Renormalization of the flavor-singlet axial-vector current and its anomaly at N^3LO in QCD

Long Chen

Institute for Theoretical Particle Physics and Cosmology,
RWTH Aachen University

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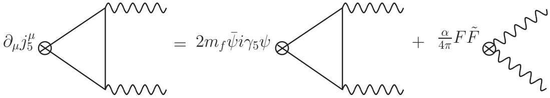
with T. Ahmed and M. Czakon, and work in progress

The Adler-Bell-Jackiw anomaly

The anomalous axial-vector divergence equation [Adler 69; Bell, Jackiw 69]

$$\partial_\mu \bar{\psi}\,\gamma^\mu \gamma_5\,\psi = 2m_f \bar{\psi}\, i\gamma_5\,\psi - \frac{\alpha}{4\pi}\,\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}\,.$$

Diagrammatically,

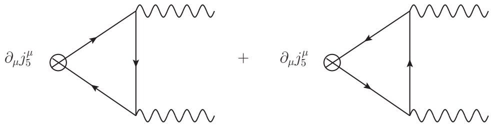


The Adler-Bardeen theorem [Adler, Bardeen 69] : “one-loop” exact

- *gauge/internal anomalies* must cancel !
  - The Standard Model is *anomaly free*
  - Anomaly matching ['t Hooft et al. 80]
- *global/external anomalies* are allowed and important
  - $\pi \to \gamma\gamma$ decay [Steinberger 49; Sutherland, Veltman 67; Adler 69; Bell, Jackiw 69]
  - $U(1)_A/\eta'$ problem [Weinberg 75; 't Hooft 76]
  - Strong CP problem and Axion [Peccei, Quinn 77] ...

Calculating the axial anomaly in DR: the γ_5 issue

The axial anomaly



"vanishes" with *translational invariant loop integrals* and an *anticommuting* γ_5.

In addition,

> A fully anticommuting γ_5 is algebraically incompatible with the Dirac algebra in a general $D \neq 4$ dimensions.

Within the Dimensional Regularization, two classes of γ_5 prescriptions:

- A non-anticommuting γ_5 (*constructively* given)
  ['t Hooft,Veltman 72; Breitenlohner,Maison 77; Larin,Vermaseren 91 ...]
- An anticommuting γ_5 (with a careful re-definition of "γ_5-trace")
  [Bardeen 72, Chanowitz et al. 79; Kreimer 90; Zerf 20 ...]

The γ_5 prescription in use

The HV/BM [72,79] prescription of γ_5 in dimensional regularization:

$$\gamma_5 = \frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$$

$$\gamma_\mu\gamma_5 \to \frac{1}{2}\left(\gamma_\mu\gamma_5 - \gamma_5\gamma_\mu\right) = \frac{i}{6}\varepsilon_{\mu\nu\rho\sigma}\gamma^\nu\gamma^\rho\gamma^\sigma\,,$$

where the $\epsilon^{\mu\nu\rho\sigma}$ is treated outside the R-operation formally in D dimensions [Larin, Vermaseren 91; Zijlstra, Neerven 92] (conveniently called Larin's prescription).

The γ_5 **no longer anticommutes** with all γ^μ in D dimensions $\Longrightarrow$ "*spurious anomalous terms*" calling for non-trivial UV renormalization [Chanowitz et al. 79; Trueman 79; Kodaira 79; Espriu,Tarrach 82; Collins 84; Larin, Vermaseren 91; Bos 92; Larin 93, ...]

