

(Non-Abelian) Englert-Brout-Higgs mechanism I

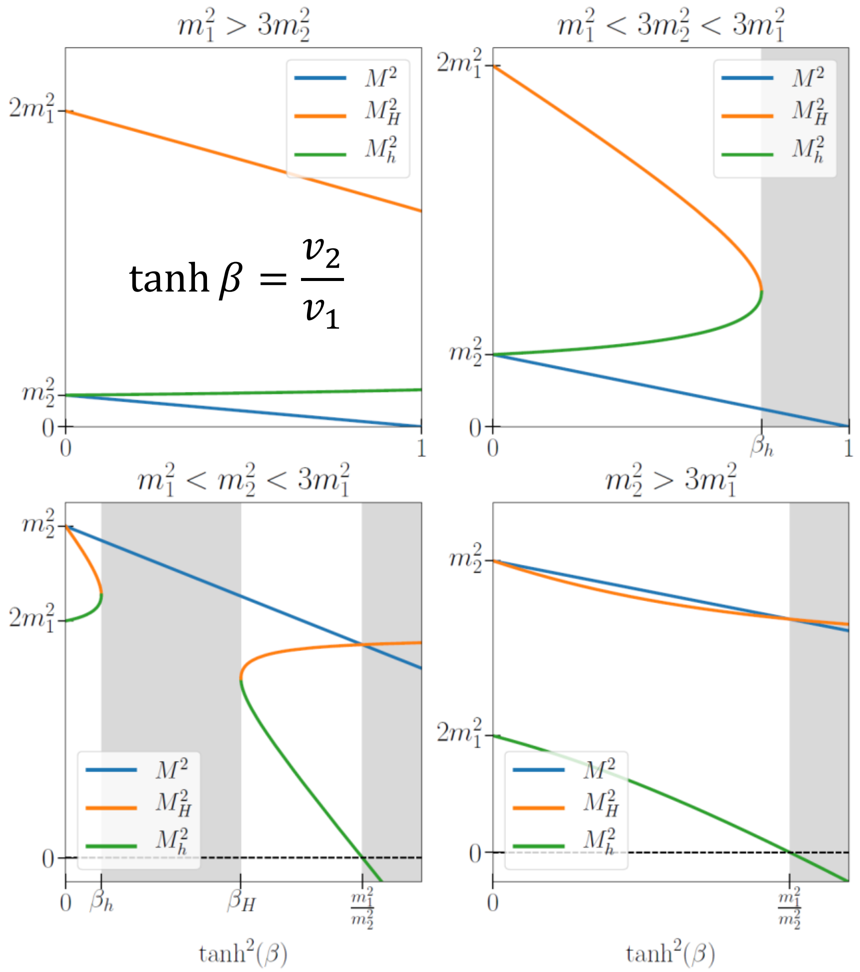




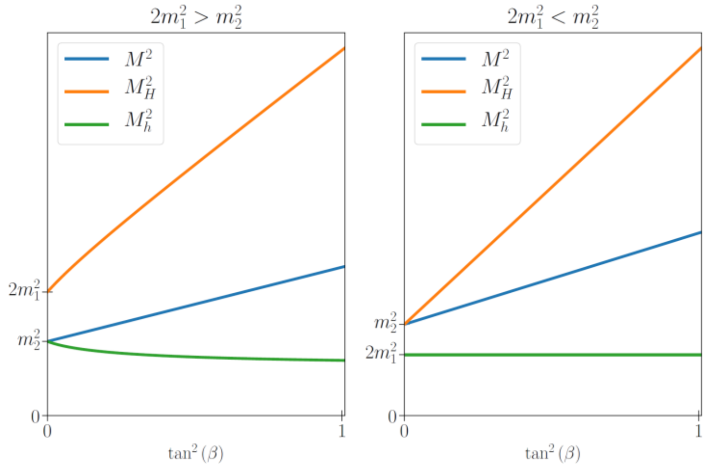
I: symmetric $SU(2) \times U(1)$ phase
II: $\mathcal{PT}$ broken phase ($M^2 < 0$ for $H^\pm, D$)
III: $\mathcal{PT}$ broken phase ($M_h^2, M_H^2 \notin \mathbb{R}$ for h, H)
Unshaded: physical SSB phase

