Maximally Non-Abelian Vortices from Self-dual Yang-Mills Fields

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1 Introduction

Non-Abelian Vortex : plays an important role in

Dual Confinement, **Cosmic String**, $\cdots$

Moduli gives **Effective Fields** on the soliton

Moduli Space describes **Dynamics of Non-Abelian Vortices**

Non-Abelian Vortices in $\boldsymbol{U(N)}$ gauge theory :

Moduli Matrix Approach powerful, but **No Exact solutions** yet

Exactly Solvable Vortex : $\boldsymbol{U(1)}$ **Vortex on a Hyperbolic Plane**

Equivalent to Instantons along a Line

Dimensional Reduction of Instantons to Hyperbolic Plane $\longrightarrow$ Vortices

Witten, Phys.Rev.Lett.**38**, 121 (1977)

Our Pourpose: Find Exact Solutions of **Non-Abelian Vortices**

2 $SO(3)$ Invariant Instantons

Pure $SU(2)$ Gauge Theory in Euclidean 4 dimensions

Instantons as Solutions of **Self-duality** Equations: $F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}$

Instantons along a line (Let's call it τ axis)

Invariant under **$SO(3)$ Rotations** around the τ axis

($SU(2)$ gauge transformations can be accompanied)

Take spherical polar coordinates r, θ, ϕ for S^3

$SO(3)$ invariant configurations : functions of τ, r (independent of θ, φ)

$$ds^2 = d\tau^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

$$= r^2\left(\frac{d\tau^2 + dr^2}{r^2} + (d\theta^2 + \sin^2\theta d\varphi^2)\right)$$

Conformally equivalent to **hyperbolic plane** and sphere $\Sigma \times S^2$

Yang-Mills Theory and Self-Duality is **Conformally invariant**

Complex coordinates: $z = \tau + ir,\ y = \tan\frac{\theta}{2} e^{i\varphi}$

$$ds^2 = (\mathrm{Im}z)^2 \left(\frac{dz d\bar{z}}{(\mathrm{Im}z)^2} + \frac{4}{(1+y\bar{y})^2} dy d\bar{y} \right)$$

Witten Ansatz : $\mathcal{A}_j = A_i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad j = \tau, r,$

$$\mathcal{A}_\theta = \begin{pmatrix} 0 & \bar{H} \\ H & 0 \end{pmatrix}, \mathcal{A}_\varphi = \begin{pmatrix} -\cos\theta & -i\bar{H}\sin\theta \\ iH\sin\theta & \cos\theta \end{pmatrix}$$

$A_i(\tau, r)$: **2 Dimensional gauge fields** for $U(1)$ (I_3 of $SU(2)$)

$H(\tau, r)$: **charged complex scalar** field

Self-Duality

$$\mathcal{F}_{\tau r} = \frac{1}{r^2 \sin\theta} \mathcal{F}_{\theta\varphi}, \quad \mathcal{F}_{\tau\theta} = \frac{1}{\sin\theta} \mathcal{F}_{\varphi r}, \quad \mathcal{F}_{r\theta} = \frac{1}{\sin\theta} \mathcal{F}_{\tau\varphi}$$

Reduces to **BPS equations for Vortices on a Hyperbolic Plane**

$$D_\tau H = i D_r H, \qquad F_{\tau r} = \frac{1}{2r^2}(1 - |H|^2)$$

$SO(3)$ invariant Instantons in $SU(2)$ are equivalent to

$U(1)$ vortices on a hyperbolic plane Σ

3 General $SO(3)$ Invariant Gauge Fields

Metric on $\Sigma \times S^2$ ($\sigma = \frac{2}{(\mathrm{Im} z)^2}$, if Σ is the hyperbolic plane)

$$ds^2 = \sigma(z, \bar{z}) dz d\bar{z} + \frac{8}{(1 + y\bar{y})^2} dy d\bar{y}$$

Field configuration should be invariant under

a combined spatial $SO(3)$ rotation and gauge $SO(3)$ rotation

General Embedding of $SO(3)$ into Non-Abelian Group G

Isotropy generator $SO(2)$ is mapped to an $SO(2)$ generator Λ in $\mathcal{G}$

Most general $SO(3)$ invariant gauge potential

$$\mathcal{A}_z = \mathrm{A}_z(z, \bar{z}), \quad \mathcal{A}_{\bar{z}} = \mathrm{A}_{\bar{z}}(z, \bar{z})$$

$$\mathcal{A}_y = \frac{1}{1 + y\bar{y}}(-\Phi(z, \bar{z}) - i\Lambda\bar{y}), \quad \mathcal{A}_{\bar{y}} = \frac{1}{1 + y\bar{y}}(\bar{\Phi}(z, \bar{z}) + i\Lambda y)$$

$SO(2) = U(1)$ invariance (generators are anti-hermitian matrix)

$$[\Lambda, \mathrm{A}_z] = [\Lambda, \mathrm{A}_{\bar{z}}] = 0$$

$$[\Lambda, \Phi] = -i\Phi \,, \quad [\Lambda, \bar{\Phi}] = i\bar{\Phi}$$

Finite Energy Solutions $\longrightarrow$ **Vacuum** ($\mathrm{F}_{z\bar{z}} = 0$) at $z \longrightarrow \infty$