The properly renormalized singlet axial current reads

$$\begin{aligned} [J_5^\mu]_R &= Z_J\,\mu^{4-D}\,\bar{\psi}_B\,\gamma^\mu\gamma_5\,\psi_B \\ &= Z_5^f\,Z_5^{ms}\,\mu^{4-D}\,\bar{\psi}_B\,\frac{-i}{3!}\epsilon^{\mu\nu\rho\sigma}\gamma_\nu\gamma_\rho\gamma_\sigma\,\psi_B \end{aligned}$$

Form factor decomposition of the AVV amplitude

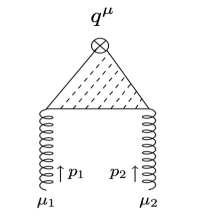
Determine UV Z_5s via computing the 2-gluon matrix elements of $\left[\partial_\mu J_5^\mu\right]_R = a_s\, n_f\, \mathrm{T}_F\, \left[F\tilde{F}\right]_R$

The 1PI AVV amplitude without external polarization vectors:

$$\Gamma^{\mu\mu_1\mu_2}_{lhs}(p_1,p_2) \equiv \int d^4x d^4y\, e^{-ip_1\cdot x - iq\cdot y}\, \langle 0|\hat{\mathrm{T}}\left[J_5^\mu(y)\, A_a^{\mu_1}(x)\, A_a^{\mu_2}(0)\right]|0\rangle|_{\mathrm{amp}}$$

Form factor decomposition:

$$\begin{aligned}\Gamma^{\mu\mu_1\mu_2}_{lhs}(p_1,p_2) &= F_1\, \epsilon^{\mu\,\mu_1\,\mu_2\,(p_2-p_1)} \\ &+ F_2\left(p_1^{\mu_1}\epsilon^{\mu\,\mu_2\,p_1\,p_2} - p_2^{\mu_2}\epsilon^{\mu\,\mu_1\,p_1\,p_2}\right) \\ &+ F_3\left(p_1^{\mu_2}\epsilon^{\mu\,\mu_1\,p_1\,p_2} - p_2^{\mu_1}\epsilon^{\mu\,\mu_2\,p_1\,p_2}\right) \\ q_\mu\Gamma^{\mu\mu_1\mu_2}_{lhs}(p_1,p_2) &= 2F_1\, \epsilon^{\mu_1\,\mu_2\,p_1\,p_2}\,.\end{aligned}$$



taking into account the odd parity and Bose symmetry w.r.t gluons ($p_1 \leftrightarrow p_2\,, \mu_1 \leftrightarrow \mu_2$).

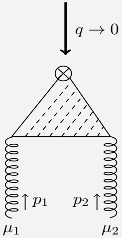
Projection at the zero momentum insertion limit

Despite $q_\mu \Gamma^{\mu\mu_1\mu_2}_{lhs}(p_1, -p_1) = 0$, the form factor F_1 is **not** zero if the p_1 is set **off-shell**.

$$\Gamma^{\mu\mu_1\mu_2}_{lhs}(p_1,-p_1) = -2F_1\, \epsilon^{\mu\,\mu_1\,\mu_2\,p_1}\,,$$

$$\mathcal{P}_{\mu\mu_1\mu_2} = -\frac{1}{6\,p_1\cdot p_1}\,\epsilon_{\mu\mu_1\mu_2\nu}\,p_1^\nu\,,$$

$$\mathcal{M}_{lhs} = \mathcal{P}_{\mu\mu_1\mu_2}\,\Gamma^{\mu\mu_1\mu_2}_{lhs}(p_1,-p_1) \propto -2F1$$



With $q = 0$ and $p_1^2 \neq 0$:

- possible IR divergences nullified owing to the *IR-rearrangement* [Vladimirov 79]
- 4-loop massless *propagator-type* master integrals available [Smirnov, Tentyukov 10, Baikov, Chetyrkin 10; Lee, Smirnov, Smirnov 11]
- gauge-dependent $\mathcal{M} \Longrightarrow$ UV renormalization of *gauge parameter* ξ !

