$\mathcal{N}$ =2 instanton effective action in Ω-background from superstrings

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Introduction

Non-perturbative calculations of $\mathcal{N} = 2$ Super Yang-Mills theory

The low-energy effective action of $\mathcal{N} = 2$ SYM is determined by some holomorphic function $\mathcal{F}$ called prepotential. (Seiberg-Witten, 1994)

$$\mathcal{F}(\phi) = \mathcal{F}_{\text{pert}}(\phi) + \mathcal{F}_{\text{inst}}(\phi), \qquad \mathcal{F}_{\text{inst}}^{SU(2)}(\phi) = \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{\phi}\right)^{4k} \phi^2$$

Introducing parameters ε , the instanton partition function for $\mathcal{N} = 2$ Super Yang-Mills theory is calculated. (Nekrasov, 2002)

$$Z_{\text{inst}}(\phi, \Lambda, \epsilon_1, \epsilon_2) = \sum_{k=0}^{\infty} \Lambda^{2Nk} \int_{\mathcal{M}_{N,k}} d\mu_{\text{inst}} \exp[-S_{\text{eff}}(\phi, \epsilon_1, \epsilon_2)]$$

$S_{\text{eff}}(\phi, \epsilon_1, \epsilon_2)$: instanton effective action deformed by ε's

$$\log Z_{\text{inst}}(\phi, \Lambda, \epsilon_1, \epsilon_2) = \frac{1}{\epsilon_1 \epsilon_2} \left(\mathcal{F}_{\text{inst}}(\phi) + (\epsilon_1 + \epsilon_2)\mathcal{F}_1(\phi) + \mathcal{O}(\epsilon^2)\right)$$

- Interpretation of deformation parameters in terms of superstring theory

In case of $\epsilon_1 = -\epsilon_2 = \hbar$ → coupling constant of topological string theory

self-dual graviphoton background (closed string approach)
[Antoniadis-Gava-Narain-Taylor], etc.

self-dual R-R 3-form background (open string approach)
[Billo-Frau-Fucito-Lerda], [Matsuura]

Find the background parameter in string theory context corresponding to general ϵ_1, ϵ_2

cf.) Case of general ϵ_1, ϵ_2

→Partition function is related to the refinement of topological string theory

[Antoniadis-Hohenegger-Narain-Taylor]

$\mathcal{N}$ =2 SYM in Ω-backgorund

6 dim. $\mathcal{N}$ =1 SYM Lagrangian (Euclidean)

$$\mathcal{L}_{6\mathrm{D}} = \sqrt{-G}\,\mathrm{Tr}\left[\frac{1}{4}G^{MP}G^{NQ}F_{MN}F_{PQ} + \frac{i}{2}\bar{\Psi}\Gamma^{M}\mathcal{D}_{M}\Psi\right]$$

with metric

$$\mathrm{d}s^2_{6\mathrm{D}} = 2\mathrm{d}z\mathrm{d}\bar{z} + (\mathrm{d}x^m + \bar{\Omega}^m\mathrm{d}z + \Omega^m\mathrm{d}\bar{z})^2$$

where

$$\Omega^m = \Omega^{mn}x_n, \qquad \bar{\Omega}^m = \bar{\Omega}^{mn}x_n$$

$$\Omega^{mn} = \frac{1}{2\sqrt{2}}\begin{pmatrix} & \epsilon_1 & & \\ -\epsilon_1 & & & \\ & & & -\epsilon_2 \\ & & \epsilon_2 & \end{pmatrix}, \bar{\Omega}^{mn} = \frac{1}{2\sqrt{2}}\begin{pmatrix} & \bar{\epsilon}_1 & & \\ -\bar{\epsilon}_1 & & & \\ & & & -\bar{\epsilon}_2 \\ & & \bar{\epsilon}_2 & \end{pmatrix}$$

constant anti-symmetric matrices

Insert R-symmetry Wilson line (gauging SU(2) R-symmetry)

$$\boldsymbol{A}^I{}_J = \bar{\mathcal{A}}^I{}_J \mathrm{d}z + \mathcal{A}^I{}_J \mathrm{d}\bar{z}$$

$\bar{\mathcal{A}}^I{}_J, \mathcal{A}^I{}_J$: constant parameter I, J : SU(2) R-symmetry indices

