Fermion Mass Hierarchies and Residual Modular Symmetries

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Modular invariance approach to the flavour problem is a relatively new, elegant and promising approach to an old and essentially unresolved fundamental problem in particle physics. It was proposed in F. Feruglio, arXiv:1706.08749 and has been intensively developed in the last three years. The first phenomenologically viable (minimal in terms of fields and parameters involved) lepton flavour model based on modular symmetry appeared in June of 2018 (J.T. Penedo, STP, arXiv:1806.11040). Since then various aspects of this approach were and continue to be extensively studied – the number of publications on the topic exceeds 100.

The talk: bottom-up approch to the flavour problem based on modular invariance.

The talk is based on the following articles.

1. P.P. Novichkov, J.T. Penedo and S.T. Petcov, "Fermion Mass Hierarchies, Large Lepton Mixing and Residual Modular Symmetries", JHEP 2104 (2021) 206 [arXiv:2102.07488].

2. P.P. Novichkov, J.T. Penedo and S.T. Petcov, "Double cover of modular S_4 for flavour model building", Nucl. Phys. B 963 (2021) 115301 [arXiv:2006.03058].

3. P.P. Novichkov, J.T. Penedo and S.T. Petcov, A.V. Titov, "Generalised CP Symmetry in Modular-Invariant Models of Flavour", JHEP 1907 (2019) 165 [arXiv:1905.11970].

4. P.P. Novichkov, S.T. Petcov and M. Tanimoto, "Trimaximal Neutrino Mixing from Modular A4 Invariance with Residual Symmetries," Phys. Lett. B 793 (2019) 247 [arXiv:1812.11289].

5. P.P. Novichkov, J.T. Penedo and S.T. Petcov, A.V. Titov, "Modular A_5 symmetry for flavour model building," JHEP 1904 (2019) 174 [arXiv:1812.02158].

6. P.P. Novichkov, J.T. Penedo and S.T. Petcov, A.V. Titov, "Modular S_4 models of lepton masses and mixing," JHEP 1904 (2019) 005 [arXiv:1811.04933].

7. J.T. Penedo and S.T. Petcov, "Lepton Masses and Mixing from Modular S_4 Symmetry," Nucl. Phys. B 939 (2019) 292 [arXiv:1806.11040].

The Flavour Problem

Understanding the origins of flavour in both quark and lepton sectors, i.e., of the patterns of quark masses and mixing, and of the charged lepton and neutrino masses and of neutrino mixing and of CP violation in the quark and lepton sector, is one of the most challenging fundamental problems in contemporary particle physics.

"Asked what single mystery, if he could choose, he would like to see solved in his lifetime, Weinberg doesnt have to think for long: he wants to be able to explain the observed pattern of quark and lepton masses."

From Model Physicist, CERN Courier, 13 October 2017.

The renewed attempts to seek new better solutions of the flavour problem than those already proposed were stimulated primarily by the remarkable progress made in the studies of neutrino oscillations, which began 23 years ago with the discovery of oscillations of atmospheric ν_μ and $\bar{\nu}_\mu$ by SuperKamiokande experiment. This lead, in particular, to the determination of the pattern of the 3-neutrino mixing, which turn out to consist of two large and one small mixing angles.

In what follows we will discuss a new approach to the flavour problem within the three family framework.

The Lepton Flavour Problem

Consists of three basic elements (sub-problems), namely, understanding:

- Why $m_{\nu_j} <<< m_{e,\mu,\tau}, m_q$, $q = u, c, t, d, s, b$ ($m_{\nu_j} \lesssim 0.5$ eV, $m_l \geq 0.511$ MeV, $m_q \gtrsim 2$ MeV);

- The origins of the patterns of
i) neutrino mixing of 2 large and 1 small angles ($\theta^l_{12} = 33.65^\circ$, $\theta^l_{23} = 47.1^\circ$, $\theta^l_{13} = 8.49^\circ$),
and of ii) Δm^2_{ij}, i.e., of $\Delta m^2_{21} \ll |\Delta m^2_{31}|$, $\Delta m^2_{21}/|\Delta m^2_{31}| \cong 1/30$.

- The origin of the hierarchical pattern of charged lepton masses:
$m_e \lll m_\mu \lll m_\tau$, $m_e/m_\mu \cong 1/200$, $m_\mu/m_\tau \cong 1/17$.

The quark Flavour Problem

Consists of two basic elements (sub-problems), namely, understanding:

- **The origin(s) of the observed patterns of up- and down-type quark masses characterized by strong hierachies.**

$$m_d \ll m_s \ll m_b\,,\ \frac{m_d}{m_s} = 5.02 \times 10^{-2}\,,\ \frac{m_s}{m_b} = 2.22 \times 10^{-2}\,,\ m_b = 4.18\ \text{GeV};$$

$$m_u \ll m_c \ll m_t\,,\ \frac{m_u}{m_c} = 1.7 \times 10^{-3}\,,\ \frac{m_c}{m_t} = 7.3 \times 10^{-3}\,,\ m_t = 172.9\ \text{GeV};$$

- **The origin of the pattern of the quark mixing: the three quark mixing angles are small and hierarchical,** $\sin\theta^q_{13} \ll \sin\theta^q_{23} \ll \sin\theta^q_{12} \ll 1$**,** $\sin\theta^q_{12} \cong 0.22$**.**

Each of the considered sub-problems of the lepton and qaurk flavour problems is by itself a formidable problem. As a consequence, solutions to each individual problem have been proposed. However, a universal "elegant and convincing" solution to the lepton and quark flavour problems is still lacking.

Considered Solutions

- $m_{\nu_j} <<< m_{e,\mu,\tau}, m_q$, $q = u, c, t, d, s, b$:

seesaw mechanism, Weinberg operator, radiative ν mass generation, extra dimensions.
However, additional input (symmetries) needed to explain the pattern of lepton mixing and to get specific testable predictions.

- The origin of the hierarchical pattern of charged lepton and quark masses.

The best qualitative explanation is arguably provided by the Frogatt-Nielsen mechanism based on $U(1)_{FN}$ flavour symmetry and its generalisations.
Problems: predictions suffer from uncertainties; most naturally accomodates small mixing angles, while two lepton mixing angles are large.

- The origins of the patterns of neutrino mixing of 2 large and 1 small angles.

Arguably the most elegant and natural explanation is obtained within the non-Abelian discrete flavour symmetry approach to the problem.
However, the symmetry breaking in the lepton and quark flavour models based on non-Abelian discrete symmetries is impressively cumbersome: it requires the introduction of a plethora of "flavon" scalar fields having elaborate potentials, which in turn require the introduction of a number of "driving fields" and large shaping symmetries to ensure the requisite breaking of the symmetry leading to correct mass and mixing patterns.

Combining the proposed individual "solutions" of the related sub-problems it is difficult, if not impossible, to avoid the drawbacks of each of the "ingredient" sub-problem "solutions". In some cases this can be achieved at the cost of severe fine-tuning.

In neutrino physics of fundamental importance are also:

• the determination of the status of lepton charge conservation and the nature - Dirac or Majorana - of massive neutrinos (which is one of the most challenging and pressing problems in present day elementary particle physics) (GERDA, CUORE, KamLAND-Zen, EXO, LEGEND, nEXO,...);

• determining the status of CP symmetry in the lepton sector (T2K, NOνA; T2HK, DUNE);

• determination of the type of spectrum neutrino masses possess, or the "neutrino mass ordering" (T2K + NOνA; JUNO; PINGU, ORCA; T2HKK, DUNE);

• determination of the absolute neutrino mass scale, or $\min(m_j)$ (KATRIN, new ideas; cosmology).

The program of research extends beyond 2035.

These are the "big questions" especially relevant to the reference 3-neutrino mixing scheme, which I am going to employ for the discussion of the lepton flavour problem.

• **BS3νRM: eV scale sterile ν's; NSI's; ChLFV processes ($\mu \to e + \gamma$, $\mu \to 3e$, $\mu^- - e^-$ conversion on (A,Z)); $\nu-$related BSM physics at the TeV scale (N_{jR}, H^{--}, H^-, etc.).**

Lepton sector: reference 3-ν mixing.

Lepton sector: reference 3-ν mixing scheme

$$\nu_{l\mathrm{L}} = \sum_{j=1}^{3} U_{lj}\, \nu_{j\mathrm{L}} \qquad l = e, \mu, \tau.$$

ν_j, $m_j \neq 0$: **Majorana particles (assumed).**

Data: 3 νs are light: $\nu_{1,2,3}$, $m_{1,2,3} \lesssim 0.5$ **eV;**
the value of $\min(m_j)$ **and the "ordering" unknown.**

Δm^2_{21}, $|\Delta m^2_{31}|$ - **known.**

The PMNS matrix U - 3×3 **unitary:** θ_{12}, θ_{13}, θ_{23} - **known; CPV phases** δ, α_{21}, α_{31} - **unknown.**

Thus, 5 known + 4 unknown parameters + MO.

"Known" = measured; "unknown" = not measured.

Global analyses after Nu2020: combine, in particular, the latest T2K and NOνA data.

Results on CPV due to δ and NO vs IO spectrum - inconclusive.

K.J. Kelly, P.A. Machado, S.J. Parke, Y.F. Perez Gonzalez and R. Zukanovich-Funchal, "Back to (Mass-)Square(d) One: The Neutrino Mass Ordering in Light of Recent Data," arXiv:2007.08526 [hep-ph].

I. Esteban, M.C. Gonzalez-Garcia, M. Maltoni, T. Schwetz and A. Zhou, "The fate of hints: updated global analysis of three-flavor neutrino oscillations," arXiv:2007.14792 [hep-ph].

Result on CPV, b.f.v.: $\delta = 197^{\circ}$, NO; $\delta = 282^{\circ}$, IO.

At 3σ: δ is found to lie in $[120^{\circ}, 369^{\circ}]$ ($[193^{\circ}, 352^{\circ}]$), NO (IO).

IO: CPV due to δ at 3σ.

IO disfavored at 1.6σ with respect to NO (2.7σ including SuperK ν_{atm} data).

Lepton and Quark Masses and Mixing

The observed patterns of the masses of up- and down-type quarks and of the charged leptons of the three families of SM are characterized by strong hierarchies:

$$m_d \ll m_s \ll m_b\,,\ \frac{m_d}{m_s} = 5.02 \times 10^{-2}\,,\ \frac{m_s}{m_b} = 2.22 \times 10^{-2}\,,\ m_b = 4.18\ \text{GeV};$$
$$m_u \ll m_c \ll m_t\,,\ \frac{m_u}{m_c} = 1.7 \times 10^{-3}\,,\ \frac{m_c}{m_t} = 7.3 \times 10^{-3}\,,\ m_t = 172.9\ \text{GeV};$$
$$m_e \ll m_\mu \ll m_\tau\,,\ \frac{m_e}{m_\mu} = 4.8 \times 10^{-3}\,,\ \frac{m_\mu}{m_\tau} = 5.95 \times 10^{-2}\,,\ m_\tau = 1776.86\ \text{MeV}.$$

The three quark mixing angles are small and hierarchical,

$$\theta^q_{12} = 12.96^\circ\,,\quad \theta^q_{23} = 2.42^\circ\,,\quad \theta^q_{13} = 0.022^\circ,$$

