A Multiple Scales Approach to the Averaging Problem in Cosmology

Barry Ginat,
Technion – Israel Institute of Technology
EPS HEP conference, 2021

Outline

- What is the cosmological back-reaction problem?
- Rapid review of multiple-scales technique
- Application to Einstein's equations
- Consistency conditions
- Summary

The Universe is Homogeneous on Large Scales

- Described by FLRW metric

$$\mathrm{d}s^2 = a^2(\eta)\left(-\mathrm{d}\eta^2 + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j\right)$$

- Obtained by solving the Einstein equations

$$G_{ab}(g_{ab}) + \Lambda g_{ab} = 8\pi G\, T_{ab}$$

for an isotropic and homogeneous matter distribution

But Very Inhomogeneous on Small Scales

- Inside galaxies the density contrast is huge
- Only upon averaging over a large scale ($\sim 100\text{Mpc}$) does T_{ab} become homogeneous and isotropic (e.g. Coley & Ellis 2020).
- But Einstein's equations are not linear, so $\overline{G_{ab}} \neq G_{ab}(\overline{g_{ab}})$.
- But we get FLRW from setting
  $G_{ab}(\overline{g_{ab}}) + \Lambda\overline{g_{ab}} = 8\pi G\overline{T_{ab}}$, as opposed to
  $$\overline{G_{ab}} + \Lambda\overline{g_{ab}} = 8\pi G\,\overline{T_{ab}}$$
- So something is possibly wrong.
- Averaging does not commute with the Einstein equations. This is the averaging (or back-reaction) problem.

Can a Background be defined?

- In cosmological perturbation theory, one splits the actual metric g_{ab} into a sum
$$g_{ab} = g_{ab}^{FLRW} + h_{ab},$$
- Perturbation theory consists of taking h_{ab} small.
- But its magnitude depends on scale and on gauge (e.g. Adamek et al. 2018).
- Is a background-perturbation split at all justified?

Solving the Problem Requires Special Techniques

- Several proposals were put forward to address this problem, including:
  - Special averaging techniques in curved space-time (e.g. Buchert 2000).
  - Special perturbative equations (Green & Wald 2011)
- However all relied on some assumptions, e.g. that there is a well-defined large-scale background metric.

Multiple Scales Technique

- The multiple scales technique addresses perturbative problems where there are more than one scale.
- Examples are homogenisation of diffusion in inhomogeneous media, Chapman-Enskog expansions,… (see, e.g., Pavliotis & Stuart, Springer, 2008).
- Take a PDE, e.g. $\nabla^2 u = f$, where $f(x) = g\left(x, \frac{x}{\varepsilon}\right)$, for $\varepsilon \ll 1$.
- E.g. let $X = x/\varepsilon$. Then the technique treats X and x independently, by changing $\frac{\partial}{\partial x} \mapsto \frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial X}$,
- Then

