Models of Lepton Flavor From Small Groups

Akın Wingerter

work done in collaboration with Krishnamohan Parattu

Indian Institute of Science
Centre for High Energy Physics

August 24, 2010

Neutrino Mixing Matrix

What we know about the mixing angles ...

$$U_{\mathrm{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{13}s_{23}c_{12}e^{i\delta} & c_{23}c_{12} - s_{13}s_{23}s_{12}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} - s_{13}c_{23}c_{12}e^{i\delta} & -s_{23}c_{12} - s_{13}c_{23}s_{12}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & e^{-i\delta}s_{13} \\ 0 & 1 & 0 \\ -e^{i\delta}s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

T. Schwetz, M. A. Tortola, and J. W. F. Valle, "Three-flavour neutrino oscillation update," *New J. Phys.* **10** (2008) 113011, 0808.2016.

Angle	1σ	2σ	3σ
θ_{12}	$32.46° - 34.82°$	$31.31° - 36.27°$	$30.00° - 37.47°$
θ_{23}	$41.55° - 49.02°$	$38.65° - 52.54°$	$36.87° - 54.94°$
θ_{13}	$0.00° - 9.28°$	$0.00° - 11.54°$	$0.00° - 13.69°$

Harrison-Perkins-Scott Matrix

Presently our best guess ...

P. F. Harrison, D. H. Perkins, and W. G. Scott, "Tri-bimaximal mixing and the neutrino oscillation data," *Phys. Lett.* **B530** (2002) 167, hep-ph/0202074.

$$U_{\rm HPS} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$$

Indicative of an underlying symmetry ...

Some groups that have been considered in the literature:

Review: G. Altarelli and F. Feruglio, "Discrete Flavor Symmetries and Models of Neutrino Mixing," 1002.0211.

$$S_3,\ D_4,\ D_7,\ A_4,\ A_5,\ \widetilde{T},\ S_4,\ (C_3\times C_3)\rtimes_\varphi C_3,\ C_7\rtimes_\varphi C_3,\ \mathrm{PSL}_2(7)$$

$\rightsquigarrow$ As a paradigm, we will consider a model with $A_4 \times C_3$ symmetry and then generalize it to other symmetry groups

Altarelli-Feruglio Model Revisited

G. Altarelli and F. Feruglio, "Tri-Bimaximal Neutrino Mixing, A4 and the Modular Symmetry," *Nucl. Phys.* **B741** (2006) 215–235, hep-ph/0512103.

❶ Symmetries of the model

$$\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \times \mathrm{U}(1)_R \times A_4 \times C_3$$

❷ Particle content and charges

Field	$\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$	$\mathrm{U}(1)_R$	A_4	C_3	$A_4 \times C_3$
L	$(\mathbf{2},\text{-}1)$	1	$\mathbf{3}$	ω	$\mathbf{3}'$
e	$(\mathbf{1},\ 2)$	1	$\mathbf{1}$	ω^2	$\mathbf{1}'$
μ	$(\mathbf{1},\ 2)$	1	$\mathbf{1}''$	ω^2	$\mathbf{1}^{(8)}$
τ	$(\mathbf{1},\ 2)$	1	$\mathbf{1}'$	ω^2	$\mathbf{1}^{(5)}$
h_u	$(\mathbf{2},\ 1)$	0	$\mathbf{1}$	1	$\mathbf{1}$
h_d	$(\mathbf{2},\text{-}1)$	0	$\mathbf{1}$	1	$\mathbf{1}$
φ_T	$(\mathbf{1},\ 0)$	0	$\mathbf{3}$	1	$\mathbf{3}$
φ_S	$(\mathbf{1},\ 0)$	0	$\mathbf{3}$	ω	$\mathbf{3}'$
ξ	$(\mathbf{1},\ 0)$	0	$\mathbf{1}$	ω	$\mathbf{1}''$

❸ Breaking the family symmetry

$$\varphi_T = (v_T, 0, 0), \quad \varphi_S = (v_S, v_S, v_S), \quad \xi = v_\xi,$$

Group Information from GAP

The GAP Group, “GAP – Groups, Algorithms, and Programming, Version 4.4.12.”, http://www.gap-system.org

“GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory.”

```
group := SmallGroup(36,11);;
Display(StructureDescription(group));
Display(GeneratorsOfGroup(group));
for el in GeneratorsOfGroup(group) do
  Display(Factorization(group, el));
od;
for el in Elements(group) do
  Display(Factorization(group, el));
od;
chartab := Irr(group);;
Display(chartab);
SizesConjugacyClasses(CharacterTable(group));
LoadPackage("repsn");;
for i in [1..Size(chartab)] do
  Display(IrreducibleAffordingRepresentation(chartab[i]));
od;
```