Treatment of the operator $F\tilde{F}$

The axial-anomaly (topological-charge density) operator $F\tilde{F}$ with the *Chern-Simons current* K^μ

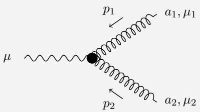
$$F\tilde{F} = \partial_\mu K^\mu = \partial_\mu \left(-4\, \epsilon^{\mu\nu\rho\sigma} \left(A^a_\nu \partial_\rho A^a_\sigma + g_s \frac{1}{3} f^{abc} A^a_\nu A^b_\rho A^c_\sigma \right) \right)$$

by the virtue of total antisymmetry of $\epsilon^{\mu\nu\rho\sigma}$ [Bardeen 74].
Unlike J^μ_5, the current K^μ is not gauge-invariant, and $[F\tilde{F}]_R = Z_{F\tilde{F}}\, [F\tilde{F}]_B + Z_{FJ}\, [\partial_\mu J^\mu_5]_B$.

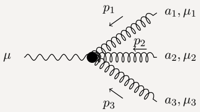
$$\Gamma^{\mu\mu_1\mu_2}_{rhs}(p_1, p_2) \equiv \int d^4x d^4y\, e^{-ip_1\cdot x - iq\cdot y} \langle 0|\hat{T}\left[K^\mu(y)\, A^{\mu_1}_a(x)\, A^{\mu_2}_a(0)\right]|0\rangle|_{amp}$$

$$\mathcal{M}_{rhs} = \mathcal{P}_{\mu\mu_1\mu_2}\, \Gamma^{\mu\mu_1\mu_2}_{rhs}(p_1, -p_1)\,.$$

The *Feynman Rules* in use:



$= 4\, \delta_{a_1a_2}\, \varepsilon_{\mu_1\mu_2\mu(p_2-p_1)}$



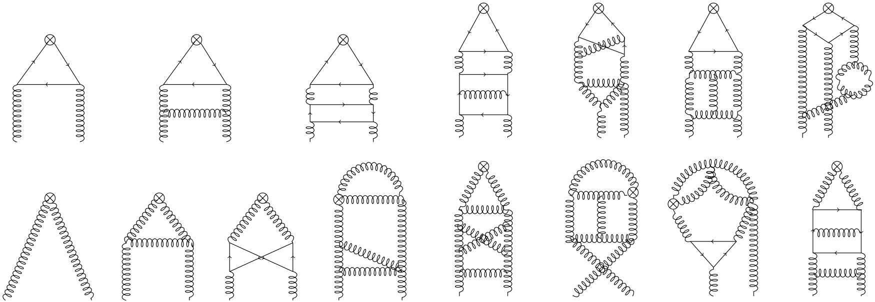
$= 8\, i\, g_s\, f^{a_1a_2a_3}\, \varepsilon_{\mu_1\mu_2\mu_3\mu}$

Feynman diagrams

The Work Flow:

- Generating Feynman diagrams
- Applying Feynman Rules, Dirac/Lorentz algebra, Color algebra
- IBP reduction of loop integrals
- Inserting Master integrals

Generator / Loop order	DiaGen		Qgraf	
	l.h.s.	r.h.s.	l.h.s.	r.h.s.
1	2	3	2	4
2	20	57	21	64
3	429	1361	447	1488
4	11302	37730	11714	40564



IBP reduction and master integrals

Loop integrals in diagrams, reduced by IBP [Tkachov 81; Chetyrkin,Tkachov 81];

Analytic results of p-master integrals, up to 4 loop [Baikov, Chetyrkin 10; Lee, Smirnov,Smirnov 12].

- DiaGen/IdSolver [Czakon] + Forcer [Ruijl,Ueda,Vermaseren]
  - **Amplitude projection**: about $3+6$ days @ 24 cores (Intel® Xeon® Silver 4116)
  - **Forcer (pre-solved IBP)**: about $12+24$ hours @ 8 cores (Intel® Xeon® E3-1275 V2)
- QGRAF [Nogueira] + FORM [Vermaseren] + Reduze 2 [Manteuffel, Studerus] + FIRE [Smirnov] combined with LiteRed [Lee]
  - **IBP (by Laporta)**: about one month @ 32 cores (Intel® Xeon® Silver 4216)
  - a few hundred GB RAM

At 4-loop: $\sim 10^5$ loop integrals in Feynman amplitudes reduced to **28** masters.