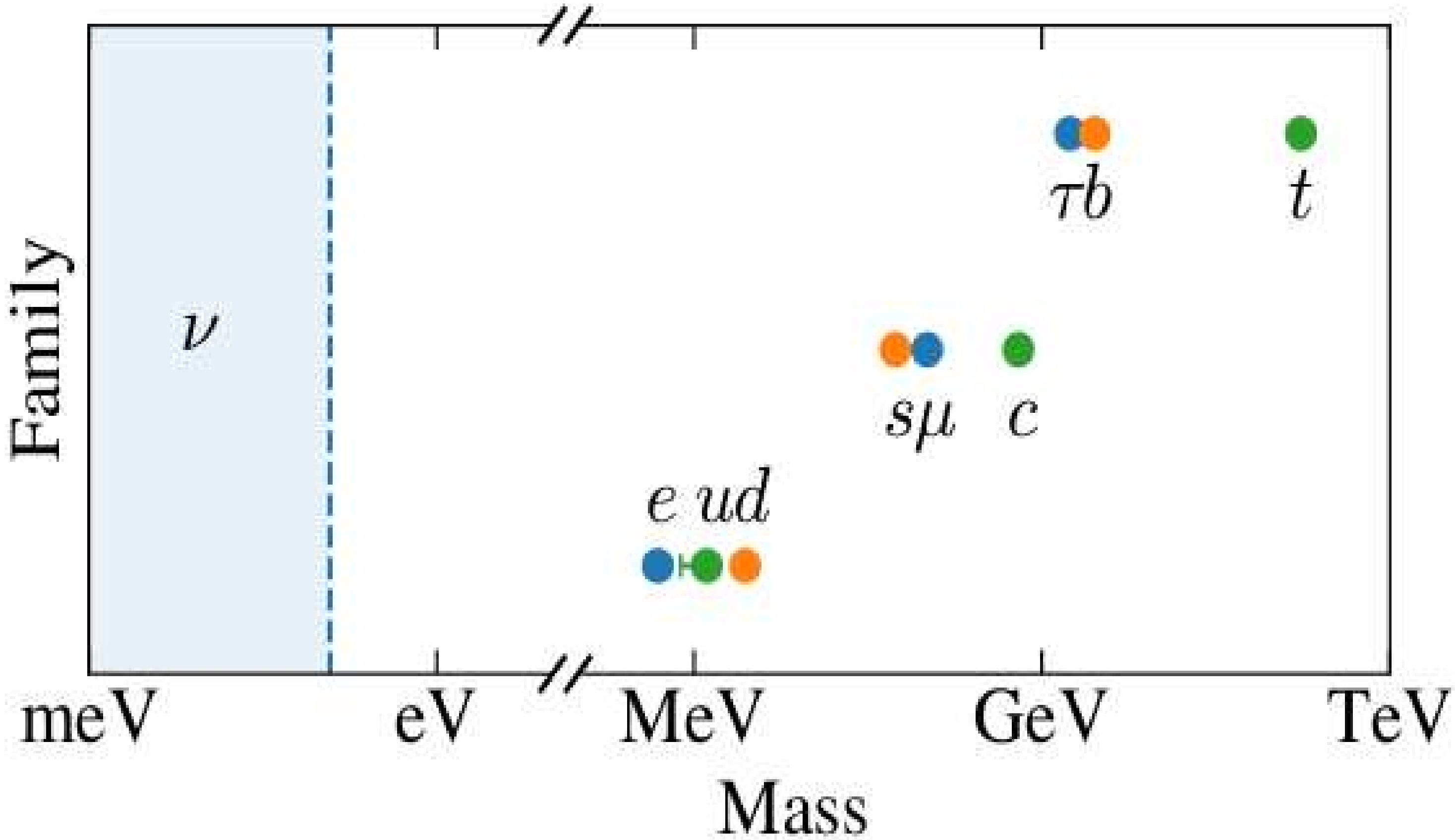
while the lepton mixing is characterized by two large and one small angles,

$$\theta^l_{12} = 33.65^\circ\,,\ \theta^l_{13} = 8.49^\circ\,,\ \theta^l_{23} = 47.1^\circ\ \ (45^\circ\ \text{within}\ 1.5\sigma)\,.$$

The quoted values correspond to the standard" parametrisations of V_{CKM} and U_{PMNS}. The Dirac CPV phases in CKM and PMNS matrices read:

$$\delta_q = (73.5 - 5.1 + 4.2)^\circ\,,\quad \delta_l = (1.37 - 0.16 + 0.18) \times 180^\circ\,(?)\,.$$

F. Capozzi et al. (Bari Group), arXiv:1804.09678.



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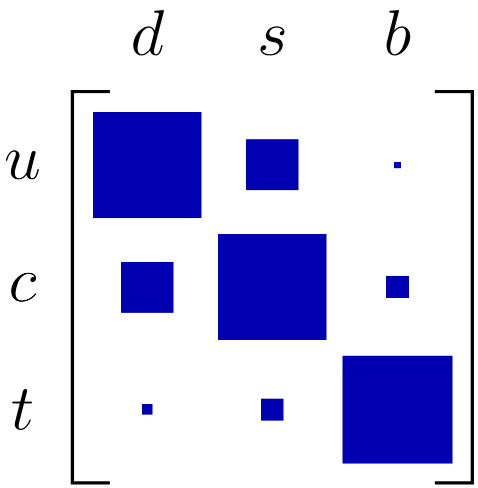
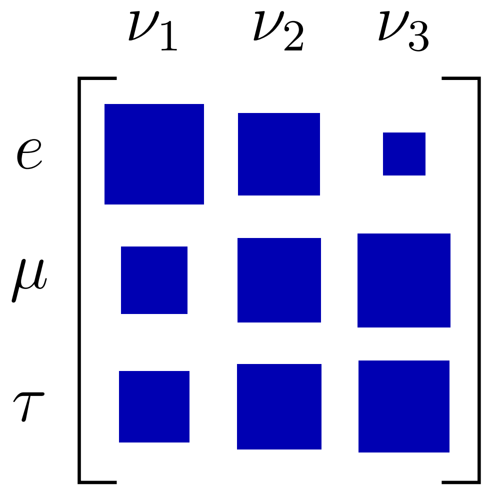
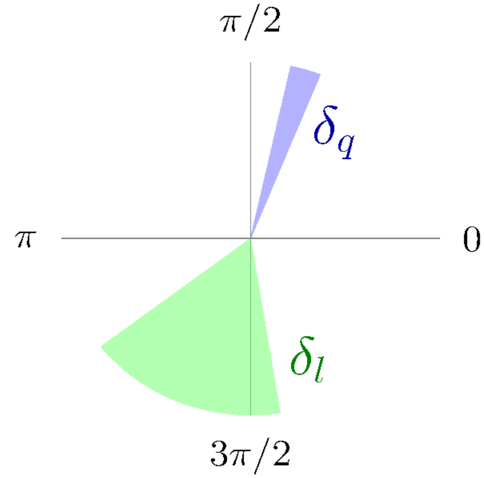






Figures by P. Novichkov

The Flavour Problem: Modular Invariance Approach

In this approach the flavour (modular) symmetry is broken by the vacuum expectation value (VEV) of a single scalar field - the modulus τ. The VEV of τ can also be the only source of violation of the CP symmetry.
Many (if not all) of the drawbacks of the widely studied alternative approaches are absent in the modular invariance approach to the flavour problem.

The present talk: bottom-up approach based on modular invariance.

Modular invariance has been investigated in the context of field and superstring theories, being a feature of a number of theoretical physics constructions (theories with extra dimensions compactified on a torus (or tori), superstring theories on tori or orbifolds, supergravity theories) [2]-[7]; it can be present in theories with global or local supersymmetry and appears to be a property of the quantum Hall effect [8]-[13]. The modular forms which are an integral part of the approach (see further) have been extensively studied by mathematicians, in particular, in connection with number theory [14].

[2] R. Blumenhagen, B. Kors, D. Lust and S. Stieberger, Phys. Rept. 445, 1 (2007). [3] L. E. Ibanez, Phys. Lett. B181, 269 (1986). [4] S. Hamidi and C. Vafa, Nucl. Phys. B279, 465 (1987). [5] S. Ferrara, D. Lust and S. Theisen, Phys. Lett. B233, 147 (1989). [6] D. Cremades, L. E. Ibanez and F. Marchesano, JHEP 0405, 079 (2004). [7] S. Ferrara, D. Lust, A. D. Shapere and S. Theisen, Phys. Lett. B225, 363 (1989). [8] C. A. Ltken and G. G. Ross, Phys. Rev. D45, 11837 (1992). [9] A. Cappelli and G. R. Zemba, Nucl. Phys. B490, 595 (1997). [10] C. P. Burgess and B. P. Dolan, Phys. Rev. B63, 155309 (2001). [11] M. Lippert, R. Meyer and A. Taliotis, JHEP 1501, 023 (2015). [12] C.A. Lutken, EPJ Web Conf. 71, 0079 (2014) 00079 (doi:10.1051/epjconf/20147100079). [13] C. A. Lutken, Phys. Rev. B99, 195152 (2019). [14] H. M. Farkas and I. Kra, Theta Constants, Riemann Surfaces and the Modular Group, Graduate Studies in Mathematics, vol. 37, American Mathematical Society (2001).

Top-Down Approach

– A. Baur, M. Kade, H.P. Nilles, S. Ramos-Snchez, P.K.S. Vaudrevange, "Completing the eclectic flavor scheme of the $\mathbb{Z}_2$ orbifold," arXiv:2104.03981 [hep-th].
– A.Baur, M. Kade, H.P. Nilles, S.Ramos-Sanchez and P.K.S. Vaudrevange, "Siegel modular flavor group and CP from string theory," Phys. Lett. B **816** (2021) 136176 [arXiv:2012.09586 [hep-th]].
– H.P. Nilles, S. Ramos-Sanchez, P.K.S. Vaudrevange, "Eclectic flavor scheme from ten-dimensional string theory - II detailed technical analysis," Nucl. Phys. B **966** (2021) 115367 [arXiv:2010.13798 [hep-th]].
– H.P. Nilles, S. Ramos-Sanchez and P.K.S. Vaudrevange, "Eclectic flavor scheme from ten-dimensional string theory I. Basic results," Phys. Lett. B **808** (2020) 135615 [arXiv:2006.03059 [hep-th]].
– H.P. Nilles, S. Ramos-Sanchez and P.K.S. Vaudrevange, "Eclectic flavor scheme from ten-dimensional string theory I. Basic results," Phys. Lett. B **808** (2020) 135615 [arXiv:2006.03059 [hep-th]].
– Y. Almumin, M.C. Chen, V. Knapp-Perez, S. Ramos-Sanchez, M. Ratz and S. Shukla, "Metaplectic Flavor Symmetries from Magnetized Tori," arXiv:2102.11286 [hep-th].
– K. Ishiguro, T. Kobayashi, H. Otsuka, "Hierarchical structure of physical Yukawa couplings from matter field Kähler metric", arXiv:2103.10240.
– S. Kikuchi, T. Kobayashi, H. Uchida, "Modular flavor symmetries of three-generation modes on magnetized toroidal orbifolds," arXiv:2101.00826.

The Modular Group and the Finite Modular Groups

The modular group $\overline{\Gamma}$ – group of linear fractional transformations γ acting on the complex variable τ belonging to the upper-half complex plane:

$$\gamma\tau = \frac{a\tau+b}{c\tau+d}\,, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,, \quad a,b,c,d \in \mathbb{Z}\,, \quad ad-bc = 1\,, \quad \mathrm{Im}\tau > 0\,.$$

$\overline{\Gamma}$ is generated by two transformations S and T satisfying

$$S^2 = (ST)^3 = I\,,$$

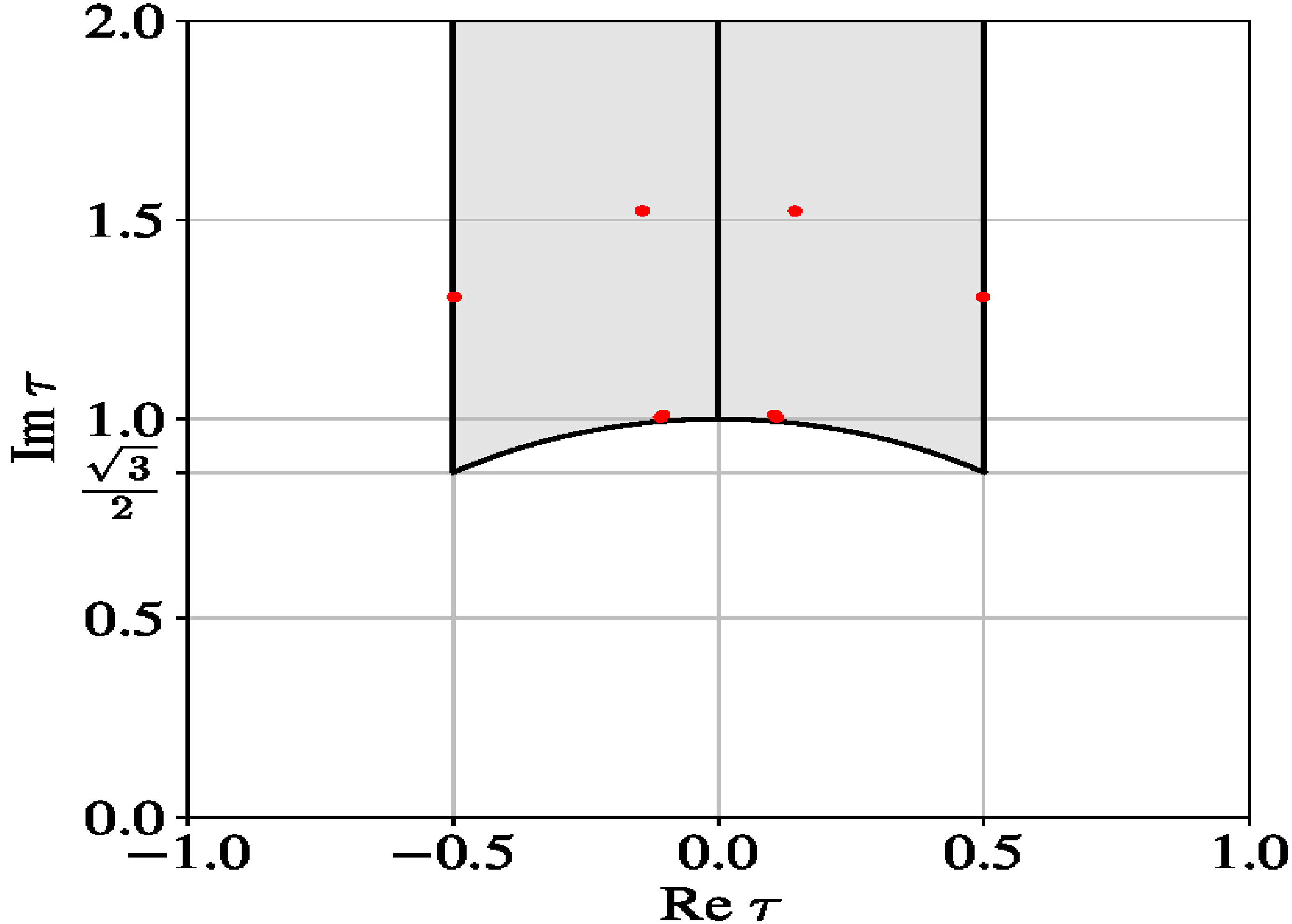
I being the identity element, and acting on τ as

$$\tau \xrightarrow{S} -\frac{1}{\tau}\,, \qquad \tau \xrightarrow{T} \tau + 1\,.$$

S and T can be represented as

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\,.$$

Complex variable τ - modulus (the VEV of complex scalar field $\tau(x)$).
$\overline{\Gamma}$ – *inhomogeneous* modular group.