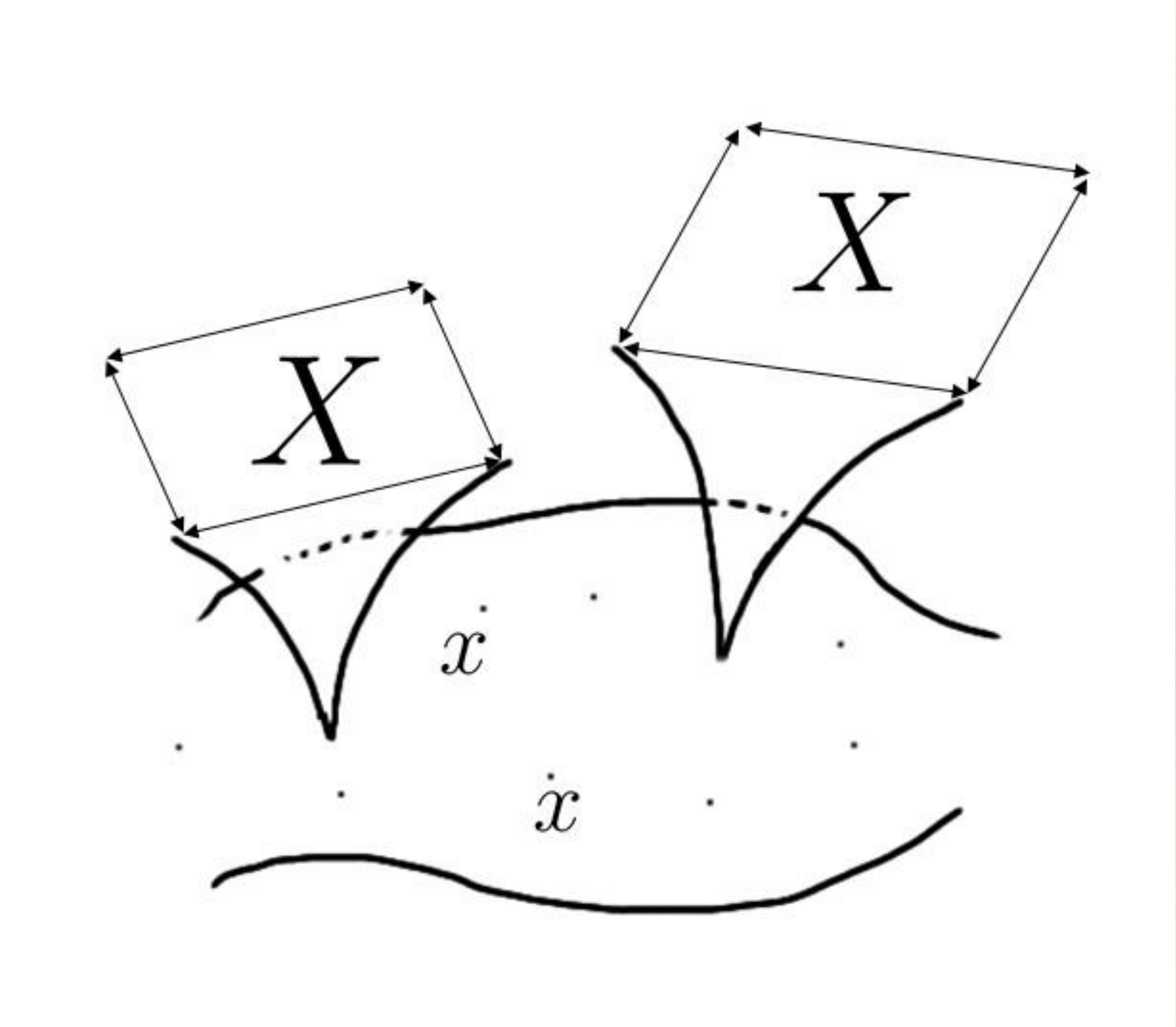
$$\left[\nabla_x^2 + \frac{2}{\varepsilon}\frac{\partial^2}{\partial x \partial X} + \frac{1}{\varepsilon^2}\nabla_X^2\right]\left(u_0(x,X) + \varepsilon u_1(x,X) + \varepsilon^2 u_2(x,X) + \cdots\right) = g(x,X)$$

Scale Splitting

- This is combined with an asymptotic expansion $u(x, X) \sim u_0(x, X) + \varepsilon u_1(x, X) + \varepsilon^2 u_2(x, X) + \cdots$
- Inserting both into the original differential equation and solving iteratively yields solutions for u_i.
- Must be combined with *consistency conditions* to ensure that the expansion remains asymptotic:
  - These arise to ensure $\varepsilon^n u_n \gg \varepsilon^{n+1} u_{n+1}$.
  - Done by setting certain integration constants to zero.
  - Freedom to do so ensured by the added degree of freedom.

Multiple-Scales Treatment of Einstein Equations

- Work in harmonic co-ordinates so that the equations are hyperbolic (e.g. Choquet-Bruhat, OUP, 2009).
- Set x as cosmological scales and X as small scales (e.g. galactic scales)
- Expand derivatives and solve at increasingly higher orders in epsilon.
- Impose consistency requirement at all orders.