[Alexandre, Ellis, PM & Seynaeve '20]

(Non-Abelian) Englert-Brout-Higgs mechanism II



$$\tanh\beta = \frac{v_2}{v_1}$$



cf. Hermitian case

[Alexandre, Ellis, PM & Seynaeve '20]

Canonical variables

$\Phi \equiv (\phi_1, \phi_2)$ evolves with H

$\Leftrightarrow$

$\dot{\Phi}^\dagger \equiv \left(\dot{\phi}_1, \dot{\phi}_2\right)^\dagger$ evolves with $H^\dagger \neq H$

But canonical variables must both evolve with **the same** H (or $H^\dagger$)!

Variational procedure revisited

The **self-consistent non-Hermitian deformation** (with the same EoMs) is
[Alexandre, Ellis & PM ’20b]

$$\mathcal{L} = \partial_\nu \tilde{\phi}_i^* \partial^\nu \phi_i - m_i^2 \tilde{\phi}_i^* \phi_i - \mu^2 \left(\tilde{\phi}_1^* \phi_2 - \tilde{\phi}_2^* \phi_1 \right)$$

The Euler-Lagrange equations are now mutually consistent:

$$\frac{\partial \mathcal{L}}{\partial \tilde{\phi}_i^*} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\nu \tilde{\phi}_i^*} = 0 \iff \frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi_i} = 0$$

The **tilde-conjugated fields** are defined via

$$\mathcal{P}\colon\ P_{ij} \tilde{\phi}_j(t, \mathcal{P}\boldsymbol{x}) = \mathcal{P} \phi_i(t, \boldsymbol{x}) \mathcal{P}^{-1}$$

But there is still a choice in the equations of motion.

Second quantisation goes through but we need two types of operator.

What about fermions?

An example: non-Hermitian extension of the **Dirac theory:**

[Bender, Jones & Rivers ’05; Alexandre, Bender & PM ‘15; Alexandre, PM & Seynaeve ‘17]

$$\mathcal{L} = \bar{\psi}\left(i\gamma^{\nu}\partial_{\nu} - m - \mu\gamma^{5}\right)\psi, \qquad \gamma^{5} = \left(\gamma^{5}\right)^{\dagger}$$

Eigenmasses: $M^2 = m^2 - \mu^2$

The **conserved current** is [Alexandre & Bender ‘15]

$$j^{\nu} = \bar{\psi}\gamma^{\nu}\left(1 + \frac{\mu}{m}\gamma^{5}\right)\psi = \left(1 - \frac{\mu}{m}\right)\psi_L^{\dagger}\bar{\sigma}^{\nu}\psi_L + \left(1 + \frac{\mu}{m}\right)\psi_R^{\dagger}\bar{\sigma}^{\nu}\psi_R$$

corresponding to

$$\psi \longrightarrow \psi' = \exp\left[i\alpha\left(1 + \frac{\mu}{m}\gamma^{5}\right)\right]\psi$$

with

$$\delta\mathcal{L} = -2\mu\bar{\psi}\gamma^{5}\delta\psi \neq 0$$

Unique phenomenology: exceptional points

- **Exceptional points** $\mu = +(-)m \implies$ a **massless theory** of right(left) chiral Weyl fermions.
  [Alexandre, Bender & PM '15, cf. Chernodub '17]
- Gauging this model, the full **vector plus axial vector gauge symmetry** is recovered at the **exceptional points**.
  [Alexandre, Bender & PM '15; Alexandre, PM & Seynaeve '17]
- Massless fermions can undergo **flavour oscillations**.
  [Jones-Smith & Mathur '14]
- The same model can be obtained from a **non-Hermitian Higgs-Yukawa theory**.
  [Alexandre, Bender & PM '15 & '17; see also Alexandre & Mavromatos '20]

$$\mathcal{L}_{\text{Yuk}} = -y_- \bar{L}_L \widetilde{H} \nu_R - y_+ \bar{\nu}_R \widetilde{H}^\dagger L_L$$

- A non-Hermitian explanation for the smallness of the **light neutrino masses**?

