Factorial cumulants from global baryon number conservation

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Outline

1. Introduction: QCD phase diagram, ALICE results, factorial cumulants
2. Calculation of proton-antiproton factorial cumulants
3. Exact results
4. Approximate and numerical results
5. Summary

Introduction

QCD phase diagram

Most of this is only an educated guess based on effective models.

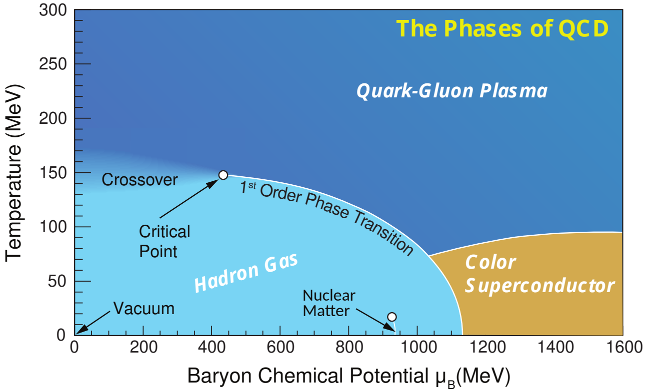


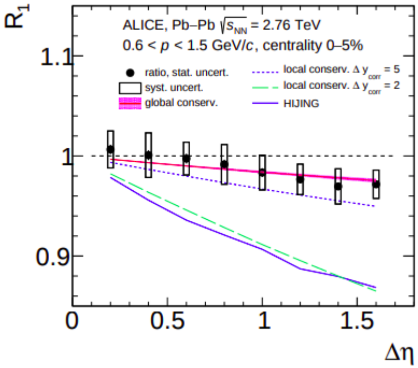
Figure: A. Bzdak, S. Esumi, V. Koch, J. Liao, M. Stephanov and N. Xu, Phys. Rept. **853**, 1-87 (2020) [arXiv:1906.00936 [nucl-th]].

Effects contributing to fluctuations

- Fluctuations may reflect the desired phase transition
- ... but they may be due to other usual effects (kind of background):
    - small fluctuations of the impact parameter (and thus the number of wounded nucleons)
    - **global baryon number conservation**

ALICE results

- "For the first time, it is shown that event-by-event baryon number conservation leads to subtle long-range correlations arising from very early interactions in the collisions."
- $R_1 = \kappa_2(n_p - n_{\bar{p}})/\langle n_p + n_{\bar{p}} \rangle$ - normalized cumulant
- S. Acharya *et al.* [ALICE], Phys. Lett. B **807**, 135564 (2020)



Factorial cumulants

- The interpretation of cumulants may be misleading:
  - they mix correlation functions of different orders
  - they can be dominated by e.g. $\langle n \rangle$.
- Factorial cumulants represent multi-particle correlation function.
- $C_2(y_1, y_2) = \varrho_2(y_1, y_2) - \varrho(y_1)\varrho(y_2)$ - two-particle density correlation function
- $\hat{C}_2 = \int dy_1 dy_2 C_2(y_1, y_2)$
- No correlations: $\hat{C}_k = 0$ $(k > 1)$
- Factorial cumulant generating function:

$$G(z) = \ln\left(\sum_{n=0}^{\infty} P(n) z^n\right)$$

- Factorial cumulants: $\hat{C}_k = \left.\frac{d^k G}{dz^k}\right|_{z=1}$

Calculation

Goal:
find the contribution of baryon number conservation in correlation functions.

MB and A. Bzdak, Phys. Rev. C **102**, no.6, 064908 (2020)

Probability distribution

- Let:
  - N_b, $\bar{N}_b$ - # baryons and antibaryons
  - B - *conserved* baryon number
- Probability distribution of observing n_p protons and $\bar{n}_p$ antiprotons is:

$$P(n_p, \bar{n}_p) = A \sum_{N_b=n_p}^{\infty} \sum_{\bar{N}_b=\bar{n}_p}^{\infty} \delta_{N_b - \bar{N}_b, B} \left[\frac{\langle N_b \rangle^{N_b}}{N_b!} e^{-\langle N_b \rangle} \right] \left[\frac{\langle \bar{N}_b \rangle^{\bar{N}_b}}{\bar{N}_b!} e^{-\langle \bar{N}_b \rangle} \right] \times \left[\frac{N_b!}{n_p!(N_b - n_p)!} p^{n_p} (1-p)^{N_b - n_p} \right] \left[\frac{\bar{N}_b!}{\bar{n}_p!(\bar{N}_b - \bar{n}_p)!} \bar{p}^{\bar{n}_p} (1-\bar{p})^{\bar{N}_b - \bar{n}_p} \right], \quad (1)$$