KK reduction of toric dim. → 4 dim. $\mathcal{N}$ =2 SYM Lagrangian

$$\begin{aligned}
\mathcal{L}_\Omega = \mathrm{Tr}\Big[& \frac{1}{4}F_{mn}F^{mn} + (D_m\varphi - gF_{mn}\Omega^n)(D_m\bar{\varphi} - gF_{mp}\bar{\Omega}^p) \\
& + \Lambda^I\sigma^m D_m\bar{\Lambda} - \frac{i}{\sqrt{2}}g\Lambda^I[\bar{\varphi}, \Lambda_I] + \frac{i}{\sqrt{2}}g\bar{\Lambda}_I[\varphi, \bar{\Lambda}^I] \\
& + \frac{g^2}{2}\left([\varphi, \bar{\varphi}] + i\Omega^m D_m\bar{\varphi} - i\bar{\Omega}^m D_m\varphi + ig\bar{\Omega}^m\Omega^n F_{mn}\right)^2 \\
& + \frac{1}{\sqrt{2}}g\bar{\Omega}^m\Lambda^I D_m\Lambda_I - \frac{1}{2\sqrt{2}}g\bar{\Omega}_{mn}\Lambda^I\sigma^{mn}\Lambda_I \\
& - \frac{1}{\sqrt{2}}g\Omega^m\bar{\Lambda}_I D_m\bar{\Lambda}^I + \frac{1}{2\sqrt{2}}g\Omega_{mn}\bar{\Lambda}_I\bar{\sigma}^{mn}\bar{\Lambda}^I \\
& - \frac{1}{\sqrt{2}}g\bar{\mathcal{A}}^J{}_I\Lambda^I\Lambda_J - \frac{1}{\sqrt{2}}g\mathcal{A}^J{}_I\bar{\Lambda}^I\bar{\Lambda}_J \Big]
\end{aligned}$$

When we choose R-sym. Wilson line as

$$\mathcal{A}^I{}_J = -\frac{1}{2}\Omega_{mn}(\bar{\sigma}^{mn})^I{}_J, \quad \bar{\mathcal{A}}^I{}_J = -\frac{1}{2}\bar{\Omega}_{mn}(\bar{\sigma}^{mn})^I{}_J$$

$\mathcal{L}_\Omega$ preserves one supersymmetry.

= scalar supercharge of topological twisted theory $\bar{Q}_\Omega$

Nilpotency of $\bar{Q}_\Omega$

$\bar{Q}_\Omega^2$ = gauge transf. by $2\sqrt{2}\varphi$ + U(1)^2 rotation by $2\sqrt{2}\Omega_{mn}$

equivariant deformation

$\bar{Q}_\Omega$ -exactness

The action can be written in exact form as

$$\int \mathrm{d}^4x \mathcal{L}_\Omega = \frac{8\pi^2 k}{g^2} + \bar{Q}_\Omega \Xi$$

(k : instanton number)

Deformed instanton effective action from string theory

N fractional D3-branes on $\mathbb{R}^2 \times (\mathbb{R}^4/\mathbb{Z}_2)$ singularity

→ $\mathcal{N} = 2$ super Yang-Mills

Embed k D(-1)-branes on D3

→ k-instanton configuration of $\mathcal{N} = 2$ super Yang-Mills

ADHM instanton moduli

→ massless modes of open string connecting D(-1)-branes

Bosonic modes	$(a'_m)_{ij}$	k × k Hermitian	"position moduli"
	$(w^{\dot{\alpha}})_{ui}$	N × k complex	"size moduli"
Fermionic modes	$(\mathcal{M}'^I_\alpha)_{ij}$	k × k Hermitian matrices	
	$(\mu^A)_{ui}$	N × k complex matrices	
Auxiliary variables	$\chi, \bar{\chi}, D^c$	k × k Hermitian	bosonic
	$\bar{\psi}^{\dot{\alpha}}_I$	k × k Hermitian	fermionic

