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The Electroweak Hamiltonian in the Gradient Flow Formalism

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The Gradient Flow

- Origin in lattice QCD
- Introduce parameter *flow time* $t \geq 0$ [Lüscher 2010]
- *Flowed fields* in $D+1$ dimensions obey differential *flow equations* like

$$\partial_t \Phi(t,x) = D_x \Phi(t,x) \quad \text{with} \quad \Phi(t,x)|_{t=0} = \phi(x)$$

- Flow equation similar to the heat equation (thermodynamics)

$$\partial_t u(t,\vec{x}) = \alpha \Delta u(t,\vec{x}) \quad \text{with} \quad \Delta = \sum_i \partial_{x_i}^2$$

- Fields at positive flow time smeared out with smearing radius $\sqrt{8t}$

⇒ Intuition: Regulates divergencies

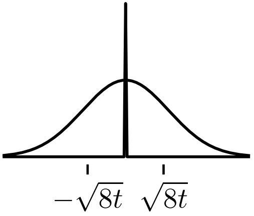

Figure: Sketch of smearing.

Gluon Flow Equation

$$\partial_t B^a_\mu = \mathcal{D}^{ab}_\nu G^b_{\nu\mu} \quad \text{with} \quad B^a_\mu(t,x)\Big|_{t=0} = A^a_\mu(x)$$

- Covariant derivative:

$$\mathcal{D}^{ab}_\mu = \delta^{ab}\partial_\mu - f^{abc}B^c_\mu$$

- Field-strength tensor:

$$G^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + f^{abc}B^b_\mu B^c_\nu$$

- Similar equations for the quarks

Applications of the Gradient Flow in Lattice QCD

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- Inherent smearing for better continuum extrapolation [Lüscher 2010]
- New strategies for scale setting [Lüscher 2010; Borsányi et al. 2012; ...]
- Composite operators do not require renormalization [Lüscher, Weisz 2011]

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⇒ No scheme matching for operators and observables in different schemes, e.g. lattice and perturbative schemes:

  - Extract parameters like α_s from lattice simulations [Fodor et al. 2012; Fritsch, Ramos 2013; ...; Harlander, Neumann 2016; ...; Artz, Harlander, **FL**, Neumann, Prausa 2019; ...]
  - *Flowed operator product expansion* [Suzuki 2013; Makino, Suzuki 2014; Monahan, Orginos 2015]:
    - Define the energy-momentum tensor of QCD on the lattice [Suzuki 2013; Makino, Suzuki 2014; Harlander, Kluth, **FL** 2018]
    - Alternative determination of vacuum polarization functions on the lattice [Harlander, **FL**, Neumann 2020]
    - Apply to electroweak Hamiltonian $\leftarrow$ this talk (see also [Suzuki, Taniguchi, Suzuki, Kanaya 2020])
    - ...

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Lagrangian

- Write Lagrangian for the gradient flow as [Lüscher, Weisz 2011; Lüscher 2013]

$$\mathcal{L} = \mathcal{L}_{\text{QCD}} + \mathcal{L}_B + \mathcal{L}_\chi,$$

$$\mathcal{L}_{\text{QCD}} = \frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} + \sum_{f=1}^{n_\mathrm{f}} \bar{\psi}_f (\not{D}^{\mathrm{F}} + m_f) \psi_f + \ldots$$

- Construct flowed Lagrangian using Lagrange multiplier fields $L^a_\mu(t,x)$:

$$\mathcal{L}_B = -2 \int_0^\infty \mathrm{d}t \, \mathrm{Tr} \left[L^a_\mu T^a \left(\partial_t B^b_\mu T^b - \mathcal{D}^{bc}_\nu G^c_{\nu\mu} T^b \right) \right]$$

- Similarly for the quarks

⇒ Flow equations automatically fulfilled

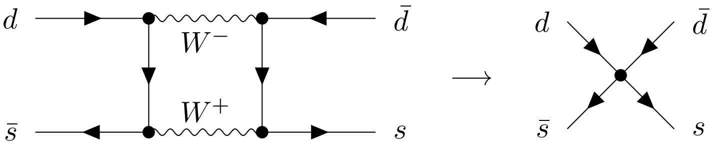
⇒ QCD Feynman rules + gradient-flow Feynman rules

The Electroweak Hamiltonian

- Effective Hamiltonian often used in flavor physics

$$\mathcal{H}_{\text{eff}} = -\frac{4G_{\text{F}}}{\sqrt{2}} V_{\text{CKM}} \sum_i C_i \mathcal{O}_i$$