SUSY embedding?

Two $\mathcal{N} = 1$ scalar chiral superfields: Φ_1, Φ_2

Superpotential: [Alexandre, Ellis & PM '20a]

$$W_\pm = \frac{1}{2} m_{11} \Phi_1^2 \mp m_{12} \Phi_1 \Phi_2 + \frac{1}{2} m_{22} \Phi_2^2, \qquad \mathcal{L} = \mathcal{L}_K + \int d^2\theta \; W_+ + \int d^2\theta^\dagger \, W_-^\dagger \neq \mathcal{L}^\dagger$$

On-shell (two complex scalars, two Majorana fermions, $a = 1{,}2$):

$$\mathcal{L}_{\mathrm{scal}} = \partial_\nu \phi_a^* \partial^\nu \phi_a - \left(m_{aa}^2 - m_{12}^2\right) \phi_a^* \phi_a - m_{12}(m_{22} - m_{11})(\phi_1^* \phi_2 - \phi_2^* \phi_1)$$

$$\mathcal{L}_{\mathrm{ferm}} = \frac{1}{2} \bar{\psi}_a i \gamma^\nu \partial_\nu \psi_a - \frac{1}{2} m_{aa} \bar{\psi}_a \psi_a - \frac{1}{2} m_{12} \left(\bar{\psi}_1 \gamma^5 \psi_2 + \bar{\psi}_2 \gamma^5 \psi_1\right), \qquad \gamma^5 = \left(\gamma^5\right)^\dagger$$

But $M^2_{\mathrm{scal},\pm} \neq M^2_{\mathrm{ferm},\pm} \;\Rightarrow$ supersymmetry breaking!

Closing remarks

- **Non-Hermiticity** may provide new avenues for **model building in particle physics.**
- **Noether's theorem**, **Goldstone theorem** and **Englert-Brout-Higgs mechanism** borne out.
- **Exceptional points** and **parametric dependence** very different to similar Hermitian models.
- **Second quantisation** of non-Hermitian field theories is subtle; what about **fermions** and **gauge fields**? [Alexandre, Ellis & PM in prep].
- Potential implications for the **neutrino sector; CP violation?** [Dale, Mason & PM in prep]
- A new possibility for **SUSY breaking.**

Thank you and stay well

Questions or **comments**?

Message me on **twitter @pwmillington** or **email** me at **p.millington@nottingham.ac.uk**.

References

Please see references therein.

Alexandre J, Bender C M 2015 *Foldy-Wouthuysen transformation for non-Hermitian Hamiltonians*, J Phys A **48** 185403 [arXiv:1501.01232]

Alexandre J, Bender C M, Millington P 2017 *Light neutrino masses from a non-Hermitian Yukawa theory*, J Phys: Conf Ser **873** 012047 [arXiv:1703.05251]

Alexandre J, Bender C M, Millington P 2015 *Non-Hermitian extension of gauge theories and implications for neutrino physics*, J High Energy Phys JHEP11(2015)111 [arXiv:1509.01203]

Alexandre J, Ellis J, Millington P 2020 *Discrete spacetime symmetries and particle mixing in non-Hermitian scalar quantum field theories*, Phys Rev D **102** 125030 [arXiv:2006.06656]

Alexandre J, Ellis J, Millington P 2020 *PT-symmetric non-Hermitian quantum field theories with supersymmetry*, Phys Rev D **101** 085015 [arXiv:2001.11996]

Alexandre J, Ellis J, Millington P, Seynaeve D 2020 *Spontaneously breaking non-Abelian gauge symmetry in non-Hermitian field theories*, Phys Rev D **101** 035008 [arXiv:1910.03985]

Alexandre J, Ellis J, Millington P, Seynaeve D 2019 *Gauge invariance and the Englert-Brout-Higgs mechanism in non-Hermitian field theories*, Phys Rev D **99** 075024 [arXiv:1808.00944]