Multiple-Scales Treatment of Einstein Equations

Work with trace-reversed equation

$$R_{ab} - \Lambda g_{ab} = 8\pi G \rho_{ab} \equiv 8\pi G \left(T_{ab} - \frac{1}{2} T g_{ab} \right)$$

One Finds:

- X space-time is flat, so okay to perform Fourier transform in it.
- Averaged Einstein equation:
  $R_{ab}(g^0_{ab}) - \Lambda g^0_{ab} = 8\pi G \langle \rho^0_{ab} \rangle_X + B_{ab}$,
  where B_{ab} corresponds, roughly, to the X-averaged Newtonian gravitational potential energy.
- Averaged equation agrees with that of Green & Wald (2011).

Orders 0 & 1

- The $ord(\varepsilon^{-2})$ equation is

$$-\frac{1}{2}g_0^{cd}\partial^2_{X^cX^d}g^0_{ab}+P_{ab}(g_0)\partial_X g_0\partial_X g_0=8\pi G\rho^{-2}_{ab}$$

- Can be solved consistently to define X-independent background $g_0(x)$ only if $\rho^{-2}_{ab}=0$, i.e. only Newtonian sources.
- Order -1 equation is a wave equation:

$$-\frac{1}{2}g_0^{cd}\partial^2_{X^cX^d}g^1_{ab}=8\pi G\rho^{-1}_{ab}$$

Second Order

- The equation for $g^2_{ab}(\cdot, X)$ contains $g^1_{ab}(\cdot, X)$ as a source for the wave-operator, due to non-linearity.
- Problem: there might be resonances, which invalidate the asymptotic expansion $g \sim g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + \cdots$.
- But $g^1_{ab}(\cdot, X)$ contains an as-yet undetermined term, $g^1_{\mathrm{hom}\ ab}(\cdot, X)$.
- This term is constrained to remove all resonant sources from the equation for $g^2_{ab}(\cdot, X)$.

Consistency Conditions

- Arise to tackle resonant terms.
- Additional equations possible (and necessary) because multiple-scales increases the number of degrees of freedom.
- At second order:

$$
\begin{aligned}
&- k^c \partial_{x^c} \hat{g}^1_{\mathrm{hom},ab}(x,k) + P^{cdefgh}_{ab}(g_0)\left(k_c \hat{g}^1_{\mathrm{hom},ef}(x,k)\partial_{x^d} g^0_{gh} + k_d \hat{g}^1_{\mathrm{hom},gh}(x,k)\partial_{x^c} g^0_{ef}\right) \\
&\qquad - \mathrm{i}\int \mathrm{d}^4\tilde{X} e^{\mathrm{i}\tilde{k}\cdot\tilde{X}} \left\{ P_{ab}(g_0)\partial_X g_1 \partial_X g_1 - \frac{1}{2} g_1^{cd} \partial^2_{X^c X^d} g^1_{ab} \right\}_{\mathrm{osc}} = -8\pi \mathrm{i} G \hat{\rho}^0_{ab}(x,k),
\end{aligned}
$$

- Turns out that for Newtonian sources they can be solved and resonances are consistently removed (Ginat 2021).

Outcome:

- The multiple scales technique enables one to:
  - Show that there is a well-defined background metric that depends only on the large-scale.
  - Derive an effective Einstein equation where small-scale gravitational potential energy gravitates on the large scale.
  - Determine the consistency conditions under which these conclusions obtain.
- Future prospects:
  - Go from asymptoticity to some sort of convergence, e.g., two-scale convergence (defined in, e.g., Allaire 1992).
  - Include isolated relativistic sources, such as black holes or neutron stars.

More to Explore

- See my paper: Y.B. Ginat, 2021, JCAP02(2021)049, arXiv: arXiv:2005.03026
- Or send me an e-mail: Barry Ginat, ginat@campus.technion.ac.il