The Fundamental Domain of $\overline{\Gamma}$ shown for $\mathrm{Im}\tau \leq 2$ (the red dots correspond to solutions of the lepton flavour problem, see further).

Figure from P.P. Novichkov, J.T. Penedo, STP, A.V. Titov, arXiv:1811.04933.

$\overline{\Gamma}$ **is isomorphic to the** *projective special linear group* $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$**,** $SL(2,\mathbb{Z})$ **is the special linear group of** 2×2 **matrices with integer elements and unit determinant, and** $\mathbb{Z}_2 = \{I, -I\}$ **is its centre.**
$SL(2,\mathbb{Z}) = \Gamma(1) \equiv \Gamma$ **contains a** *series of infinite normal subgroups* $\Gamma(N)$**,**

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}\,,\ N = 1,2,3,\ldots\,,$$

called the *principal congruence subgroups*. **For** $N=1$ **and** 2**, we define the groups** $\overline{\Gamma}(N) \equiv \Gamma(N)/\{I,-I\}$ **with** $\overline{\Gamma}(1) \equiv \overline{\Gamma}$**. For** $N > 2$**,** $\overline{\Gamma}(N) \equiv \Gamma(N)$ **since** $\Gamma(N)$ **does not contain the subgroup** $\{I,-I\}$**.**

The *quotient groups* $\Gamma_N \equiv \overline{\Gamma}/\overline{\Gamma}(N)$ *are called (inhomogeneous) finite modular groups*. **Remarkably, for** $N \leq 5$**,** Γ_N **are isomorphic to non-Abelian discrete groups widely used in flavour model building:**
$\Gamma_2 \simeq S_3$**,** $\Gamma_3 \simeq A_4$**,** $\Gamma_4 \simeq S_4$ **and** $\Gamma_5 \simeq A_5$**.**
Γ_N **is presented by two generators** S **and** T **satisfying:**

$$S^2 = (ST)^3 = T^N = I\,.$$

The group theory of $\Gamma_2 \simeq S_3$**,** $\Gamma_3 \simeq A_4$**,** $\Gamma_4 \simeq S_4$ **and** $\Gamma_5 \simeq A_5$ **is summarized, e.g., in P.P. Novichkov** *et al.***, JHEP 07 (2019) 165, arXiv:1905.11970.**

One can consider also:

$\Gamma \simeq SL(2,\mathbb{Z})$ − *homogeneous* **modular group,** $\Gamma(N)$ **and the quotient groups** $\Gamma'_N \equiv \Gamma/\Gamma(N)$ **(***homogeneous* **finite modular groups). For** $N = 3,4,5$**,** Γ'_N **are isomorphic to the double covers of the corresponding non-Abelian discrete groups:**
$\Gamma'_3 \simeq A'_4 \equiv T'$**,** $\Gamma'_4 \simeq S'_4$ **and** $\Gamma'_5 \simeq A'_5$**.**
Γ'_N **is presented by two generators** S **and** T **satisfying:**

$$S^4 = (ST)^3 = T^N = I\,,\ \ S^2\,T = T\,S^2 \quad (S^2 = R)\,.$$

The group theory of $\Gamma'_3 \simeq A'_4$**,** $\Gamma'_4 \simeq S'_4$ **and** $\Gamma'_5 \simeq A'_5$ **for flavour model building was developed in X.-G. Liu, G.-J. Ding, arXiv:1907.01488 (**A'_4**);**
P.P. Novichkov et al., arXiv:2006.03058 (S'_4**); C.-Y. Yao et al., arXiv:2011.03501 (**A'_5**).**

Relevant sub-groups of Γ'_N

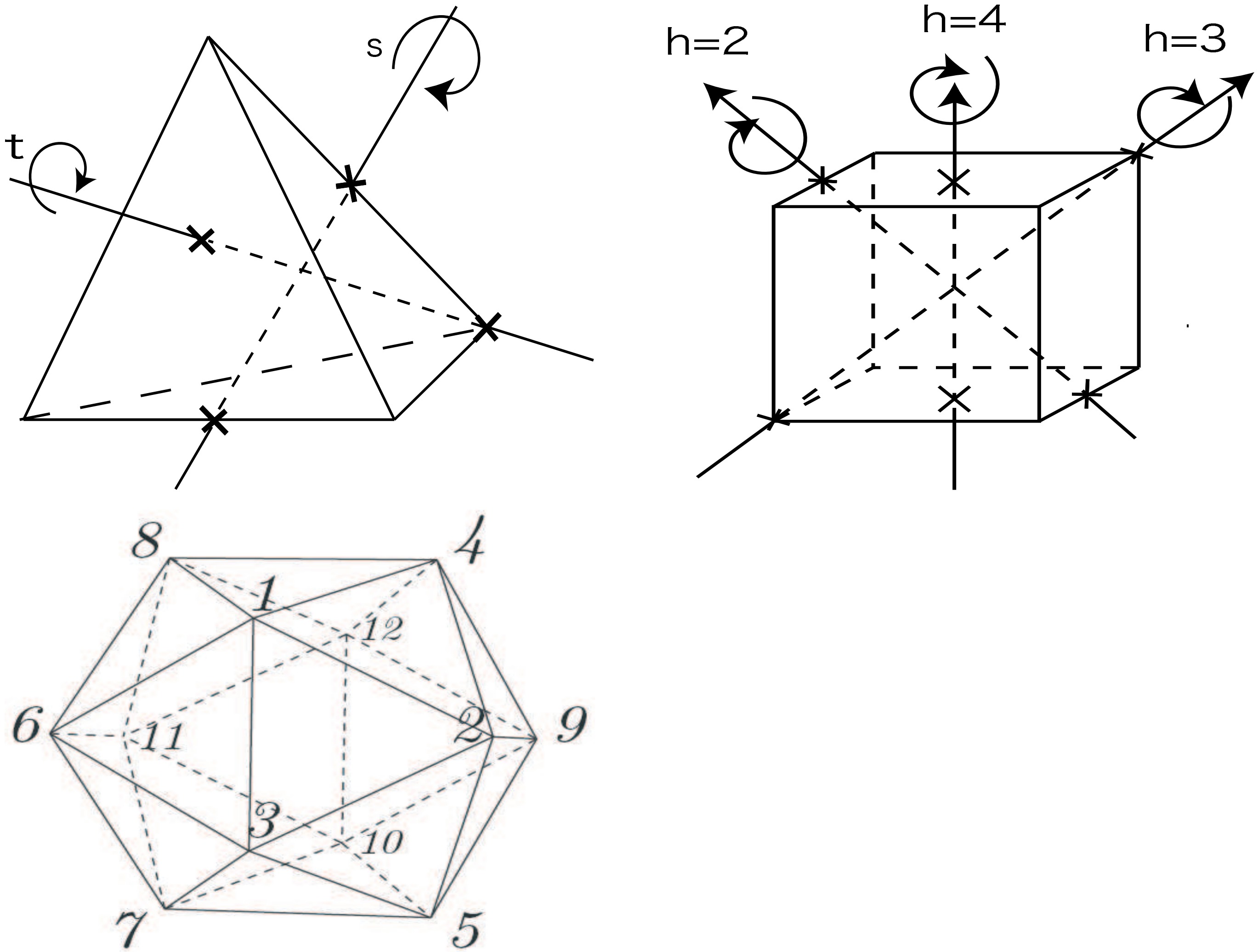
$$\mathbb{Z}_4^S = \{I, S, S^2, S^3\} \quad (R^2 = I,\ \mathbb{Z}_2^R = \{I, R\})$$

$$\mathbb{Z}_3^{ST} = \{I, ST, (ST)^2\}$$

$$\mathbb{Z}_N^T = \{I, T, (T)^2, ..., T^{N-1}\}$$

Group	Number of elements	Generators	Irreducible representations
S_4	24	S, T (U)	$\mathbf{1}$, $\mathbf{1'}$, $\mathbf{2}$, $\mathbf{3}$, $\mathbf{3'}$
S'_4	48	S, T (R)	$\mathbf{1}$, $\mathbf{1'}$, $\mathbf{2}$, $\mathbf{3}$, $\mathbf{3'}$, $\hat{\mathbf{1}}$, $\hat{\mathbf{1}}'$, $\hat{\mathbf{2}}$, $\hat{\mathbf{3}}$, $\hat{\mathbf{3}}'$
A_4	12	S, T	$\mathbf{1}$, $\mathbf{1'}$, $\mathbf{1''}$, $\mathbf{3}$
T'	24	S, T (R)	$\mathbf{1}$, $\mathbf{1'}$, $\mathbf{1''}$, $\mathbf{2}$, $\mathbf{2'}$, $\mathbf{2''}$, $\mathbf{3}$
A_5	60	$\tilde{S}$, $\tilde{T}$	$\mathbf{1}$, $\mathbf{3}$, $\mathbf{3'}$, $\mathbf{4}$, $\mathbf{5}$
A'_5	120	$\tilde{S}$, $\tilde{T}$	$\mathbf{1}$, $\mathbf{3}$, $\mathbf{3'}$, $\mathbf{4}$, $\mathbf{5}$, $\hat{\mathbf{2}}$, $\hat{\mathbf{2}}'$, $\hat{\mathbf{4}}$, $\hat{\mathbf{6}}$.

Number of elements, generators and irreducible representations of S_4, S'_4, A_4, $A'_4 \equiv T'$, A_5 and A'_5 discrete groups.



Examples of symmetries: A_4, S_4, A_5.

From M. Tanimoto et al., arXiv:1003.3552

Residual Symmetries

The breakdown of modular symmetry is parameterised by the VEV of τ.
There is no value of τ's VEV which preserves the full symmetry $\Gamma^{(\prime)}$ ($\Gamma_N^{(\prime)}$).
At certain "symmetric points" $\tau = \tau_{\text{sym}}$, $\Gamma^{(\prime)}$ ($\Gamma_N^{(\prime)}$) is only partially broken, with the unbroken generators giving rise to residual symmetries.
The R generator is unbroken for any value of τ, thus a $\mathbb{Z}_2^R$ symmetry is always preserved.

There are only 3 inequivalent symmetric points in $\mathcal{D}$:

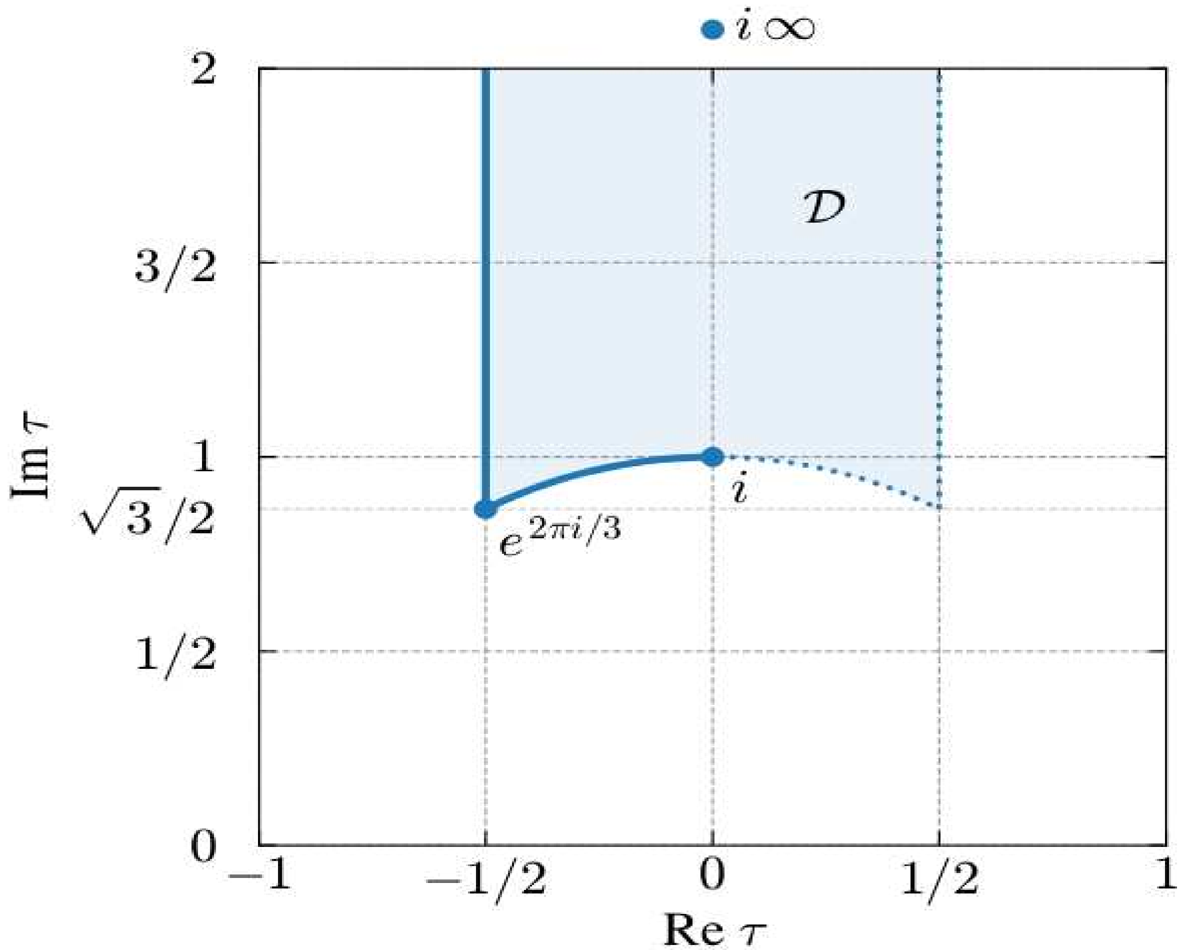
- $\tau_{\text{sym}} = i\infty$, invariant under T, preserving $\mathbb{Z}_N^T \times \mathbb{Z}_2^R$;
- $\tau_{\text{sym}} = i$, invariant under S, preserving $\mathbb{Z}_4^S$ (recall that $S^2 = R$);
- $\tau_{\text{sym}} = \omega \equiv \exp(2\pi i/3)$, "the left cusp", invariant under ST, preserving $\mathbb{Z}_3^{ST} \times \mathbb{Z}_2^R$.

P.P. Novichkov et al., arXiv:1811.04933 and arXiv:2006.03058

These symmetric values of τ preserve the CP ($\mathbb{Z}_2^{CP}$) symmetry of a CP- and modular-invariant theory (e.g. a modular theory where the couplings satisfy a reality condition).

P.P. Novichkov et al., arXiv:1911.04933 and arXiv:2006.03058

The CP ($\mathbb{Z}_2^{\text{CP}}$) symmetry is preserved for $\text{Re}\,\tau = 0$ or for τ lying on the border of the fundamental domain $\mathcal{D}$, but is broken at generic values of τ.



The fundamental domain $\mathcal{D}$ of the modular group Γ and its three symmetric points $\tau_{\text{sym}} = i\infty, i, \omega$. At the solid and dotted lines (which include the three points) CP is also preserved. The value of τ can always be restricted to $\mathcal{D}$ by a suitable modular transformation.

Figure from P.P. Novichkov et al., arXiv:2006.03058

Matter Fields and Modular Forms

The matter(super)fields (charged lepton, neutrino, quark) transform under $\overline{\Gamma}$ (Γ) as "weighted" multiplets:

$$\psi_i = (c\tau + d)^{-k_\psi} \rho_{ij}(\gamma)\psi_j \,, \quad \gamma \in \overline{\Gamma} \quad (\gamma \in \Gamma) \,,$$

k_ψ is the weight and $\rho(\gamma)$ is a unitary representation of $\overline{\Gamma}$ (Γ); k_ψ can be positive integer, or negative integer, or 0: $k \in \mathbb{Z}$.
$\rho(\gamma)$ is the identity matrix whenever $\gamma \in \overline{\Gamma}(N)$ ($\gamma \in \Gamma(N)$).
Thus, effectively, $\rho(\gamma)$ is a unitary representation of the finite modular group Γ_N (Γ'_N).

F. Feruglio, arXiv:1706.08749; S. Ferrara et al., Phys.Lett. B233 (1989) 147, B225 (1989) 363

As we have indicated in brackets, one can consider also the case of Γ and $\gamma \in \Gamma(N)$. Then $\rho(\gamma)$ will be a unitary representation of the *homogeneous* finite modular group Γ'_N.

Modular Forms

Within the considered framework the elements of the Yukawa coupling and fermion mass matrices in the Lagrangian of the theory are expressed in terms of modular forms of a certain level N and weight k_f.
The modular forms are functions of a single complex scalar field – the modulus τ – and have specific transformation properties under the action of the modular group.
Both the Yukawa couplings and the matter fields (supermultiplets) are assumed to transform in representations of an inhomogeneous (homogeneous) finite modular group $\Gamma^{(\prime)}_N$.
Once τ acquires a VEV, the modular forms and thus the Yukawa couplings and the form of the mass matrices get fixed, and a certain flavour structure arises.
Quantitatively and barring fine-tuning, the magnitude of the values of the non-zero elements of the fermion mass matrices and therefore the fermion mass ratios are determined by the modular form values (which in turn are functions of the τ's VEV).

Modular Forms (contd.)

The key elements of the considered framework are modular forms $f(\tau)$ of *weight* k_f *and level* N **– holomorphic functions of τ, which transform under $\overline{\Gamma}$ (Γ) as follows:**

$$f\,(\gamma\tau) = (c\tau + d)^{k_f}\; f(\tau)\,, \quad \gamma \in \overline{\Gamma}\ (\gamma \in \Gamma)\,,$$

In the case of $\overline{\Gamma}$ (Γ) *non-trivial* **modular forms exist only for positive even integer (positive integer) weight k_f.**
For given k, N (N is a natural number), the modular forms span a linear space of finite dimension:
of weight k and level 3, $\mathcal{M}_k(\Gamma_3^{(\prime)} \simeq A_4^{(\prime)})$, is $k+1$;
of weight k and level 4, $\mathcal{M}_k(\Gamma_4^{(\prime)} \simeq S_4^{(\prime)})$, is $2k+1$;
of weight k and level 5, $\mathcal{M}_k(\Gamma_5^{(\prime)} \simeq A_5^{(\prime)})$, is $5k+1$.
Thus, $\dim\mathcal{M}_1(\Gamma_3' \simeq A_4') = 2$, $\dim\mathcal{M}_1(\Gamma_4' \simeq S_4') = 3$, $\dim\mathcal{M}_1(\Gamma_5' \simeq A_5') = 6$.
One can find a basis $F(\tau) \equiv (f_1(\tau), f_2(\tau), \ldots)^T$ in each of these spaces such that for any $\gamma \in \overline{\Gamma}$ ($\gamma \in \Gamma$), $F(\gamma\tau)$ belongs to the same space and transforms according to a unitary irreducible representation $\mathbf{r}$ of Γ_N (Γ'_N):

$$F\,(\gamma\tau) = (c\tau + d)^{k_F}\,\rho_{\mathbf{r}}(\gamma) F(\tau)\,, \quad \gamma \in \overline{\Gamma}\ \ (\gamma \in \Gamma)\,.$$

This result is at the basis of the modular invariance approach to the flavour problem proposed in F. Feruglio, arXiv:1706.08749.

The Framework

$\mathcal{N}=1$ **rigid (global) SUSY, the matter action** $\mathcal{S}$ **reads:**

$$\mathcal{S} = \int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\overline{\theta}\;\; K(\tau,\overline{\tau},\psi,\overline{\psi}) + \left(\int \mathrm{d}^4x\,\mathrm{d}^2\theta\;\; W(\tau,\psi) + \mathrm{h.c.}\right)\,,$$

K **is the Kähler potential,** W **is the superpotential,** ψ **denotes a set of chiral supermultiplets** ψ_i**,** θ **and** $\overline{\theta}$ **are Grassmann variables;**
τ **is the modulus chiral superfield, whose lowest component is the complex scalar field acquiring a VEV (we use in what follows the same notation** τ **for the lowest complex scalar component of the modulus superfield and call this component also "modulus").**
τ **and** ψ_i **transform under the action of** $\overline{\Gamma}$ **(**Γ**) in a certain way (S. Ferrara et al., PL B225 (1989) 363 and B233 (1989) 147). Assuming that** $\psi_i = \psi_i(x)$ **transform in a certain irrep** $\mathbf{r}_i$ **of** Γ_N **(**Γ'_N**), the transformations read:**

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma}\ (\Gamma):\qquad \begin{cases} \tau \to \dfrac{a\tau+b}{c\tau+d}\,, \\ \psi_i \to (c\tau+d)^{-k_i}\,\rho_{\mathbf{r}_i}(\gamma)\,\psi_i\,. \end{cases}$$

ψ_i **is not a modular form multiplet, the integer** $(-k_i)$ **can be** >0**,** <0**, 0.**
Invariance of $\mathcal{S}$ **under these transformations implies (global SUSY):**

$$W(\tau,\psi) \to W(\tau,\psi)\,,$$

The superpotential can be expanded in powers of ψ_i:

$$W(\tau,\psi) = \sum_n \sum_{\{i_1,\ldots,i_n\}} \sum_s g_{i_1\ldots i_n,s}\left(Y_{i_1\ldots i_n,s}(\tau)\,\psi_{i_1}\ldots\psi_{i_n}\right)_{\mathbf{1},s}\,,$$

$\mathbf{1}$ stands for an invariant singlet of Γ_N (Γ'_N). For each set of n fields $\{\psi_{i_1},\ldots,\psi_{i_n}\}$, the index s labels the independent singlets. Each of these is accompanied by a coupling constant $g_{i_1\ldots i_n,s}$ and is obtained using a modular multiplet $Y_{i_1\ldots i_n,s}$ of the requisite weight. To ensure invariance of W under Γ_N (Γ'_N), $Y_{i_1\ldots i_n,s}(\tau)$ must transform as:

$$Y(\tau) \xrightarrow{\gamma} (c\tau+d)^{k_Y}\rho_{\mathbf{r}_Y}(\gamma)\,Y(\tau)\,,$$

$\mathbf{r}_Y$ is a representation of Γ_N (Γ'_N), and k_Y and $\mathbf{r}_Y$ are such that

$$k_Y = k_{i_1} + \cdots + k_{i_n}\,, \tag{1}$$

$$\mathbf{r}_Y \otimes \mathbf{r}_{i_1} \otimes \ldots \otimes \mathbf{r}_{i_n} \supset \mathbf{1}\,. \tag{2}$$

Thus, $Y_{i_1\ldots i_n,s}(\tau)$ represents a multiplet of weight k_Y and level N modular forms transforming in the representation $\mathbf{r}_Y$ of Γ_N (Γ'_N).

Mass Matrices

Consider the bilinear (i.e., mass term)

$$\psi_i^c \, M(\tau)_{ij} \, \psi_j \,,$$

where the superfields ψ and ψ^c transform as

$$\psi \xrightarrow{\gamma} (c\tau + d)^{-k} \rho_r(\gamma) \, \psi \quad (\rho(\gamma)\,, \; \Gamma_N^{(\prime)}\,, \; N = 2, 3, 4, 5)\,,$$
$$\psi^c \xrightarrow{\gamma} (c\tau + d)^{-k^c} \rho^c_{r^c}(\gamma) \, \psi^c\,, \quad (\rho^c(\gamma)\,, \; \Gamma_N^{(\prime)})\,.$$

Modular invariance: $M(\tau)_{ij}$ must be modular form of level N and weight $K \equiv k + k^c$.

It is of crucial importance for model building to find the basis of modular forms of the lowest weight 2 (weight 1) transforming in irreps of Γ_N (Γ'_N).
Multiplets of Γ_N (Γ'_N) of higher weight modular forms can be constructed from tensor products of the lowest weight 2 (weigh 1) multiplets (they represent homogeneous polynomials of the lowest weight modular forms).

The modular forms of level $N = 2,3,4$ for $\Gamma_{2,3,4} \simeq S_3, A_4, S_4$ have been constructed by use of the Dedekind eta function, $\eta(\tau)$,

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)\,, \qquad q = e^{i2\pi\tau}\,.$$

Modular forms of level $N = 4$ for $\Gamma'_4 \simeq S'_4$ – in terms of $\theta(\tau)$ and $\varepsilon(\tau)$:

$$\theta(\tau) \equiv \frac{\eta^5(2\tau)}{\eta^2(\tau)\eta^2(4\tau)} = \Theta_3(2\tau)\,,\ \varepsilon(\tau) \equiv \frac{2\,\eta^2(4\tau)}{\eta(2\tau)} = \Theta_2(2\tau)\,.$$

$\Theta_2(\tau)$ and $\Theta_3(\tau)$ are the Jacobi theta constants, $\eta(a\tau)$, $a = 1,2,4$, is the Dedekind eta function.

For ($\Gamma_3 \simeq A_4$), the generating (basis) modular forms of weight 2 were shown to form a 3 of A_4 (expressed in terms of the Dedekind eta function).