Alexandre J, Ellis J, Millington P, Seynaeve D 2018 *Spontaneous symmetry breaking and the Goldstone theorem in non-Hermitian field theories*, Phys Rev D **98** 045001 [arXiv:1805.06380]

Alexandre J, Mavromatos N E 2020 On the consistency of a non-Hermitian Yukawa interaction, Phys Lett B **807** 135562 [arXiv:2004.03669]

Alexandre J, Millington P, Seynaeve D 2017 *Symmetries and conservation laws in non-Hermitian field theories*, Phys Rev D **96** 065027 [arXiv:1707.01057]

Bender C M, Boettcher S 1998 *Real spectra in non-Hermitian Hamiltonians having PT symmetry*, Phys Rev Lett **80** 5243 [arXiv:physics/9712001]

Bender C M, Brody D, Jones H F 2002 Complex extension of quantum mechanics, Phys Rev Lett **92** 119902 [arXiv:quant-ph/0208076]

Bender C M, Jones H F, Rivers R J 2005 *Dual PT-symmetric quantum field theories*, Phys Lett B **625** 333 [arXiv:hep-th/0508105]

Chernodub M 2017 *The Nielsen-Ninomiya theorem, PT-invariant non-Hermiticity and single 8-shaped Dirac cone*, J Phys A **50** 385001 [arXiv:1701.07426]

Fring A, Taira T 2020 *Massive gauge particles versus Goldstone bosons in non-Hermitian non-Abelian gauge theory*, preprint arXiv:2004.00723

Fring A, Taira T 2020 *Pseudo-Hermitian approach to Goldstone's theorem in non-Abelian non-Hermitian quantum field theories*, Phys Rev D **101** 045014 [arXiv:1911.01405]

Fring A, Taira T 2020 *Goldstone bosons in different PT-regimes of non-Hermitian scalar quantum field theories*, Nucl Phys B **950** 114834 [arXiv:1906.05738]

Jones-Smith K, Mathur H 2014 *Relativistic non-Hermitian quantum mechanics*, Phys Rev D **89** 125014 [arXiv:0908.4257]

Mannheim P D 2019 *Goldstone bosons and the Englert-Brout-Higgs mechanism in non-Hermitian theories*, Phys Rev D **99** 045006 [arXiv:1808.00437]

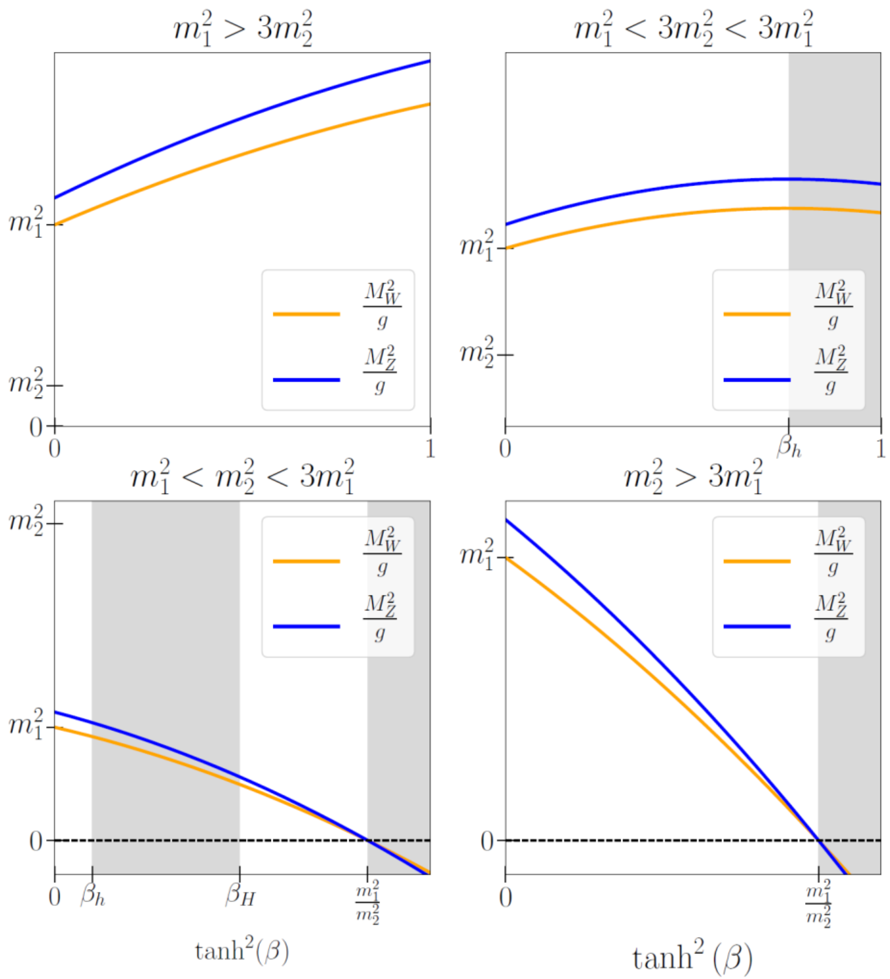
Mannheim P D 2018 *Appropriate Inner Product for PT-Symmetric Hamiltonians*, Phys Rev D **97** 045001 [arXiv:1708.01247]