where

$$A = \frac{\left(\frac{\langle \bar{N}_b \rangle}{\langle N_b \rangle} \right)^{\frac{B}{2}} e^{\langle N_b \rangle + \langle \bar{N}_b \rangle}}{I_B \left(2\sqrt{\langle N_b \rangle \langle \bar{N}_b \rangle} \right)}$$ -normalization constant, $p = \langle n_p \rangle / \langle N_b \rangle$, $\bar{p} = \langle \bar{n}_p \rangle / \langle \bar{N}_b \rangle$,

I_B - modified Bessel function

baryon number conservation

baryons, antibaryons number $\sim$ Poisson (no correlations)

binomial acceptance

Factorial cumulant generating function

General formula for 2 variables:

$$G(x,\bar{x}) = \ln\left[\sum_{n_p=0}^{\infty}\sum_{\bar{n}_p=0}^{\infty} x^{n_p}\bar{x}^{\bar{n}_p} P(n_p,\bar{n}_p)\right]. \qquad (2)$$

In our case:

$$G(x,\bar{x}) = \ln\left[\left(\frac{px+1-p}{\bar{p}\bar{x}+1-\bar{p}}\right)^{\frac{B}{2}} \frac{I_B\left(2z\sqrt{(px+1-p)(\bar{p}\bar{x}+1-\bar{p})}\right)}{I_B(2z)}\right], \qquad (3)$$

where

- $z = \sqrt{\langle N_b\rangle\langle \bar{N}_b\rangle}$,
- $\langle N_b\rangle$, $\langle \bar{N}_b\rangle$ - mean numbers of baryons and antibaryons *before* baryon number conservation,
- $\langle N_b\rangle_c = z\frac{I_{B-1}(2z)}{I_B(2z)}$, $\quad \langle \bar{N}_b\rangle_c = z\frac{I_{B+1}(2z)}{I_B(2z)}$ - mean numbers of baryons and antibaryons *with* baryon number conservation.

Factorial cumulants

Having

$$G(x,\bar{x}) = \ln\left[\left(\frac{px+1-p}{\bar{p}\bar{x}+1-\bar{p}}\right)^{\frac{B}{2}} \frac{I_B\left(2z\sqrt{(px+1-p)(\bar{p}\bar{x}+1-\bar{p})}\right)}{I_B(2z)}\right],$$

we can calculate proton-antiproton factorial cumulants:

$$\hat{C}^{(n,m)} = \frac{\partial^n}{\partial x^n}\frac{\partial^m}{\partial \bar{x}^m} G(x,\bar{x})\Big|_{x=\bar{x}=1} \tag{4}$$

for n protons and m antiprotons.
Note: no correlations: $\hat{C}^{(n,m)} = 0$ (except $\hat{C}^{(1,0)}$, $\hat{C}^{(0,1)}$)

Exact results

Results

- order 1:
  - $\hat{C}^{(1,0)} = p\langle N_b\rangle_c$
- order 2:
  - $\hat{C}^{(2,0)} = -p^2\left(\langle N_b\rangle_c + \Delta\right)$
  - $\hat{C}^{(1,1)} = -p\bar{p}\Delta$
- order 3:
  - $\hat{C}^{(3,0)} = p^3\left[2!\left(\langle N_b\rangle_c + \Delta + \frac{1}{2}\gamma\right)\right]$
  - $\hat{C}^{(2,1)} = p^2\bar{p}\,\gamma$
- order 4:
  - $\hat{C}^{(4,0)} = -p^4\left[3!\left(\langle N_b\rangle_c + \Delta + \frac{1}{2}\gamma\right) + \beta\right]$
  - $\hat{C}^{(3,1)} = -p^3\bar{p}\beta$
  - $\hat{C}^{(2,2)} = -p^2\bar{p}^2\left(\beta - \gamma\right)$
- Results are $\neq 0 \Rightarrow$ global baryon number conservation introduces correlations.
- Easy to obtain formulae with $p \leftrightarrow \bar{p}$

Introduced notation:

- $z_c = \sqrt{\langle N_b\rangle_c\langle \bar{N}_b\rangle_c}$,
- $\langle N\rangle_c = \langle N_b\rangle_c + \langle \bar{N}_b\rangle_c$,
- $\Delta = z_c^2 - z^2$,
- $\gamma = z_c^2 + \Delta\langle N\rangle_c$,
- $\beta = \gamma(\langle N\rangle_c + 2) + 2\Delta^2$.
- They all depend on B and $z = \sqrt{\langle N_b\rangle\langle \bar{N}_b\rangle}$ only.