F. Feruglio, arXiv:1706.08749

For ($\Gamma_4 \simeq S_4$), the 5 basis modular forms of weight 2 were shown to form a 2 and a $3'$ of S_4 (expressed in terms of the Dedekind eta function).

J. Penedo, STP, arXiv:1806.11040

For ($\Gamma_5 \simeq A_5$), the 11 basis modular forms of weight 2 were shown to form a 3, a $3'$ and a 5 of A_5 (expressed in terms of the Jacobi theta functions).

P.P. Novichkov, J. Penedo, STP, A.V. Titov, arXiv:1812.02158

For ($\Gamma_2 \simeq S_3$), the 2 basis modular forms of weight 2 were shown to form a 2 of S_3 (expressed in terms of the Dedekind eta function).

T. Kobayashi, K. Tanaka, T.H. Tatsuishi, arXiv:1803.10391

Multiplets of higher weight modular forms have been also constructed from tensor products of the lowest weight 2 multiplets:
i) for $N = 4$ (i.e., S_4), multiplets of weight 4 (weight $k \leq 10$) were derived in arXiv:1806.11040 (arXiv:1811.04933);
ii) for $N = 3$ (i.e., A_4) multiplets of weight $k \leq 6$ were found in arXiv:1706.08749;
iii) for $N = 5$ (i.e., A_5), multiplets of weight $k \leq 10$ were derived in arXiv:1812.02158.

For ($\Gamma'_3 \simeq A'_4$), the generating (basis) modular forms of weight 1 were shown to form a 2 of A'_4 (expressed in terms of the Dedekind eta function).

X.-G. Liu, G.-J. Ding, arXiv:1907.01488

For ($\Gamma'_4 \simeq S'_4$), the 3 basis modular forms of weight 1 were shown to form a $\hat{3}$ and of S'_4 (expressed in terms of two Jacobi constant functions).

P.P. Novichkov et al., arXiv:2006.03058

For ($\Gamma'_5 \simeq A'_5$), the 6 basis modular forms of weight 1 were shown to form a $\hat{6}$ of A'_5.

C.-Y. Yao et al., arXiv:2011.03501

In each of three cases of A'_4, S'_4 and A'_5 the lowest weight 1 modular forms, and thus all higher weight modular forms, icluding those (of even weight) associated with A_4, S_4 and A_5, constructed from tensor products of the lowest weight 1 multiplets, were shown to be expressed in term of only two independent functions of τ.
These pairs of functions are different for the three different groups; but they all are related (in a different way) to the Dedekind eta function and have similar $q-$expansions, i.e., power series expansions in $q = e^{2\pi i\tau}$.

Example: S'_4

P.P. Novichkov, J.T. Penedo. S.T.P., arXiv:2006.03058

Weight 1 modular forms furnishing a $\hat{3}$ of S'_4:

$$Y^{(1)}_{\hat{\mathbf{3}}}(\tau) = \begin{pmatrix} \sqrt{2}\,\varepsilon\,\theta \\ \varepsilon^2 \\ -\theta^2 \end{pmatrix}$$

Modular S_4 lowest-weight 2 multiplets furnish a 2 and a $3'$ irreducible representations of S_4 (S'_4) and are given by: :

$$Y^{(2)}_{\mathbf{2}}(\tau) = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\theta^4 + \varepsilon^4\right) \\ -\sqrt{6}\,\varepsilon^2\,\theta^2 \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \qquad Y^{(2)}_{\mathbf{3'}}(\tau) = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\theta^4 - \varepsilon^4\right) \\ -2\,\varepsilon\,\theta^3 \\ -2\,\varepsilon^3\,\theta \end{pmatrix} = \begin{pmatrix} Y_3 \\ Y_4 \\ Y_5 \end{pmatrix}.$$

At weight $k = 3$, a non-trivial singlet and two triplets exclusive to S'_4 arise:

$$Y^{(3)}_{\hat{\mathbf{1}}'}(\tau) = \sqrt{3}\left(\varepsilon\,\theta^5 - \varepsilon^5\,\theta\right),$$

$$Y^{(3)}_{\hat{\mathbf{3}}}(\tau) = \begin{pmatrix} \varepsilon^5\,\theta + \varepsilon\,\theta^5 \\ \frac{1}{2\sqrt{2}}\left(5\,\varepsilon^2\,\theta^4 - \varepsilon^6\right) \\ \frac{1}{2\sqrt{2}}\left(\theta^6 - 5\,\varepsilon^4\,\theta^2\right) \end{pmatrix}, \qquad Y^{(3)}_{\hat{\mathbf{3}}'}(\tau) = \frac{1}{2}\begin{pmatrix} -4\sqrt{2}\,\varepsilon^3\,\theta^3 \\ \theta^6 + 3\,\varepsilon^4\,\theta^2 \\ -3\,\varepsilon^2\,\theta^4 - \varepsilon^6 \end{pmatrix}.$$

At weight $k = 4$ one again recovers the S_4 result: the modular forms furnish a 1, 2, 3 and $3'$ irreducible representations of S_4 (S'_4).

$$Y^{(4)}_{\mathbf{1}}(\tau) = \frac{1}{2\sqrt{3}}\left(\theta^8 + 14\,\varepsilon^4\,\theta^4 + \varepsilon^8\right), \qquad Y^{(4)}_{\mathbf{2}}(\tau) = \begin{pmatrix} \frac{1}{4}\left(\theta^8 - 10\,\varepsilon^4\,\theta^4 + \varepsilon^8\right) \\ \sqrt{3}\left(\varepsilon^2\,\theta^6 + \varepsilon^6\,\theta^2\right) \end{pmatrix},$$

$$Y^{(4)}_{\mathbf{3}}(\tau) = \frac{3}{2\sqrt{2}}\begin{pmatrix} \sqrt{2}\left(\varepsilon^2\,\theta^6 - \varepsilon^6\,\theta^2\right) \\ \varepsilon^3\,\theta^5 - \varepsilon^7\,\theta \\ -\varepsilon\,\theta^7 + \varepsilon^5\,\theta^3 \end{pmatrix}, \qquad Y^{(4)}_{\mathbf{3'}}(\tau) = \begin{pmatrix} \frac{1}{4}\left(\theta^8 - \varepsilon^8\right) \\ \frac{1}{2\sqrt{2}}\left(\varepsilon\,\theta^7 + 7\,\varepsilon^5\,\theta^3\right) \\ \frac{1}{2\sqrt{2}}\left(7\,\varepsilon^3\,\theta^5 + \varepsilon^7\,\theta\right) \end{pmatrix},$$

The functions $\theta(\tau)$ and $\varepsilon(\tau)$ are given by:

$$\theta(\tau) \equiv \frac{\eta^5(2\tau)}{\eta^2(\tau)\eta^2(4\tau)} = \Theta_3(2\tau)\,, \qquad \varepsilon(\tau) \equiv \frac{2\,\eta^2(4\tau)}{\eta(2\tau)} = \Theta_2(2\tau)\,.$$

$\Theta_2(\tau)$ and $\Theta_3(\tau)$ are the Jacobi theta constants, $\eta(a\tau)$, $a = 1,2,4$, is the Dedekind eta function.

The functions $\theta(\tau)$ and $\varepsilon(\tau)$ admit the following q-expansions - power series expansions in $q_4 \equiv \exp(i\pi\tau/2)$ ($\mathrm{Im}(\tau) \geq \sqrt{3}/2$, $|q_4| \lesssim 0.26$) :

$$\theta(\tau) = 1 + 2\sum_{k=1}^{\infty} q_4^{(2k)^2} = 1 + 2\,q_4^4 + 2\,q_4^{16} + \ldots\,,$$

$$\varepsilon(\tau) = 2\sum_{k=1}^{\infty} q_4^{(2k-1)^2} = 2\,q_4 + 2\,q_4^9 + 2\,q_4^{25} + \ldots\,.$$

In the "large volume" limit $\mathrm{Im}\,\tau \to \infty$, $\theta \to 1$, $\varepsilon \to 0$.
In this limit $\varepsilon \sim 2\,q_4$ and $\mathcal{E}$ can be used as an expansion parameter instead of q_4.
Due to quadratic dependence in the exponents of q_4, the $q-$expansion series converge rapidly in the fundamental domain of the modular group, where $\mathrm{Im}(\tau) \geq \sqrt{3}/2$ and $|q_4| \leq \exp(-\pi\sqrt{3}/4) \simeq 0.26$.

Similar conclusions are valid for the pair of functions in terms of which the lowest weight 1 modular forms, and thus all higher weight modular forms of A'_4 and A'_5 are expressed.

Example: A'_5

C.-Y. Yao et al., arXiv:2011.03501

Weight 1 modular forms furnishing a $\hat{6}$ of A'_5:

$$Y^{(1)}_{\hat{\mathbf{6}}}(\tau) = \left(2\,\varepsilon_5^5 + \theta_5^5, 2\,\theta_5^5 - \varepsilon_5^5, 5\,\varepsilon_5\,\theta_5^4, 5\sqrt{2}\,\varepsilon_5^2\,\theta_5^3, -5\sqrt{2}\,\varepsilon_5^3\,\theta_5^2, 5\,\varepsilon_5^4\,\theta_5\right)^T .$$

The functions $\theta_5(\tau)$ and $\varepsilon_5(\tau)$ are related to the Dedekind eta function and have the following $q-$expansions:

$$\begin{aligned}\theta_5(\tau) &= 1 + \frac{3}{5}\,q_5^5 + \frac{2}{25}\,q_5^{10} - \frac{28}{125}q_5^{15} + \ldots\,,\\ \varepsilon_5(\tau) &= q_5\left(1 - \frac{2}{5}\,q_5^5 + \frac{12}{25}\,q_5^{10} + + \frac{37}{125}q_5^{15} + \ldots\right)\,,\;\; q_5 \equiv \exp(i2\pi\tau/5)\,.\end{aligned}$$

In the "large volume" limit $\mathrm{Im}\,\tau \to \infty$, similar to the S'_4 two functions, $\theta_5 \to 1$, $\varepsilon_5 \to 0$.
In this limit $\varepsilon_5 \sim q_5$ and ε_5 can be used as an expansion parameter instead of q_5.
The q_5-expansion series converge rapidly in the fundamental domain of the modular group, where $\mathrm{Im}(\tau) \geq \sqrt{3}/2$ and $|q_5| \leq \exp(-\pi\sqrt{3}/5) \simeq 0.34$.

Example: Lepton Flavour Models Based on S_4 (Seesaw Models without Flavons)

P.P. Novichkov et al., arXiv:1811.04933

We assume that neutrino masses originate from the (supersymmetric) type I seesaw mechanism.

We assume further:

- Higgs doublets H_u and H_d transform trivially under Γ_4, $\rho_u = \rho_d \sim \mathbf{1}$, and $k_u = k_d = 0$;
- lepton $SU(2)$ doublets L_1, L_2, L_3 furnish a 3-dim. irrep of S_4, i.e., $\rho_L \sim \mathbf{3}$ or $\mathbf{3}'$, and carry weight $k_L = 2$;
- neutral lepton gauge singlets N_1^c, N_2^c, N_3^c transform as a triplet of Γ_4, $\rho_N \sim \mathbf{3}$ or $\mathbf{3}'$, and carry weight $k_N = 0$;
- charged lepton $SU(2)$ singlets E_1^c, E_2^c, E_3^c transform as singlets of Γ_4, $\rho_{1,2,3} \sim \mathbf{1}', \mathbf{1}, \mathbf{1}'$ and carry weights $k_{1,2.3} = 0, 2, 2$.

We work in a basis in which the S_4 generators S and T are represented by symmetric matrices for all irreducible representations $\mathbf{r}$. In this basis the triplet irreps of S and T to be used in this section read:

$$S = \pm\frac{1}{3}\begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2\omega & 2 & -\omega^2 \\ 2\omega^2 & -\omega & 2 \end{pmatrix}, \quad T = \pm\frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega & 2\omega^2 & -1 \\ 2\omega^2 & -1 & 2\omega \end{pmatrix},$$

$\omega = e^{i2\pi\tau/3}$. The plus (minus) corresponds to the irrep $\mathbf{3}$ ($\mathbf{3'}$) of S_4.

In the employed basis we have:

$$ST = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

We assume that neutrino masses originate from the (supersymmetric) type I seesaw mechanism. The superpotential in the lepton sector reads

$$W = \alpha \left(E^c L H_d f_E(Y)\right)_{\mathbf{1}} + g \left(N^c L H_u f_N(Y)\right)_{\mathbf{1}} + \Lambda \left(N^c N^c f_M(Y)\right)_{\mathbf{1}} \,,$$

a sum over all independent invariant singlets with the coefficients $\alpha = (\alpha, \alpha', \dots)$, $g = (g, g', \dots)$ and $\Lambda = (\Lambda, \Lambda', \dots)$ is implied. $f_{E,N,M}(Y)$ denote the modular form multiplets required to ensure modular invariance.