The analytical results were found to be identical between the two set-ups.

UV renormalization

$$\overline{\text{MS}}: \quad \hat{a}_s\, S_\epsilon = Z_{a_s}(\mu^2)\, a_s(\mu^2)\, \mu^{2\epsilon}\,, \quad a_s \equiv \frac{\alpha_s}{4\pi} = \frac{g_s^2}{16\pi^2}\,, \quad S_\epsilon = (4\pi)^\epsilon\, e^{-\epsilon\gamma_E}$$

By the multiplicative renormalizability of the QCD Lagrangian and of J_5^μ and K^μ:

$$\begin{aligned}
\mathcal{M}_{lhs} &= Z_J\, Z_3\, \hat{\mathcal{M}}_{lhs}\,(\hat{a}_s\,,\,\hat{\xi}) \\
&= Z_5^f\, Z_5^{ms}\, Z_3\, \hat{\mathcal{M}}_{lhs}\,(Z_{a_s}\, a_s\,,\, 1 - Z_3 + Z_3\,\xi) \equiv Z_5^f\, \bar{\mathcal{M}}_{lhs}
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_{rhs} &= Z_{F\tilde{F}}\, Z_3\, \hat{\mathcal{M}}_{rhs}\,(\hat{a}_s\,,\,\hat{\xi}) + Z_{FJ}\, Z_3\, \hat{\mathcal{M}}_{lhs}\,(\hat{a}_s\,,\,\hat{\xi}) \\
&= Z_{F\tilde{F}}\, Z_3\, \hat{\mathcal{M}}_{rhs}\,(Z_{a_s}\, a_s\,,\, 1 - Z_3 + Z_3\,\xi) + Z_{FJ}\, Z_3\, \hat{\mathcal{M}}_{lhs}\,(Z_{a_s}\, a_s\,,\, 1 - Z_3 + Z_3\,\xi)
\end{aligned}$$

where the QCD gauge-fixing parameter ξ is defined via $\frac{i}{k^2}\left(-g^{\mu\nu} + \xi\,\frac{k^\mu k^\mu}{k^2}\right)$.

- The renormalization $1 - \hat{\xi} = Z_3(1 - \xi)$ is crucial here.
- The overall wavefunction Z_3 is not necessary (for determining Z_5).

Results on $Z_J \equiv Z_5^{ms} Z_5^f$ and $Z_{F\tilde{F}}$

The Z_5^{ms} extracted from the ϵ poles of the 4-loop $\hat{\mathcal{M}}_{lhs}$:

$$
\begin{aligned}
Z_5^{ms} &= 1 + a_s^2 \Big\{ C_A C_F \Big(\frac{22}{3\epsilon}\Big) + C_F n_f \Big(\frac{5}{3\epsilon}\Big) \Big\} \\
&+ a_s^3 \Big\{ C_A^2 C_F \Big(\frac{3578}{81\epsilon} - \frac{484}{27\epsilon^2}\Big) + C_A C_F n_f \Big(\frac{149}{81\epsilon} - \frac{22}{27\epsilon^2}\Big) + C_A C_F^2 \Big(-\frac{308}{9\epsilon}\Big) \\
&+ C_F^2 n_f \Big(-\frac{22}{9\epsilon}\Big) + C_F n_f^2 \Big(\frac{20}{27\epsilon^2} + \frac{26}{81\epsilon}\Big) \Big\}.
\end{aligned}
$$

agreement with [Larin,Vermaseren, 91] was found.