Approximate and numerical results

B=0 special case

- Observe in results that each $\hat{C}^{(n,m)} \sim p^n \bar{p}^m$.
- So let's define:

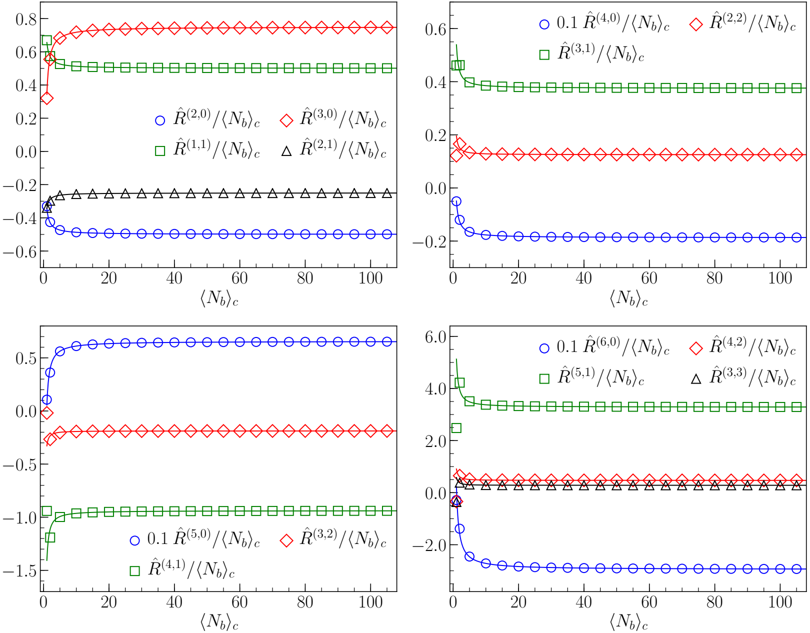
$$\hat{R}^{(n,m)} = \frac{\hat{C}^{(n,m)}}{p^n \bar{p}^m}\,, \tag{5}$$

- For $B = 0$ we have $z_c = \langle N_b \rangle_c = \langle \bar{N}_b \rangle_c = \langle N \rangle_c / 2$
- Using asymptotic expansion of I_B one can show that

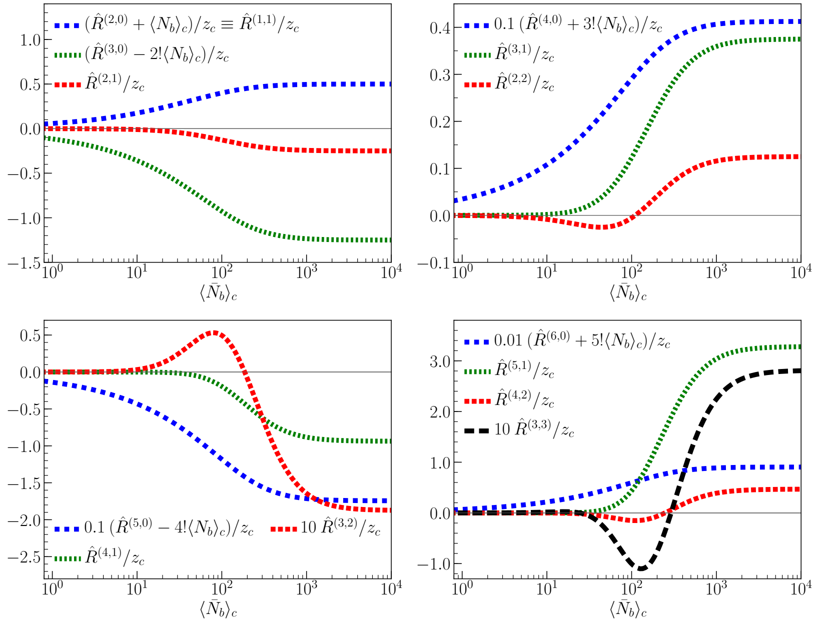
$$\hat{R}^{(n,m)}(z_c) \sim a_1 z_c + a_0 + a_{-1} z_c^{-1} + \ldots\,, \tag{6}$$

  linear term dominates + some corrections for small $z_c = \langle N_b \rangle_c$.
- So $\hat{R}^{(n,m)} / \langle N_b \rangle_c$ should be const. despite small $\langle N_b \rangle_c$.