Ohlsson T, Zhou S 2020 *Transition probabilities in the two-level quantum system with PT-symmetric non-Hermitian Hamiltonians*, J Math Phys **61** 052104 [arXiv:1906.01567]

Ohlsson T, Zhou S 2021 *Density matrix formalism for PT-symmetric non-Hermitian Hamiltonians with the Lindblad equation*, Phys Rev A **103** 022218 [arXiv:2006.02445]

Back up slides

(Non-Abelian) Englert-Brout-Higgs mechanism III



[Alexandre, Ellis, PM & Seynaeve '20]

Second quantisation I

In the flavour basis, the matrix-valued energy is non-Hermitian, and we need:

[see Alexandre, Ellis & PM ’20b, also for full details of how the discrete symmetry properties $\mathcal{C}, \mathcal{C}', \mathcal{P}, \mathcal{T}$ are borne out.]

$$\hat{\phi}_i(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} [2E(\boldsymbol{p})]_{ij}^{-1/2} \left[\left(e^{-ip\cdot x}\right)_{jk} \hat{a}_{k,\boldsymbol{p}}(0) + \left(e^{ip\cdot x}\right)_{jk} \check{c}_{k,\boldsymbol{p}}^{\dagger}(0)\right]$$

$$\check{\phi}_i^{\dagger}(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} [2E(\boldsymbol{p})]_{ji}^{-1/2} \left[\left(e^{-ip\cdot x}\right)_{kj} \hat{c}_{k,\boldsymbol{p}}(0) + \left(e^{ip\cdot x}\right)_{kj} \check{a}_{k,\boldsymbol{p}}^{\dagger}(0)\right]$$

The **hatted** (ˆ) and **checked** (ˇ) **fields** are related via **parity**:

$$P_{ij}\check{\phi}_j(\mathcal{P}x) = \hat{\mathcal{P}}\hat{\phi}_i(x)\hat{\mathcal{P}}^{-1}$$

The **second-quantised Lagrangian** is

$$\hat{\mathcal{L}} = \partial_\nu \check{\phi}_i^{\dagger} \partial^\nu \hat{\phi}_i - m_i^2 \check{\phi}_i^{\dagger} \hat{\phi}_i - \mu^2 \left(\check{\phi}_1^{\dagger} \hat{\phi}_2 - \check{\phi}_2^{\dagger} \hat{\phi}_1\right)$$

Second quantisation II

Why this doubling?