We assume further:

- Higgs doublets H_u and H_d transform trivially under Γ_4, $\rho_u = \rho_d \sim \mathbf{1}$, and $k_u = k_d = 0$;
- lepton $SU(2)$ doublets L_1, L_2, L_3 furnish a 3-dim. irrep of S_4, i.e., $\rho_L \sim \mathbf{3}$ or $\mathbf{3}'$, and carry weight $k_L = 2$;
- neutral lepton gauge singlets N_1^c, N_2^c, N_3^c transform as a triplet of Γ_4, $\rho_N \sim \mathbf{3}$ or $\mathbf{3}'$, and carry weight $k_N = 0$;
- charged lepton $SU(2)$ singlets E_1^c, E_2^c, E_3^c transform as singlets of Γ_4, $\rho_{1,2,3} \sim \mathbf{1}'\,, \mathbf{1}\,, \mathbf{1}'$ and carry weights $k_{1,2.3} = 0, 2, 2$.

With these assumptions, we can rewrite the superpotential as

$$W = \sum_{i=1}^{3} \alpha_i \left(E_i^c \, L \, f_{E_i}(Y)\right)_{\mathbf{1}} H_d + g \left(N^c \, L \, f_N(Y)\right)_{\mathbf{1}} H_u + \Lambda \left(N^c \, N^c \, f_M(Y)\right)_{\mathbf{1}}$$

By specifying the weights of the matter fields one obtains the weights of the relevent modular forms.
After modular symmetry breaking, the matrices of charged lepton and neutrino Yukawa couplings, λ and $\mathcal{Y}$, as well as the Majorana mass matrix M for heavy neutrinos, are generated:

$$W = \lambda_{ij}\, E_i^c\, L_j\, H_d + \mathcal{Y}_{ij}\, N_i^c\, L_j\, H_u + \frac{1}{2}\, M_{ij}\, N_i^c\, N_j^c\,,$$

a sum over $i, j = 1, 2, 3$ is assumed. After integrating out N^c and after EWS breaking, the charged lepton mass matrix M_e and the light neutrino Majorana mass matrix M_ν are generated (we work in the L-R convention for the charged lepton mass term and the R-L convention for the light and heavy neutrino Majorana mass terms):

$$M_e = v_d\, \lambda^\dagger\,, \quad v_d \equiv H_d^0\,,$$
$$M_\nu = -v_u^2\, \mathcal{Y}^T M^{-1} \mathcal{Y}\,, \quad v_u \equiv H_u^0\,.$$

The Majorana mass term for heavy neutrinos

Assume $k_\Lambda = 0$, i.e., no non-trivial modular forms are present in $\Lambda\,(N^c\,N^c\,f_M\,(Y))_{\mathbf{1}}$, $k_N = 0$, and for both $\rho_N \sim \mathbf{3}$ or $\rho_N \sim \mathbf{3'}$

$$(N^c\,N^c)_{\mathbf{1}} = N^c_1\,N^c_1 + N^c_2\,N^c_3 + N^c_3\,N^c_2\,,$$

leading to the following mass matrix for heavy neutrinos:

$$M = 2\,\Lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \textbf{for} \quad k_\Lambda = 0\,.$$

The spectrum of heavy neutrino masses is degenerate; the only free parameter is the overall scale Λ, which can be rendered real. The Majorana mass term conserves a "non-standard" lepton charge and two of the three heavy Majorana neutrinos with definite mass form a Dirac pair.

C.N. Leung, STP, 1983

The neutrino Yukawa couplings

The lowest non-trivial weight, $k_L = 2$, leads to

$$g\left(N^c\,L\,Y_{\mathbf{2}}^{(2)}\right)_{\mathbf{1}} H_u + g'\left(N^c\,L\,Y_{\mathbf{3'}}^{(2)}\right)_{\mathbf{1}} H_u\,.$$

There are 4 possible assignments of ρ_N and ρ_L we consider. Two of them, namely $\rho_N = \rho_L \sim \mathbf{3}$ and $\rho_N = \rho_L \sim \mathbf{3'}$ give the following form of $\mathcal{Y}$:

$$\mathcal{Y} = g\left[\begin{pmatrix} 0 & Y_1 & Y_2 \\ Y_1 & Y_2 & 0 \\ Y_2 & 0 & Y_1 \end{pmatrix} + \frac{g'}{g}\begin{pmatrix} 0 & Y_5 & -Y_4 \\ -Y_5 & 0 & Y_3 \\ Y_4 & -Y_3 & 0 \end{pmatrix}\right], \quad \text{for} \quad k_L + K_N = 2 \quad \text{and} \quad \rho_N = \rho_L\,.$$

The two remaining combinations, $(\rho_N, \rho_L) \sim (\mathbf{3}, \mathbf{3'})$ and $(\mathbf{3'}, \mathbf{3})$, lead to:

$$\mathcal{Y} = g\left[\begin{pmatrix} 0 & -Y_1 & Y_2 \\ -Y_1 & Y_2 & 0 \\ Y_2 & 0 & -Y_1 \end{pmatrix} + \frac{g'}{g}\begin{pmatrix} 2Y_3 & -Y_5 & -Y_4 \\ -Y_5 & 2Y_4 & -Y_3 \\ -Y_4 & -Y_3 & 2Y_5 \end{pmatrix}\right], \quad \text{for} \quad k_L + k_N = 2 \quad \text{and} \quad \rho_N \neq \rho_L\,.$$

In both cases, up to an overall factor, the matrix $\mathcal{Y}$ depends on one complex parameter g'/g and the VEV τ.

$$Y_{\mathbf{2}}^{(2)}(\tau) = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\theta^4 + \varepsilon^4\right) \\ -\sqrt{6}\,\varepsilon^2\,\theta^2 \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \qquad Y_{\mathbf{3'}}^{(2)}(\tau) = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\theta^4 - \varepsilon^4\right) \\ -2\,\varepsilon\,\theta^3 \\ -2\,\varepsilon^3\,\theta \end{pmatrix} = \begin{pmatrix} Y_3 \\ Y_4 \\ Y_5 \end{pmatrix}.$$

The charged lepton Yukawa couplings

In the minimal (in terms of weights) viable possibility for $L_{1,2,3}$ furnishing a 3-dim. irrep of S_4, i.e., $\rho_L \sim \mathbf{3}$ or $\mathbf{3'}$, and carrying a weight $k_L = 2$, and $E^c_{1,2,3}$ transforming as singlets of Γ_4, $\rho_{1,2,3} \sim 1', 1, 1'$ (up to permutations) and carrying weights $k_{1,2.3} = 0, 2, 2$, the relevant part of W, W_e, can take 6 different forms which lead to the same matrix U_e diagonalising $M_e M_e^\dagger = v_d^2\, \lambda^\dagger \lambda$, and thus do not lead to new results for the PMNS matrix. We give just one of these 6 forms corresponding to $\rho_L = \mathbf{3}$, $\rho_1 = \mathbf{1'}$, $\rho_2 = \mathbf{1}$, $\rho_3 = \mathbf{1'}$:

$$\alpha \left(E_1^c\, L\, Y^{(2)}_{\mathbf{3'}}\right)_{\mathbf{1}} H_d + \beta \left(E_2^c\, L\, Y^{(4)}_{\mathbf{3}}\right)_{\mathbf{1}} H_d + \gamma \left(E_3^c\, L\, Y^{(4)}_{\mathbf{3'}}\right)_{\mathbf{1}} H_d\,.$$

This leads leads to

$$\lambda = \begin{pmatrix} \alpha\, Y_3 & \alpha\, Y_5 & \alpha\, Y_4 \\ \beta\,(Y_1Y_4 - Y_2Y_5) & \beta\,(Y_1Y_3 - Y_2Y_4) & \beta\,(Y_1Y_5 - Y_2Y_3) \\ \gamma\,(Y_1Y_4 + Y_2Y_5) & \gamma\,(Y_1Y_3 + Y_2Y_4) & \gamma\,(Y_1Y_5 + Y_2Y_3) \end{pmatrix},$$

In this "minimal" example the matrix λ depends on 3 free parameters, α, β and γ, which can be rendered real by re-phasing of the charged lepton fields.

We recall that

$$M_e = v_d\, \lambda^\dagger\,, \qquad v_d \equiv H_d^0\,,$$
$$M_\nu = -v_u^2\, \mathcal{Y}^T M^{-1} \mathcal{Y}\,, \qquad v_u \equiv H_u^0\,.$$

Parameters of the model: α, β, γ, g^2/Λ **– real;** g' **and VEV of** τ **– complex, i.e., 6 real parameters + 2 phases for description of 12 observables (3 charged lepton masses, 3 neutrino masses, 3 mixing angles and 3 CPV phases). Excellent description of the data is obtained also for real** g' **(i.e., 6 real parameters + 1 phase, employing** gCP**).**

The 3 real parameters $v_d\alpha$, β/α, γ/α **– fixed by fitting** m_e, m_μ **and** m_τ**.**
The remaining 3 real parameters and 2 (1) phases – $v_u^2 g^2/\Lambda$, $|g'/g|$, $|\tau|$ **and** $\arg(g'/g)$, $\arg\tau$ **(**$\arg\tau$**) – describe the 9** ν **observables, 3** ν **masses, 3 mixing angles and 3 CPV phases.**

The model considered leads to testable predictions for $\min(m_j)$ **(**$\sum_i m_i$**), type of the** ν **mass spectrum (NO or IO), the CPV Dirac and Majorana phases,** $|\langle m\rangle|$, θ_{23}**, as well as of correlations between different observables.**

Numerical Analysis

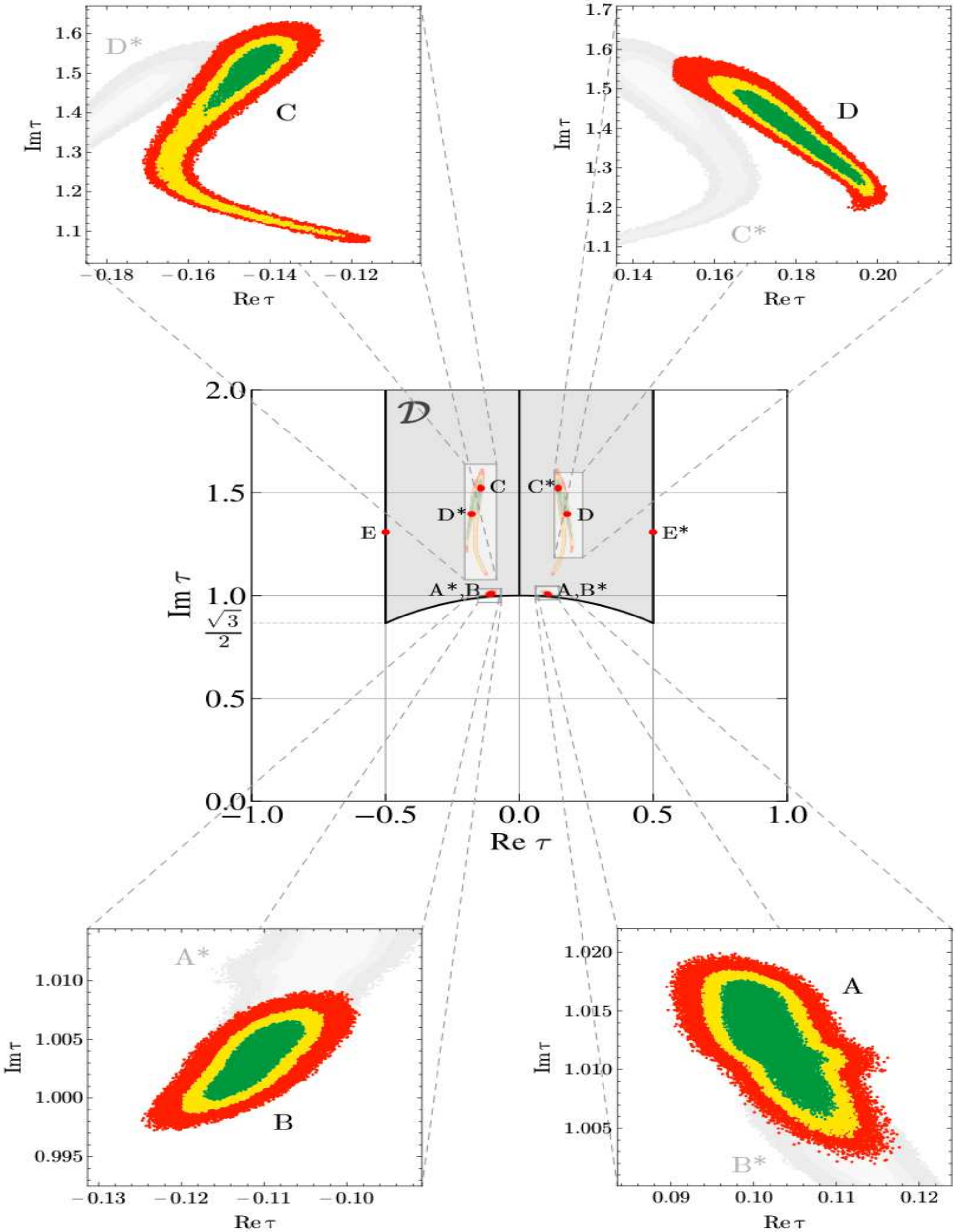
Each model depends on a set of dimensionless parameters

$$p_i = (\tau,\, \beta/\alpha,\, \gamma/\alpha,\, g'/g,\, \ldots,\, \Lambda'/\Lambda,\, \ldots)\,,$$

which determine dimensionless observables (mass ratios, mixing angles and phases), and two overall mass scales: $v_d\,\alpha$ for M_e and $v_u^2\,g^2/\Lambda$ for M_ν. Phenomenologically viable models are those that lead to values of observables which are in close agreement with the experimental results summarized in the Table below. We assume also to be in a regime in which the running of neutrino parameters is negligible.