Vacuum value Φ_0 of Φ forms $SO(3)$ algebra

$$[\Lambda, \Phi_0] = -i\Phi_0, \quad [\Lambda, \bar{\Phi}_0] = i\bar{\Phi}_0, \quad [\Phi_0, \bar{\Phi}_0] = 2i\Lambda$$

Boundary Condition at $r = \mathrm{Im} z = 0$: Fields approach vacuum values

4 Maximally Non-Abelian Vortices

Take $SU(2N)$ gauge group : Λ can be taken in Cartan subalgebra

$$\Lambda = i \begin{pmatrix} \Lambda_1 & & & \\ & \Lambda_2 & & \\ & & \ddots & \\ & & & \Lambda_{2N} \end{pmatrix}, \quad \sum \Lambda_\alpha = 0$$

$$[\Lambda, \Phi] = -i\Phi \to \Lambda_\beta - \Lambda_\alpha = 1 \ \text{ if } \ \Phi_{\alpha\beta} \neq 0$$

Maximally Non-Abelian case : $\Lambda = \frac{i}{2} \begin{pmatrix} \mathbf{1}_N & 0 \\ 0 & -\mathbf{1}_N \end{pmatrix}$

$$SU(2N) \to SU(N) \times \widetilde{SU(N)} \times U(1) \text{ gauge symmetry}$$

$SO(3)$ invariant gauge fields on $\Sigma \times S^2$

$$\mathcal{A}_z = \begin{pmatrix} A_z & 0 \\ 0 & \tilde{A}_z \end{pmatrix}, \mathcal{A}_{\bar{z}} = \begin{pmatrix} A_{\bar{z}} & 0 \\ 0 & \tilde{A}_{\bar{z}} \end{pmatrix}, \ \Phi = \begin{pmatrix} 0 & 0 \\ H & 0 \end{pmatrix}, \bar{\Phi} = \begin{pmatrix} 0 & H^\dagger \\ 0 & 0 \end{pmatrix}$$

$\boldsymbol{A_z}, (\boldsymbol{\tilde{A}_z})$: $\boldsymbol{SU(N)}$ ($\widetilde{\boldsymbol{SU(N)}}$) gauge field

$\boldsymbol{H}$: A Higgs scalar in **Bi-fundamental** of $\boldsymbol{SU(N) \times \widetilde{SU(N)}}$

Bogomolny equations for non-Abelian Vortices on Hyperbolic Plane

$$\boldsymbol{D_z H^\dagger = 0, \quad D_{\bar{z}} H = 0}$$

$$\boldsymbol{F_{z\bar{z}} = \frac{\sigma}{8}\left(-1_N + H^\dagger H\right), \quad \widetilde{F}_{z\bar{z}} = \frac{\sigma}{8}\left(1_N - H H^\dagger\right)}$$

$$\boldsymbol{D_{\bar{z}} H = \partial_{\bar{z}} H + \widetilde{A}_{\bar{z}} H - H A_{\bar{z}}, \quad F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + [A_z, A_{\bar{z}}]}$$

Vacuum Solutions

$$\boldsymbol{H = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}, \quad A_z = 0, \quad \widetilde{A}_z = 0}$$

Unbroken local gauge symmetry : $\boldsymbol{SU(N)_{\mathrm{d}}}$ diagonal gauge group

If $\boldsymbol{SU(2N) \to SU(N_1) \times SU(N_2) \times U(1), \; N_1 \neq N_2}$,

$\boldsymbol{F_{z\bar{z}} = 0}$ vacuum does not exist

Exact Vortex Solutions

$$\boldsymbol{H} = \begin{pmatrix} \boldsymbol{h}^{(1)} & & & \\ & \mathbf{1} & & \\ & & \ddots & \\ & & & \mathbf{1} \end{pmatrix}, \quad \boldsymbol{A}_{\bar{z}} = -\widetilde{\boldsymbol{A}}_{\bar{z}} = \begin{pmatrix} \boldsymbol{i}\boldsymbol{a}_{\bar{z}}^{(1)} & & & \\ & \mathbf{0} & & \\ & & \ddots & \\ & & & \mathbf{0} \end{pmatrix}$$

Bogomolny eq. reduce to exactly solved **Witten's equation**

We found **exact solutions** in the diagonal $\boldsymbol{U(1)^N}$ subgroup

Genuine **non-Abelian** vortices (fractional $\boldsymbol{U(1)}$ and $\boldsymbol{SU(N)}$ winding)

Solutions with complete orientational moduli remain to be worked out

5 Conclusion

1. $\boldsymbol{SO(3)}$ symmetric instantons gives **non-Abelian vortices** on a **hyperbolic plane**.

2. **Maximally non-Abelian** case gives non-Abelian vortices in $\boldsymbol{SU(N) \times \widetilde{SU(N)} \times U(1)}$ gauge group from $\boldsymbol{SU(2N)}$ instantons.

3. **Exact solutions** of $\boldsymbol{U(1)^N}$ subgroup are completely obtained, but the **orientational moduli** remain to be worked out.