- For example for $K^0 - \bar{K}^0$ mixing:



- C_i computed perturbatively
- $\langle \mathcal{O}_i \rangle$ computed on the lattice
- Problem: Scheme matching between lattice and perturbative results is a large source of uncertainty

Flowed Operator Product Expansion

- Small flow-time expansion [Lüscher, Weisz 2011]:

$$\tilde{\mathcal{O}}_i(t,x) = \sum_j \zeta_{ij}(t)\mathcal{O}_j(x) + O(t)$$

- Invert to express operators through flowed operators [Suzuki 2013; Makino, Suzuki 2014; Monahan, Orginos 2015]:

Flowed OPE

$$T = \sum_i C_i \mathcal{O}_i = \sum_{i,j} C_i \zeta_{ij}^{-1} \tilde{\mathcal{O}}_j \equiv \sum_j \tilde{C}_j \tilde{\mathcal{O}}_j$$

- T defined in regular QCD expressed through better behaved flowed operators $\tilde{\mathcal{O}}_j$, which do not require renormalization
- Can relate T in lattice and perturbative schemes without scheme transformation

Flowed OPE for the Electroweak Hamiltonian

- Write electroweak Hamiltonian as

$$\mathcal{H}_{\mathrm{eff}} = -\frac{4G_{\mathrm{F}}}{\sqrt{2}} V_{\mathrm{CKM}} \sum_i C_i \mathcal{O}_i = -\frac{4G_{\mathrm{F}}}{\sqrt{2}} V_{\mathrm{CKM}} \sum_{i,j} C_i \zeta_{ij}^{-1} \tilde{\mathcal{O}}_j$$

- Operators (without penguins for now):

$$\mathcal{O}_1 = -\left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu T^a \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu T^a \psi_{4,\mathrm{L}}\right) \quad \Rightarrow \quad \tilde{\mathcal{O}}_1 = -\mathring{Z}_\chi^2 \left(\bar{\chi}_{1,\mathrm{L}}\gamma_\mu T^a \chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_\mu T^a \chi_{4,\mathrm{L}}\right)$$

$$\mathcal{O}_2 = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu \psi_{4,\mathrm{L}}\right) \quad \Rightarrow \quad \tilde{\mathcal{O}}_2 = \mathring{Z}_\chi^2 \left(\bar{\chi}_{1,\mathrm{L}}\gamma_\mu \chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_\mu \chi_{4,\mathrm{L}}\right)$$

- Reminder: $\tilde{\mathcal{O}}_i$ do not require renormalization

$\Rightarrow$ Combine without scheme matching:
  - C_i known perturbatively through NNLO [Gorbahn, Haisch 2004]
  - ζ_{ij}^{-1} has to be computed $\leftarrow$ this talk (see also [Suzuki, Taniguchi, Suzuki, Kanaya 2020])
  - $\langle \tilde{\mathcal{O}}_j \rangle$ have to be computed on the lattice

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Method of Projectors

- Define projectors [Gorishny, Larin, Tkachov 1983; Gorishny, Larin 1987]

$$P_k[\mathcal{O}_i(x)] \equiv D_k \langle 0|\mathcal{O}_i(x)|k\rangle = \delta_{ik} + O(\alpha_s)$$

- Apply to small flow-time expansion:

$$P_k[\tilde{\mathcal{O}}_i(t,x)] = \sum_j \zeta_{ij}(t) P_k[\mathcal{O}_j(x)]$$

- $\zeta_{ij}(t)$ only depend on t

$\Rightarrow$ Set all other scales to zero

$\Rightarrow$ No perturbative corrections to $P_k[\mathcal{O}_j(x)]$, because all loop integrals are scaleless