Observable	Best fit value and 1σ range	
m_e/m_μ	0.0048 ± 0.0002	
m_μ/m_τ	0.0565 ± 0.0045	
	NO	IO
$\delta m^2/(10^{-5}\ \text{eV}^2)$	$7.34^{+0.17}_{-0.14}$	
$\lvert\Delta m^2\rvert/(10^{-3}\ \text{eV}^2)$	$2.455^{+0.035}_{-0.032}$	$2.441^{+0.033}_{-0.035}$
$r \equiv \delta m^2/\lvert\Delta m^2\rvert$	0.0299 ± 0.0008	0.0301 ± 0.0008
$\sin^2\theta_{12}$	$0.304^{+0.014}_{-0.013}$	$0.303^{+0.014}_{-0.013}$
$\sin^2\theta_{13}$	$0.0214^{+0.0009}_{-0.0007}$	$0.0218^{+0.0008}_{-0.0007}$
$\sin^2\theta_{23}$	$0.551^{+0.019}_{-0.070}$	$0.557^{+0.017}_{-0.024}$
δ/π	$1.32^{+0.23}_{-0.18}$	$1.52^{+0.14}_{-0.15}$

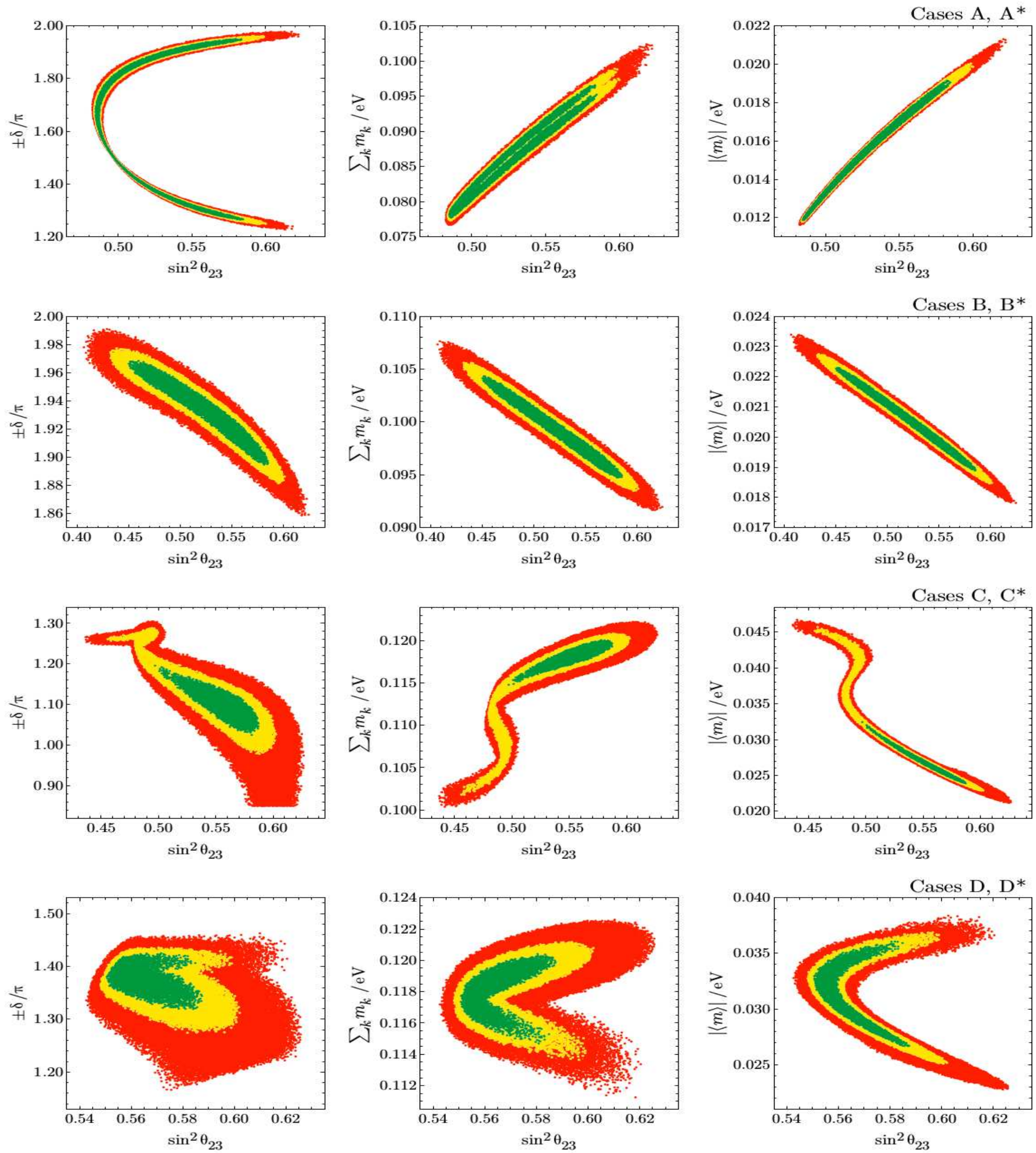
Best fit values and 1σ ranges for neutrino oscillation parameters, obtained in the global analysis of F. Capozzi et al., arXiv:1804.09678, and for charged-lepton mass ratios, given at the scale 2×10^{16} GeV with the $\tan\beta$ averaging described in F. Feruglio, arXiv:1706.08749 obtained from G.G. Ross and M. Serna, arXiv:0704.1248. The parameters entering the definition of r are $\delta m^2 \equiv m_2^2 - m_1^2$ and $\Delta m^2 \equiv m_3^2 - (m_1^2 + m_2^2)/2$. The best fit value and 1σ range of δ did not drive the numerical searches here reported.



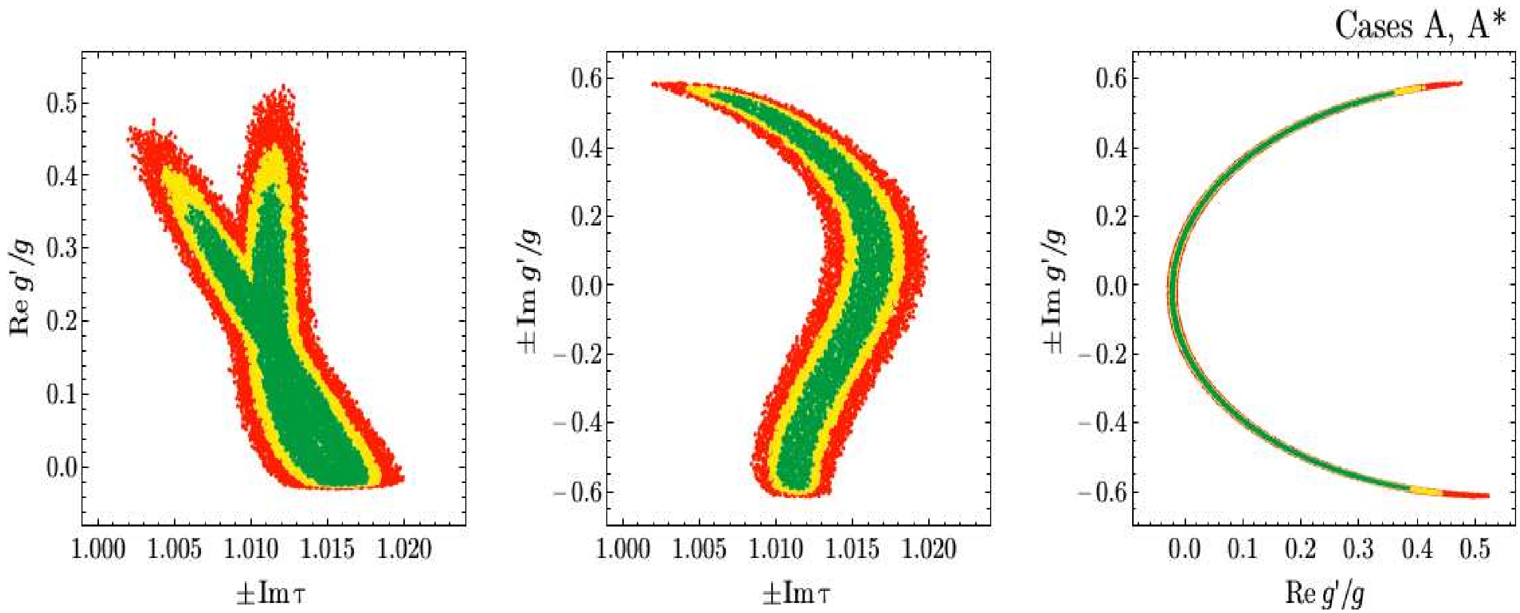
P.P. Novichkov, J.T. Penedo, STP, A.V. Titov, arXiv:1811.04933

	Best fit value	2σ range	3σ range
$\mathrm{Re}\,\tau$	± 0.1045	$\pm(0.09597 - 0.1101)$	$\pm(0.09378 - 0.1128)$
$\mathrm{Im}\,\tau$	1.01	1.006 – 1.018	1.004 – 1.018
β/α	9.465	8.247 – 11.14	7.693 – 12.39
γ/α	0.002205	0.002032 – 0.002382	0.001941 – 0.002472
$\mathrm{Re}\,g'/g$	0.233	$-0.02383 - 0.387$	$-0.02544 - 0.4417$
$\mathrm{Im}\,g'/g$	± 0.4924	$\pm(-0.592 - 0.5587)$	$\pm(-0.6046 - 0.5751)$
$v_d\,\alpha$ [MeV]	53.19		
$v_u^2\,g^2/\Lambda$ [eV]	0.00933		
m_e/m_μ	0.004802	0.004418 – 0.005178	0.00422 – 0.005383
m_μ/m_τ	0.0565	0.048 – 0.06494	0.04317 – 0.06961
r	0.02989	0.02836 – 0.03148	0.02759 – 0.03224
δm^2 [10^{-5} eV^2]	7.339	7.074 – 7.596	6.935 – 7.712
$\lvert\Delta m^2\rvert$ [10^{-3} eV^2]	2.455	2.413 – 2.494	2.392 – 2.513
$\sin^2\theta_{12}$	0.305	0.2795 – 0.3313	0.2656 – 0.3449
$\sin^2\theta_{13}$	0.02125	0.01988 – 0.02298	0.01912 – 0.02383
$\sin^2\theta_{23}$	0.551	0.4846 – 0.5846	0.4838 – 0.5999
Ordering	NO		
m_1 [eV]	0.01746	0.01196 – 0.02045	0.01185 – 0.02143
m_2 [eV]	0.01945	0.01477 – 0.02216	0.01473 – 0.02307
m_3 [eV]	0.05288	0.05099 – 0.05405	0.05075 – 0.05452
$\sum_i m_i$ [eV]	0.0898	0.07774 – 0.09661	0.07735 – 0.09887
$\lvert\langle m\rangle\rvert$ [eV]	0.01699	0.01188 – 0.01917	0.01177 – 0.02002
δ/π	± 1.314	$\pm(1.266 - 1.95)$	$\pm(1.249 - 1.961)$
α_{21}/π	± 0.302	$\pm(0.2821 - 0.3612)$	$\pm(0.2748 - 0.3708)$
α_{31}/π	± 0.8716	$\pm(0.8162 - 1.617)$	$\pm(0.7973 - 1.635)$
$N\sigma$	0.02005		

Best fit values along with 2σ and 3σ ranges of the parameters and observables in cases A and A*, (which refer to $(k_\Lambda, k_g) = (0, 2)$ and $\tau = \pm 0.1045 + i\,1.01$).



P.P. Novichkov et al., arXiv:1811.04933



P.P. Novichkov et al., arXiv:1811.04933

Fermion Mass Hierarchies without Fine-Tuning

The $l-$ and $q-$ mass hierarchies in all modular flavour models proposed so far in the literarture − obtained with fine-tuning.

Fine-tuning:
i) high sensitivity of observables to model parameters, and/or
ii) unjustified hierarchies between model's parameters.

The flavour structure of the fermion mass matrices M_F can be severely constrained by the residual symmetries present at each of the 3 symmetry points,
$\tau_{\text{sym}} = i$,
$\tau_{\text{sym}} = \omega \equiv \exp(i\,2\pi/3) = -1/2 + i\sqrt{3}/2$, and
$\tau_{\text{sym}} = i\infty$:
residual symmetries may enforce the presence of multiple zeros in M_F.

As τ moves away from τ_{sym}, the zero entries in M_F will become non-zero. Their magnitude will be controlled by the size of the departure ϵ from τ_{sym} and by the field transformation properties under the residual symmetry group.