Computing disk amplitudes of their vertex operators

→Effective action for D(-1)-branes

$$S^0_{\text{eff}} = 2\pi^2 \text{tr}_k \Big[-2[\bar{\chi}, a'_m][\chi, a'_m] - \frac{i}{2\sqrt{2}} \mathcal{M}'^{\alpha I}[\bar{\chi}, \mathcal{M}'_{\alpha I}] + \frac{i}{\sqrt{2}} \bar{\mu}^I (\mu_I \bar{\chi} - \bar{\phi}^0 \mu_I)$$
$$+ (\bar{\chi} \bar{w}^{\dot{\alpha}} - \bar{w}^{\dot{\alpha}} \bar{\phi}^0)(w_{\dot{\alpha}} \chi - \phi^0 w_{\dot{\alpha}}) + (\chi \bar{w}^{\dot{\alpha}} - \bar{w}^{\dot{\alpha}} \phi^0)(w_{\dot{\alpha}} \bar{\chi} - \bar{\phi}^0 w_{\dot{\alpha}})$$
$$- i \bar{\psi}^{\dot{\alpha}}_I (\bar{\mu}^I w_{\dot{\alpha}} + \bar{w}_{\dot{\alpha}} \mu^I + [\mathcal{M}'^{\alpha I}, a'_{\alpha\dot{\alpha}}]) + i D^c (\tau^c)^{\dot{\alpha}}{}_{\dot{\beta}} (\bar{w}^{\dot{\beta}} w_{\dot{\alpha}} + \bar{a}'^{\dot{\beta}\alpha} a'_{\alpha\dot{\alpha}}) \Big]$$

$\phi^0, \bar{\phi}^0$:VEVs of gauge adjoint scalar

Inserting R-R background

We consider constant R-R field strength $\mathcal{F}^{\alpha\beta AB}, \mathcal{F}^{\dot\alpha\dot\beta}{}_{AB}$

$\alpha, \dot\alpha$: 4 dim. 2-spinor $\qquad A$: 6 dim. 4-spinor (transverse to D3)

Corresponding vertex operator

$$V_{\mathcal{F}+} = (2\pi\alpha')\mathcal{F}^{\alpha\beta AB} S_\alpha S_A \, \mathrm{e}^{-\frac{1}{2}\phi}(z) S_\beta S_B \, \mathrm{e}^{-\frac{1}{2}\phi}(\bar z)$$
$$V_{\mathcal{F}-} = (2\pi\alpha')\mathcal{F}^{\dot\alpha\dot\beta}{}_{AB} S_{\dot\alpha} S^A \, \mathrm{e}^{-\frac{1}{2}\phi}(z) S_{\dot\beta} S^B \, \mathrm{e}^{-\frac{1}{2}\phi}(\bar z)$$

Here we focus on 3-form part

$$\mathcal{F}^{(\alpha\beta)[AB]} = \mathcal{F}^{+}_{mna}(\sigma^{mn})^{\alpha\beta}(\Sigma^a)^{AB}$$
$$\mathcal{F}^{[\alpha\beta](AB)} = \mathcal{F}_{abc}\epsilon^{\alpha\beta}(\Sigma^{[a}\bar\Sigma^b\Sigma^{c]})^{AB}$$

After orbifold projection, the remaining components are

$$\mathcal{F}^{+}_{mn} \sim (\sigma_{mn})_{\alpha\beta}\mathcal{F}^{(\alpha\beta)[12]}, \qquad \bar{\mathcal{F}}^{+}_{mn} \sim (\sigma_{mn})_{\alpha\beta}\mathcal{F}^{(\alpha\beta)[34]}$$

$$\mathcal{F}^{-}_{mn} \sim (\bar{\sigma}_{mn})_{\dot{\alpha}\dot{\beta}}\mathcal{F}^{(\dot{\alpha}\dot{\beta})}{}_{[34]}, \qquad \bar{\mathcal{F}}^{-}_{mn} \sim (\bar{\sigma}_{mn})_{\dot{\alpha}\dot{\beta}}\mathcal{F}^{(\dot{\alpha}\dot{\beta})}{}_{[12]}$$

$$\mathcal{F}^{(IJ)} \sim \epsilon_{\alpha\beta}\mathcal{F}^{[\alpha\beta](IJ)}, \qquad \mathcal{F}^{(I'J')} \sim \epsilon_{\alpha\beta}\mathcal{F}^{[\alpha\beta](I'J')}$$

$$\bar{\mathcal{F}}_{(IJ)} \sim \epsilon_{\dot{\alpha}\dot{\beta}}\mathcal{F}^{[\dot{\alpha}\dot{\beta}]}{}_{(IJ)}, \qquad \bar{\mathcal{F}}_{(I'J')} \sim \epsilon_{\dot{\alpha}\dot{\beta}}\mathcal{F}^{[\dot{\alpha}\dot{\beta}]}{}_{(I'J')}$$

Here

$$I, J = 1, 2, \qquad I', J' = 3, 4$$

In obtaining the D(-1) effective action, we take the field theory limit.

$$\alpha' \to 0, \qquad (2\pi\alpha')^{\frac{1}{2}}\mathcal{F} \equiv C \quad \text{(finite)}$$