- To ensure a **consistent variational procedure**, and to construct **canonical conjugate variables**.
- In terms of the **mass "eigenfields"** or the **similarity transformation** to the Hermitian "frame":

$$\hat{\xi}_i = R_{ij}\hat{\phi}_j \not\Leftrightarrow \hat{\xi}_i^\dagger = \hat{\phi}_j^\dagger R_{ji}^{-1}$$

instead:

$$\hat{\xi}_i = R_{ij}\hat{\phi}_j \Leftrightarrow \hat{\xi}_i^\dagger = \check{\phi}_i^\dagger R_{ji}^{-1}$$

Flavour oscillations I

Mass eigenstates: $|\boldsymbol{p}, +(-), t\rangle, \quad (|\boldsymbol{p}, +(-), t\rangle)^{\S}, \quad \S \equiv \mathcal{C}'\mathcal{P}\mathcal{T} \circ \mathrm{T}$

Flavour eigenstates: [Alexandre, Ellis & PM '20b]

$$|\breve{\boldsymbol{p}}, 1(2), t\rangle = N\left\{\eta|\boldsymbol{p}, +(-), t\rangle - \left[1 - \sqrt{1-\eta^2}\right]|\boldsymbol{p}, -(+), t\rangle\right\}$$

$$\langle\widehat{\boldsymbol{p}}, 1(2), t| = N\left\{\eta(|\boldsymbol{p}, +(-), t\rangle)^{\S} + \left[1 - \sqrt{1-\eta^2}\right](|\boldsymbol{p}, -(+), t\rangle)^{\S}\right\}$$

Orthonormality:

$$\langle\widehat{\boldsymbol{p}}, i, t|\breve{\boldsymbol{p}}', j, t\rangle = (2\pi)^3\delta_{ij}\delta^3(\boldsymbol{p} - \boldsymbol{p}')$$

$$(|\boldsymbol{p}, \pm, t\rangle)^{\S}|\boldsymbol{p}', \pm, t\rangle = (2\pi)^3\delta^3(\boldsymbol{p} - \boldsymbol{p}')$$

$$(|\boldsymbol{p}, \pm, t\rangle)^{\S}|\boldsymbol{p}', \mp, t\rangle = 0$$

Flavour oscillations II

Transition "probability":

$$\Pi_{i\to j}(t) = \frac{1}{V}\int \frac{\mathrm{d}^3\boldsymbol{p}'}{(2\pi)^3}\langle\hat{\boldsymbol{p}}, j, t|\breve{\boldsymbol{p}}', i, 0\rangle\langle\hat{\boldsymbol{p}}', i, 0|\breve{\boldsymbol{p}}, j, t\rangle, \qquad V \equiv (2\pi)^3\delta^3(\boldsymbol{0})$$

Oscillation and **"survival probabilities":** [Alexandre, Ellis & PM '20b]

$$\Pi_{1(2)\to 2(1)}(t) = -\frac{\eta^2}{1-\eta^2}\sin^2\left[\frac{1}{2}\big(E_+(\boldsymbol{p}) - E_-(\boldsymbol{p})\big)t\right] \notin [0,1]$$

$$\Pi_{1(2)\to 1(2)}(t) = 1 + \frac{\eta^2}{1-\eta^2}\sin^2\left[\frac{1}{2}\big(E_+(\boldsymbol{p}) - E_-(\boldsymbol{p})\big)t\right] \notin [0,1]$$

Unitarity: $\Pi_{1(2)\to 1(2)}(t) + \Pi_{1(2)\to 2(1)}(t) = 1$

[cf. the similar issue found in Ohlsson & Zhou '20 & '21]

Flavour oscillations III

Resolution: experimental **observables** are **scattering matrix elements**.

[Alexandre, Ellis & PM '20b]

We must source states consistent with the $\mathcal{PT}$ symmetry:

$$\mathcal{L}_{\rm int} = J_A \check{\phi}_1^\dagger + J_A^\dagger \hat{\phi}_1 - J_B \check{\phi}_2^\dagger + J_B^\dagger \hat{\phi}_2$$

"Squared" matrix element for $A \longrightarrow B$:

$$\mathcal{M}_{A\longrightarrow B}^{\mathcal{C}'\mathcal{PT}} \mathcal{M}_{A\longrightarrow B} = VT(2\pi)^4 \delta^4(p_A - p_B) \frac{\mu^4}{\left(p_A^2 - M_+^2\right)^2 \left(p_A^2 - M_-^2\right)^2} > 0$$

remaining real and perturbative all the way up to the exceptional point ($M_+^2 = M_-^2$).