Thus, fine-tuning might be avoided in the vicinity of τ_{sym} as $l-$ and $q-$ mass hierarchies would follow from the properties of the modular forms present in the corresponding M_F rather than being determined by the values of the accompanying constants also present in M_F.

The successful technical realisation of this idea:
P.P. Novichkov, J.T. Penedo, STP, arXiv:2102.07488.

A'_5 Model with $L \sim \mathbf{3}$, $E^c \sim \mathbf{3'}$, $N^c \sim \hat{\mathbf{2}}'$

$L \sim (\mathbf{3}, k_L = 3)$, $E^c \sim (\mathbf{3'}, k_E = 1)$, $N^c \sim (\hat{\mathbf{2}}', k_N = 2)$; vicinity of $\tau = i\infty$.

We consider first the most 'structured' series of hierarchical models, i.e. the case with both fields L, E^c furnishing complete irreps of the finite modular group.

At level $N = 5$ the only such possibility arises in the vicinity of $\tau = i\infty$ when L and E^c are different triplets of A'_5.
For neutrino masses generated via a type I seesaw, we have considered gauge-singlets N^c furnishing a complete irrep of dimension 2 or 3.

We performed a detailed search for a model which
i) is phenomenologically viable in the regime of interest,
ii) produces a charged-lepton spectrum which is not fine-tuned,
iii) involvs at most 8 effective parameters (including τ).

An observable O is typically considered fine-tuned with respect to some parameter p if **BG** $\equiv |\partial \ln O / \partial \ln p| \gtrsim 10$.

G. Giudice and R. Barbieri, 1987

Found one model satisfying these requirements:
$L \sim (\mathbf{3}, k_L = 3)$, $E^c \sim (\mathbf{3'}, k_E = 1)$, $N^c \sim (\hat{\mathbf{2}}', k_N = 2)$.
The charged-lepton mass matrix has the following structure:

$$M_e^\dagger \sim \begin{pmatrix} 1 & \epsilon^4 & \epsilon \\ \epsilon^3 & \epsilon^2 & \epsilon^4 \\ \epsilon^2 & \epsilon & \epsilon^3 \end{pmatrix}, \quad \epsilon \simeq q_5\,, \quad q_5 = \exp(i2\pi\tau/5)\,.$$

The predicted charged-lepton mass pattern is $(m_\tau, m_\mu, m_e) \sim (1, \epsilon, \epsilon^4)$.

S_4' Model with $L \sim \hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$, $E^c \sim \hat{\mathbf{3}}'$, $N^c \sim \mathbf{3}$

$L \sim (\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}, k_L = 2)$, $E^c \sim (\hat{\mathbf{3}}', k_E = 2)$, $N^c \sim (\mathbf{3}, k_N = 1)$; vicinity of $\tau = i\infty$.

In the second most 'structured' case, one of the fields L, E^c is an irreducible triplet, while the other decomposes into a doublet and a singlet of the finite modular group.
This possibility is realised at level $N = 4$ in the vicinity of $\tau = i\infty$.
For definiteness, we take $L = L_{12} \oplus L_3$ with $L_{12} \sim (\hat{\mathbf{2}}, k_L)$, $L_3 \sim (\hat{\mathbf{1}}, k_L)$, and $E^c \sim (\hat{\mathbf{3}}', k_E)$.

We have performed a systematic scan restricting ourselves to models involving at most 8 effective parameters (including τ) with no no limit on modular form weights.
Models predicting $m_e = 0$ are rejected.
N^c (when present) furnish a complete irrep of dimension 2 or 3.

Out of the 60 models thus identified, we have selected the only one which
i) is viable in the regime of interest and
ii) produces a charged-lepton spectrum which is not fine-tuned.
This model turns out to be consistent with the experimental bound on the Dirac CPV phase. It corresponds to $k_L = k_E = 2$ and $N^c \sim (\mathbf{3}, 1)$.

Using as expansion parameter $\epsilon \equiv \varepsilon/\theta \simeq 2q$, $\mathbf{q} = \exp(i\pi\tau/2)$, $M_e^\dagger$ is approximately given by:

$$M_e^\dagger \sim v_d \begin{pmatrix} \epsilon^2 & \epsilon & \epsilon^3 \\ 1 & \epsilon^3 & \epsilon \\ \epsilon^2 & \epsilon & \epsilon^3 \end{pmatrix} \; ; \; M_e^\dagger \simeq \frac{\sqrt{3}}{2} v_d \alpha_1 \theta^8 \begin{pmatrix} \epsilon^2 & \frac{\left(\tilde{\alpha}_2 + \sqrt{3}\right)}{2\sqrt{6}}\epsilon & \frac{\left(7\tilde{\alpha}_2 - \sqrt{3}\right)}{2\sqrt{6}}\epsilon^3 \\ -\frac{\tilde{\alpha}_2}{6} & \frac{\left(7\sqrt{3}\tilde{\alpha}_2 + 9\right)}{6\sqrt{6}}\epsilon^3 & \frac{\left(\sqrt{3}\tilde{\alpha}_2 - 9\right)}{6\sqrt{6}}\epsilon \\ \tilde{\alpha}_3\epsilon^2 & -\frac{\tilde{\alpha}_3}{\sqrt{2}}\epsilon & \frac{\tilde{\alpha}_3}{\sqrt{2}}\epsilon^3 \end{pmatrix} \; , \; \tilde{\alpha}_{2(3)} \equiv \alpha_{2(3)}/\alpha_1 \, .$$

The charged-lepton mass pattern is predicted to be $(m_\tau, m_\mu, m_e) \sim (1, \epsilon, \epsilon^3)$.

One can also find approximate expressions for the charged-lepton mass ratios:

$$\frac{m_e}{m_\mu} \simeq 18\sqrt{3}\frac{\left|\tilde{\alpha}_3(\tilde{\alpha}_2^2 - 3)\right|}{|\tilde{\alpha}_2|\left((\tilde{\alpha}_2+\sqrt{3})^2 + 12\tilde{\alpha}_3^2\right)}|\epsilon|^2\,,$$

$$\frac{m_\mu}{m_\tau} \simeq \sqrt{\frac{3}{2}}\frac{\sqrt{(\tilde{\alpha}_2+\sqrt{3})^2 + 12\tilde{\alpha}_3^2}}{|\tilde{\alpha}_2|}|\epsilon|\,.$$

These expressions isolate viable (ϵ-independent) regions in the plane of $\tilde{\alpha}_2^{-1} = \alpha_1/\alpha_2$ and $\tilde{\alpha}_3/\tilde{\alpha}_2 = \alpha_3/\alpha_2$.
These regions are shown in the next figure including contours quantifying the degree of fine-tuning involved in the relation between $l-$ mass ratios and constant parameters.
The model best-fit point corresponds to a small value of $\max(\mathbf{BG}) \simeq 0.74$.

Model	A'_5	S'_4	S'_4
Re τ	$-0.47^{+0.037}_{-0.096}$	$0.0235^{+0.0019}_{-0.002}$	$-0.496^{+0.009}_{-0.016}$
Im τ	$3.11^{+0.26}_{-0.19}$	$2.65^{+0.05}_{-0.04}$	$0.877^{+0.0023}_{-0.024}$
α_2/α_1	$1.33^{+0.20}_{-0.18}$	$-7.43^{+2.76}_{-12.2}$	—
α_3/α_1	$3.07^{+0.21}_{-0.15}$	$2.76^{+5.27}_{-1.33}$	$2.45^{+0.44}_{-0.42}$
α_4/α_1	—	—	$-2.37^{+0.36}_{-0.3}$
α_5/α_1	—	—	$1.01^{+0.06}_{-0.06}$
g_2/g_1	$-0.0781^{+0.0228}_{-0.0346}$	$-0.407^{+0.0002}_{-0.0003}$	$1.5^{+0.15}_{-0.14}$
g_3/g_1	$0.57^{+0.0023}_{-0.0017}$	$0.321^{+0.02}_{-0.043}$	$2.22^{+0.17}_{-0.15}$
$v_d\,\alpha_1$, GeV	$0.404^{+0.303}_{-0.149}$	$1.73^{+1.8}_{-1.15}$	$4.61^{+1.32}_{-1.33}$
$v_u^2\,g_1/\Lambda$, eV	$0.778^{+1.13}_{-0.477}$	$42.5^{+9.88}_{-5.2}$	$0.268^{+0.057}_{-0.063}$
$\epsilon(\tau)$	$0.0998^{+0.0267}_{-0.0274}$	$0.0313^{+0.0021}_{-0.0022}$	$0.0186^{+0.0028}_{-0.0023}$
CL mass pattern	$(1,\epsilon,\epsilon^4)$	$(1,\epsilon,\epsilon^3)$	$(1,\epsilon,\epsilon^2)$
max(BG)	5.579	0.738	0.848
m_e/m_μ	$0.00474^{+0.00062}_{-0.0005}$	$0.00479^{+0.00058}_{-0.00056}$	$0.00475^{+0.00061}_{-0.00052}$
m_μ/m_τ	$0.0573^{+0.0111}_{-0.0137}$	$0.0574^{+0.0117}_{-0.013}$	$0.0556^{+0.0136}_{-0.0116}$
r	$0.0297^{+0.0021}_{-0.0021}$	$0.0298^{+0.0019}_{-0.0023}$	$0.0298^{+0.00196}_{-0.0023}$
δm^2, 10^{-5} eV^2	$7.33^{+0.39}_{-0.4}$	$7.38^{+0.34}_{-0.44}$	$7.38^{+0.35}_{-0.44}$
$\lvert\Delta m^2\rvert$, 10^{-3} eV^2	$2.47^{+0.04}_{-0.04}$	$2.48^{+0.05}_{-0.04}$	$2.48^{+0.05}_{-0.04}$
$\sin^2\theta_{12}$	$0.306^{+0.036}_{-0.028}$	$0.301^{+0.044}_{-0.034}$	$0.304^{+0.039}_{-0.036}$
$\sin^2\theta_{13}$	$0.0222^{+0.0021}_{-0.0018}$	$0.0223^{+0.0017}_{-0.0022}$	$0.0221^{+0.0019}_{-0.002}$
$\sin^2\theta_{23}$	$0.55^{+0.044}_{-0.097}$	$0.548^{+0.045}_{-0.107}$	$0.539^{+0.0522}_{-0.099}$
m_1, eV	$0.0493^{+0.00041}_{-0.00046}$	$0.0204^{+0.00042}_{-0.00035}$	0
m_2, eV	$0.05^{+0.00037}_{-0.00042}$	$0.0221^{+0.0003}_{-0.00028}$	$0.0086^{+0.0002}_{-0.00026}$
m_3, eV	0	$0.0542^{+0.00054}_{-0.00046}$	$0.0502^{+0.00046}_{-0.00043}$
$\Sigma_i m_i$, eV	$0.0993^{+0.0008}_{-0.0009}$	$0.0967^{+0.0013}_{-0.001}$	$0.0588^{+0.0002}_{-0.0002}$
$\lvert\langle m\rangle\rvert$, eV	$0.0197^{+0.002}_{-0.0031}$	$0.0181^{+0.0004}_{-0.0003}$	$0.00144^{+0.00035}_{-0.00033}$
δ/π	$1.88^{+0.37}_{-0.13}$	$1.44^{+0.01}_{-0.01}$	$1\pm\mathcal{O}(10^{-6})$
α_{21}/π	$0.91^{+0.28}_{-0.09}$	$1.77^{+0.01}_{-0.01}$	0
α_{31}/π	0	$1.86^{+0.02}_{-0.02}$	$1\pm\mathcal{O}(10^{-5})$
$N\sigma$	0.431	0.649	0.563

Conclusions.

• Understanding the origin of quark and lepton flavours, i.e., of the patterns of quark, charged lepton and neutrino masses, of quark and lepton (neutrino) mixing and of the CP vilation in the quark and lepton sectors, is one of the most challenging fundamental problems in particle physics.

• The modular invariance (finite modular group symmetries) is a new elegant and promising approach to the flavour problem. It has been successfully applied to the lepton flavour problem. Encouraging attempts have been made to treat also the quark flavour problem as well as both the quark and lepton flavour problems.

• In its minimal version the approach involves just one complex scalar field – the modulus τ, and a certain rather small number of constant parameters. The modular symmetry is broken by the the VEV of τ, which can also be the only source of CP symmetry breaking.

• The lepton and quark flavour models based of finite modular symmetries proposed in the literature suffer from fine-tuning.

• In P.P. Novichkov, J.T. Penedo, S.T.P., arXiv:2102.07499 we have developed the formalism allowing to construct non fine-tuned modular invariant flavour models.

• The models of lepton flavour based of finite modular symmetries, lead to testable predictions for $\min(m_j)$, type of the neutrino mass spectrum (NO or IO), $\sum_i m_i$, the CPV Dirac and Majorana phases, $|\langle m \rangle|$, θ_{23}, as well as of correlations between different observables.

• The modular invariance approach to the flavour problem is still at the stage of its development at which there are still many aspects to be understood.