“Master Formula”

$$\zeta_{ij}(t) = P_j[\tilde{\mathcal{O}}_i(t,x)]\Big|_{p=m=0}$$

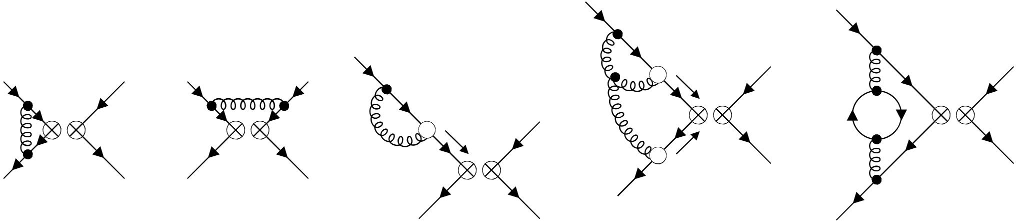
Projectors and Example Diagrams

- Projectors for $\mathcal{O}_1$ and $\mathcal{O}_2$ (schematically):

$$P_1[\mathcal{O}] = -\frac{1}{16 T_{\mathrm{R}}^2 N_{\mathrm{A}}} \mathrm{Tr}_{\mathrm{line}\,1} \mathrm{Tr}_{\mathrm{line}\,2} \, \langle 0| \left(\psi_{4,\mathrm{L}} T^b \gamma_\nu \bar{\psi}_{3,\mathrm{L}}\right) \left(\psi_{2,\mathrm{L}} T^b \gamma_\nu \bar{\psi}_{1,\mathrm{L}}\right) \mathcal{O} |0\rangle \Big|_{p=m=0},$$

$$P_2[\mathcal{O}] = \frac{1}{16 N_{\mathrm{C}}^2} \mathrm{Tr}_{\mathrm{line}\,1} \mathrm{Tr}_{\mathrm{line}\,2} \, \langle 0| \left(\psi_{4,\mathrm{L}} \gamma_\nu \bar{\psi}_{3,\mathrm{L}}\right) \left(\psi_{2,\mathrm{L}} \gamma_\nu \bar{\psi}_{1,\mathrm{L}}\right) \mathcal{O} |0\rangle \Big|_{p=m=0}$$

- Sample diagrams:



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Preliminary Results

- For $N_\mathrm{C} = 3$, $T_\mathrm{R} = \frac{1}{2}$ in the $\overline{\mathrm{MS}}$ scheme with $\mu = \frac{e^{-\gamma_\mathrm{E}/2}}{\sqrt{2t}}$:

$$\bar{\zeta}_\mathrm{R}^{-1} = \begin{pmatrix} 1 + \frac{\alpha_\mathrm{s}}{4\pi} \cdot \frac{2}{3} + \left(\frac{\alpha_\mathrm{s}}{4\pi}\right)^2 \{73.9706 - 16.7794 n_\mathrm{f}\} & -\frac{\alpha_\mathrm{s}}{4\pi} \cdot \frac{10}{3} + \left(\frac{\alpha_\mathrm{s}}{4\pi}\right)^2 \{-18.5480 + 2.52121 n_\mathrm{f}\} \\ -\frac{\alpha_\mathrm{s}}{4\pi} \cdot 15 + \left(\frac{\alpha_\mathrm{s}}{4\pi}\right)^2 \{-128.466 + 11.3454 n_\mathrm{f}\} & 1 - \frac{\alpha_\mathrm{s}}{4\pi} \cdot \frac{4}{3} + \left(\frac{\alpha_\mathrm{s}}{4\pi}\right)^2 \{54.3985 - 12.1643 n_\mathrm{f}\} \end{pmatrix}$$

- Note: algebraic expressions at $O\left(\left(\frac{\alpha_\mathrm{s}}{4\pi}\right)^2\right)$ too long for this slide
- Checks:
  - Finite after QCD + field renormalization
  - Independent of QCD gauge parameter

Conclusion and Outlook

- Gradient flow useful in lattice QCD (scale setting, smearing, non-renormalization)
- Cross-fertilization between lattice and perturbative QCD
- Apply flowed operator product expansion to the electroweak Hamiltonian to avoid scheme matching
- Status for

$$\mathcal{H}_{\text{eff}} = -\frac{4G_{\text{F}}}{\sqrt{2}} V_{\text{CKM}} \sum_{i,j} C_i \zeta_{ij}^{-1} \tilde{\mathcal{O}}_j :$$

  - C_i ✓(NNLO) [Gorbahn, Haisch 2004]
  - ζ_{ij}^{-1} (NNLO) in progress:
    - Non-trivial comparison with NLO result of [Suzuki, Taniguchi, Suzuki, Kanaya 2020] (different basis and different scheme for γ_5) to be done
    - For $\Delta F = 2$ (Kaon and B meson mixing) one operator becomes Fierz evanescent: basis transformation required
    - Extend to $\Delta F = 1$ basis (penguins!)
  - $\langle \tilde{\mathcal{O}}_j \rangle$ have to be computed on the lattice