The case with only $C^{+}_{mn}, \bar{C}^{+}_{mn}$ already considered in [Billo-Frau-Fucito-Lerda]

The deformed D(-1)-brane effective action

$$S^0_{\rm eff} = 2\pi^2 {\rm tr}_k\Big[-2\Big([\bar\chi, a'_m] + (\bar C^+_{mn} + \bar C^-_{mn}) a'^n\Big)\Big([\chi, a'_m] + (C^{+mp} + C^{-mp}) a'_p\Big)$$
$$-\frac{i}{2\sqrt2}\mathcal{M}'^{\alpha I}\Big([\bar\chi, \mathcal{M}'_{\alpha I}] + \frac12 \bar C^+_{mn}(\sigma^{mn})_\alpha{}^\beta \mathcal{M}'_{\beta I} - \frac12 \bar C^-_{mn}(\bar\sigma^{mn})^J{}_I \mathcal{M}'_{\alpha J}\Big)$$
$$+\frac{i}{\sqrt2}\bar\mu^I\Big(\mu_I\bar\chi - \bar\phi^0\mu_I + \frac12 \bar C^J{}_I \mu_J\Big)$$
$$+\Big(\bar\chi\bar w^{\dot\alpha} - \bar w^{\dot\alpha}\bar\phi^0 + \frac12 \bar C^-_{mn}(\sigma^{\bar m n})^{\dot\alpha}{}_{\dot\beta}\bar w^{\dot\beta}\Big)\Big(w_{\dot\alpha}\chi - \phi^0 w_{\dot\alpha} + \frac12 C^-_{mn}(\sigma^{\bar m n})^{\dot\gamma}{}_{\dot\alpha} w_{\dot\gamma}\Big)$$
$$+\Big(\chi\bar w^{\dot\alpha} - \bar w^{\dot\alpha}\phi^0 + \frac12 C^-_{mn}(\sigma^{\bar m n})^{\dot\alpha}{}_{\dot\beta}\bar w^{\dot\beta}\Big)\Big(w_{\dot\alpha}\bar\chi - \bar\phi^0 w_{\dot\alpha} + \frac12 \bar C^-_{mn}(\sigma^{\bar m n})^{\dot\gamma}{}_{\dot\alpha} w_{\dot\gamma}\Big)$$
$$-i\bar\psi^{\dot\alpha}_I(\bar\mu^I w_{\dot\alpha} + \bar w_{\dot\alpha}\mu^I + [\mathcal{M}'^{\alpha I}, a'_{\alpha\dot\alpha}]) + iD^c(\tau^c)^{\dot\alpha}{}_{\dot\beta}(\bar w^{\dot\beta} w_{\dot\alpha} + \bar a'^{\dot\beta\alpha} a'_{\alpha\dot\alpha})\Big]$$

This action agrees with the deformed instanton effective action when we identify

$$C_{mn} = -i\Omega_{mn}, \quad \bar C_{mn} = -i\bar\Omega_{mn}, \quad \bar C^I{}_J = -i\bar{\mathcal{A}}^I{}_J$$

and preserves one supersymmetry (explicitly shown in our paper)

= Nekrasov's equivariant BRST operator

Conclusions & Future Works

- We computed the D(-1)-brane effective action in general (not self-dual) R-R 3-form background

→Reproduced the instanton eff. action in general Ω-background

- We revealed the correspondence between R-R 3-form and Ω-background

$$\mathcal{F}^{(\alpha\beta)[IJ]}, \mathcal{F}^{(\alpha\beta)[I'J']} \leftrightarrow \Omega^{+}_{mn}, \bar{\Omega}^{+}_{mn}$$

$$\mathcal{F}^{(\dot{\alpha}\dot{\beta})}{}_{[I'J']}, \mathcal{F}^{(\dot{\alpha}\dot{\beta})}{}_{[IJ]} \leftrightarrow \Omega^{-}_{mn}, \bar{\Omega}^{-}_{mn}$$

$$\mathcal{F}^{[\dot{\alpha}\dot{\beta}]}{}_{(IJ)} \leftrightarrow \bar{A}_{IJ} \left(= -\frac{1}{2}\bar{\Omega}^{-}_{mn}(\bar{\sigma}^{mn})_{IJ} \right)$$

Our new results!!

- We explicitly showed one preserved supersymmetry of deformed action

→ Coincides with Nekrasov's BRST operator

For future works

- $\mathcal{N}$ = 4 and $\mathcal{N}$ = 2* case → more parameters in Ω-background
- Higher dimensional instanton calculus