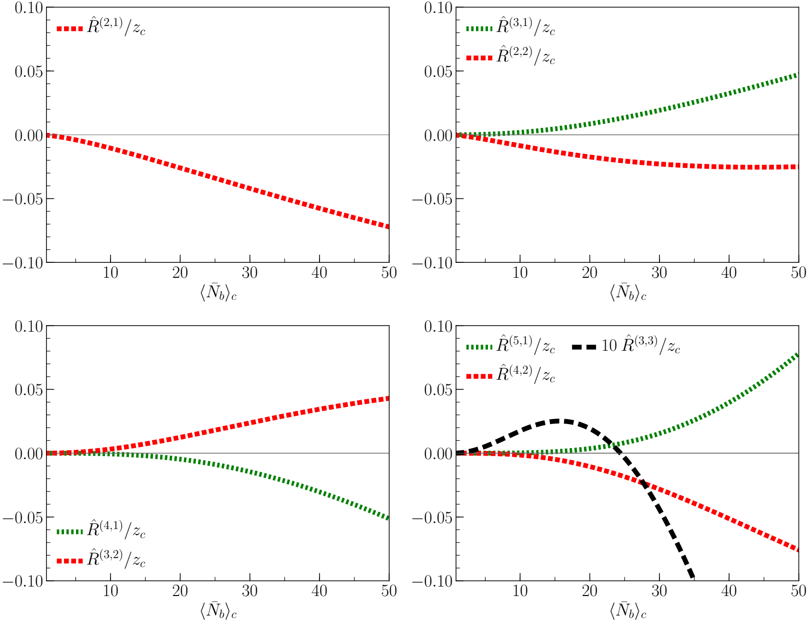
B=0. Markers - exact, lines - asymptotic



B=300, exact. For $m = 0$ adjusted to be invariant under $p \leftrightarrow \bar{p}$



B=300, exact, linear scale



Summary

- Factorial cumulants are integrated correlation functions.
- Proton, antiproton and mixed proton-antiproton $\hat{C}^{(n,m)}$ are calculated assuming the global baryon number conservation is the only source of correlations.
- At $B = 0$ the factorial cumulants $\hat{C}^{(n,m)}$ are practically proportional to $p^n \bar{p}^m$ only.
- Factorial cumulants can be measured experimentally.

Thank you :)

Backup

Results - protons-antiprotons exchange

- Recall: $\hat{C}^{(n,m)}$ is factorial cumulant (integrated correlation function) for n protons and m antiprotons.
- How does it change for m protons and n antiprotons ($p \leftrightarrow \bar{p}$)?
- $p \leftrightarrow \bar{p}$, $\langle N_b \rangle_c \leftrightarrow \langle \bar{N}_b \rangle_c$
- Examples:
  - $\hat{C}^{(2,1)} = p^2 \bar{p}\, \gamma \Rightarrow \hat{C}^{(1,2)} = p \bar{p}^2 \gamma$
  - $\hat{C}^{(1,0)} = p \langle N_b \rangle_c \Rightarrow \hat{C}^{(0,1)} = \bar{p} \langle \bar{N}_b \rangle_c$

Results - continuation

- Order 5 and 6 become slightly more complex.
- order 5:
  - $\hat{C}^{(5,0)} = p^5 \left[4! \left(\langle N_b \rangle_c + \Delta + \frac{1}{2}\gamma\right) + (\langle N \rangle_c + 7)\beta + 6\gamma\Delta\right]$
  - $\hat{C}^{(4,1)} = p^4 \bar{p} \left[(\langle N \rangle_c + 3)\beta + 6\gamma\Delta\right]$
  - $\hat{C}^{(3,2)} = p^3 \bar{p}^2 \left[(\langle N \rangle_c + 1)\beta + 6\gamma\Delta\right]$
- order 6:
  - $\hat{C}^{(6,0)} = -p^6 \left[5! \left(\langle N_b \rangle_c + \Delta + \frac{1}{2}\gamma\right) + \{(\langle N \rangle_c + 5)(\langle N \rangle_c + 7) + 12\}\beta + 6\gamma^2 + 16\Delta^3 + 2\gamma\Delta(7\langle N \rangle_c + 35)\right]$
  - $\hat{C}^{(5,1)} = -p^5 \bar{p} \left[(\langle N \rangle_c + 3)(\langle N \rangle_c + 4)\beta + 6\gamma^2 + 16\Delta^3 + 2\gamma\Delta(7\langle N \rangle_c + 20)\right]$
  - $\hat{C}^{(4,2)} = -p^4 \bar{p}^2 \left[(\langle N \rangle_c + 1)(\langle N \rangle_c + 3)\beta + 6\gamma^2 + 16\Delta^3 + 2\gamma\Delta(7\langle N \rangle_c + 11)\right]$
  - $\hat{C}^{(3,3)} = -p^3 \bar{p}^3 \left[(\langle N \rangle_c + 1)(\langle N \rangle_c + 2)\beta + 6\gamma^2 + 16\Delta^3 + 2\gamma\Delta(7\langle N \rangle_c + 8)\right]$