The ***finite*** DR-independent Z_5^f:

$$
\begin{aligned}
Z_5^f &= 1 + a_s \Big\{ -4 C_F \Big\} + a_s^2 \Big\{ C_A C_F \Big(-\frac{107}{9}\Big) + C_F^2 \Big(22\Big) + C_F n_f \Big(\frac{31}{18}\Big) \Big\} \\
&+ a_s^3 \Big\{ C_A^2 C_F \Big(56\zeta_3 - \frac{2147}{27}\Big) + C_A C_F^2 \Big(\frac{5834}{27} - 160\zeta_3\Big) + C_A C_F n_f \Big(\frac{110}{3}\zeta_3 - \frac{133}{81}\Big) \\
&+ C_F^3 \Big(96\zeta_3 - \frac{370}{3}\Big) + C_F^2 n_f \Big(\frac{497}{54} - \frac{104}{3}\zeta_3\Big) + C_F n_f^2 \Big(\frac{316}{81}\Big) \Big\}.
\end{aligned}
$$

The first application of the new result:

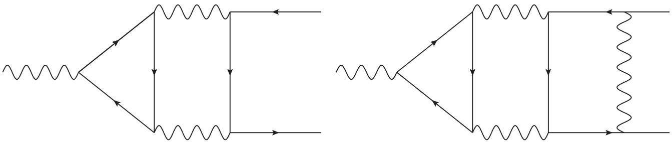
the 3-loop singlet contribution to the massless axial quark form factor [Gehrmann, Primo 21]

In addition, $Z_{F\tilde{F}} = Z_{a_s}$ is verified to 4-loop order for the first time (recently proved! [Lüscher, Weisz 21])

The singlet contributions to the axial-FF at 3 loops

QCD corrections to γ^*/W-mediated Drell-Yan (DY) processes have been recently computed to N^3LO [Duhr, Dulat, Mistlberger 20; Chen, Gehrmann, Glover, Huss, Yang, Zhu 21]

The so-called *pure-singlet* contributions, needed for Z-DYs, start from 2-loop order:



The IR-subtracted finite remainder at 3-loop order [Gehrmann, Primo 21]:

$$\begin{aligned}\mathcal{F}_{A,\mathrm{PS}}^{(3,\mathrm{finite})} = & \; C_F^2\left(\frac{409}{2} - 66L_\mu - \frac{16\pi^2}{3} - \frac{8\pi^4}{15}\right) \\ & + C_F n_l\left(\frac{469}{9} - \frac{76}{3}L_\mu + \frac{8\pi^2}{9}L_\mu + 4L_\mu^2 - \frac{40\pi^2}{27}\right) \\ & + C_F C_A\left(-\frac{7403}{18} + \frac{538}{3}L_\mu - \frac{44\pi^2}{9}L_\mu - 22L_\mu^2 + 62\zeta_3 + \frac{385\pi^2}{27} - \frac{17\pi^4}{90}\right),\end{aligned}$$

in $n_l = 5$ massless QCD (where $L_\mu = \ln(-q^2/\mu^2)$).

Contributions from the **top**-loop diagrams must be included to cancel the "unphysical" L_μ...

Summary and Outlook

- We have described a set-up for computing the renormalization constants of axial-vector currents in QCD with a non-anticommuting γ_5 in dimensional regularization.
- We have extended the result of Z_J, and in particular, of the finite non-$\overline{\mathrm{MS}}$ factor Z_5^f, of the flavor-singlet axial-vector current to $\mathcal{O}(\alpha_s^3)$.
- We have verified explicitly up to 4-loop order $Z_{F\tilde{F}} = Z_{\alpha_s}$ in the $\overline{\mathrm{MS}}$ scheme, expected based on the non-Abelian extension of the Adler-Bardeen theorem.
- Our result has found its first practical application in the computation of the 3-loop singlet contribution to the massless axial quark form factor [Gehrmann, Primo 21].
- It could be used also in places such as singlet contributions to the polarized structure functions in deep-inelastic scattering...

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THANK YOU