Relations

- Observe in results that each $\hat{C}^{(n,m)} \sim p^n \bar{p}^m$.
- So let's define:

$$\hat{R}^{(n,m)} = \frac{\hat{C}^{(n,m)}}{p^n \bar{p}^m}\,, \tag{7}$$

- Such ratios seem to satisfy the relation:

$$\hat{R}^{(n+1,m)} = \hat{R}^{(n,m+1)} - (n-m)\hat{R}^{(n,m)}$$

which have been verified up to $n + m < 9$

What is I_B?

- I_B - modified Bessel function
- Defined by series:

$$I_B(x) = \sum_{m=0}^{\infty} \frac{1}{m!(m+B)!} \left(\frac{x}{2}\right)^{2m+B} , \tag{8}$$

assuming B - natural.

Asymptotic expansion of the modified Bessel function

- For large x:

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n \prod_{i=1}^n (4\nu^2 - (2i-1)^2)}{n!(8x)^n} \right).$$

- Asymptotic ratios (for $B = 0$) become as simple as
  - $\hat{R}^{(2,0)}(z_c) \sim -\frac{1}{2} z_c + \frac{1}{8} + \frac{1}{32} z_c^{-1} + \ldots$
  - $\hat{R}^{(1,1)}(z_c) \sim \frac{1}{2} z_c + \frac{1}{8} + \frac{1}{32} z_c^{-1} + \ldots$
  - $\hat{R}^{(3,0)}(z_c) \sim \frac{3}{4} z_c - \frac{5}{16} - \frac{3}{32} z_c^{-1} + \ldots$
  - $\hat{R}^{(2,1)}(z_c) \sim -\frac{1}{4} z_c - \frac{1}{16} - \frac{1}{32} z_c^{-1} + \ldots$
  - $\hat{R}^{(4,0)}(z_c) \sim -\frac{15}{8} z_c + \frac{33}{32} + \frac{45}{128} z_c^{-1} + \ldots$
  - $\hat{R}^{(3,1)}(z_c) \sim \frac{3}{8} z_c + \frac{3}{32} + \frac{9}{128} z_c^{-1} + \ldots$
  - $\hat{R}^{(2,2)}(z_c) \sim \frac{1}{8} z_c + \frac{1}{32} + \frac{5}{128} z_c^{-1} + \ldots$
  - etc.

Cumulants

- Cumulant generating function for random variable n:

$$K(t) = \ln\left(\sum_{n=0}^{\infty} P(n)e^{tn}\right)$$

- Cumulants: $\kappa_i = \left.\frac{d^i K}{dt^i}\right|_{t=0}$
- $\kappa_1 = \langle n \rangle$ (mean)
  $\kappa_2 = \mu_2 = \sigma^2$ (variance)
  $\kappa_3 = \mu_3 = S\sigma^3$ (scaled skewness)
  $\kappa_4 = \mu_4 - 3\mu_2^2$ (kurtosis and variance), $\mu_k = \langle (n - \langle n \rangle)^k \rangle$
- Cumulants are broadly studied in heavy ion collisions, e.g.:
  M. A. Stephanov, Phys. Rev. Lett. **102**, 032301 (2009) [0809.3450]
  N. K. Behera [ALICE], Nucl. Phys. A **982**, 851-854 (2019) [1807.06780]
  L. Adamczyk *et al.* [STAR], Phys. Lett. B **785**, 551-560 (2018) [1709.00773]
  A. Adare *et al.* [PHENIX], Phys. Rev. C **93**, no.1, 011901 (2016) [1506.07834]
  P. Braun-Munzinger, A. Rustamov and J. Stachel, Nucl. Phys. A **960**, 114-130 (2017) [1612.00702]