Group Information from GAP

The GAP Group, "GAP – Groups, Algorithms, and Programming, Version 4.4.12.", http://www.gap-system.org

"GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory."

```
group := SmallGroup(36,11);;
Display(StructureDescription(group));
Display(GeneratorsOfGroup(group));
for el in GeneratorsOfGroup(group) do
  Display(Factorization(group, el));
od;
for el in Elements(group) do
  Display(Factorization(group, el));
od;
chartab := Irr(group);;
Display(chartab);
SizesConjugacyClasses(CharacterTable(group));
LoadPackage("repsn");;
for i in [1..Size(chartab)] do
  Display(IrreducibleAffordingRepresentation(chartab[i]));
od;
```

➢ Specify the group that we will work with

Group Information from GAP

The GAP Group, "GAP – Groups, Algorithms, and Programming, Version 4.4.12.", http://www.gap-system.org

"GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory."

```
group := SmallGroup(36,11);;
Display(StructureDescription(group));
Display(GeneratorsOfGroup(group));
for el in GeneratorsOfGroup(group) do
  Display(Factorization(group, el));
od;
for el in Elements(group) do
  Display(Factorization(group, el));
od;
chartab := Irr(group);;
Display(chartab);
SizesConjugacyClasses(CharacterTable(group));
LoadPackage("repsn");;
for i in [1..Size(chartab)] do
  Display(IrreducibleAffordingRepresentation(chartab[i]));
od;
```

➢ The "human readable" name of the group

Group Information from GAP

The GAP Group, "GAP – Groups, Algorithms, and Programming, Version 4.4.12.", http://www.gap-system.org

"GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory."

```
group := SmallGroup(36,11);;
Display(StructureDescription(group));
Display(GeneratorsOfGroup(group));
for el in GeneratorsOfGroup(group) do
  Display(Factorization(group, el));
od;
for el in Elements(group) do
  Display(Factorization(group, el));
od;
chartab := Irr(group);;
Display(chartab);
SizesConjugacyClasses(CharacterTable(group));
LoadPackage("repsn");;
for i in [1..Size(chartab)] do
  Display(IrreducibleAffordingRepresentation(chartab[i]));
od;
```

➢ The generators of the group

Group Information from GAP

The GAP Group, "GAP – Groups, Algorithms, and Programming, Version 4.4.12.", http://www.gap-system.org

"GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory."

```
group := SmallGroup(36,11);;
Display(StructureDescription(group));
Display(GeneratorsOfGroup(group));
for el in GeneratorsOfGroup(group) do
  Display(Factorization(group, el));
od;
for el in Elements(group) do
  Display(Factorization(group, el));
od;
chartab := Irr(group);;
Display(chartab);
SizesConjugacyClasses(CharacterTable(group));
LoadPackage("repsn");;
for i in [1..Size(chartab)] do
  Display(IrreducibleAffordingRepresentation(chartab[i]));
od;
```

➢ Express all group elements in terms of the generators

Group Information from GAP

The GAP Group, "GAP – Groups, Algorithms, and Programming, Version 4.4.12.", http://www.gap-system.org

"GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory."

```
group := SmallGroup(36,11);;
Display(StructureDescription(group));
Display(GeneratorsOfGroup(group));
for el in GeneratorsOfGroup(group) do
  Display(Factorization(group, el));
od;
for el in Elements(group) do
  Display(Factorization(group, el));
od;
chartab := Irr(group);;
Display(chartab);
SizesConjugacyClasses(CharacterTable(group));
LoadPackage("repsn");;
for i in [1..Size(chartab)] do
  Display(IrreducibleAffordingRepresentation(chartab[i]));
od;
```

➢ The character table

Group Information from GAP

The GAP Group, "GAP – Groups, Algorithms, and Programming, Version 4.4.12.", http://www.gap-system.org

"GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory."

```
group := SmallGroup(36,11);;
Display(StructureDescription(group));
Display(GeneratorsOfGroup(group));
for el in GeneratorsOfGroup(group) do
  Display(Factorization(group, el));
od;
for el in Elements(group) do
  Display(Factorization(group, el));
od;
chartab := Irr(group);;
Display(chartab);
SizesConjugacyClasses(CharacterTable(group));
LoadPackage("repsn");;
for i in [1..Size(chartab)] do
  Display(IrreducibleAffordingRepresentation(chartab[i]));
od;
```

➢ Dimensions of the conjugacy classes

Group Information from GAP

The GAP Group, "GAP – Groups, Algorithms, and Programming, Version 4.4.12.", http://www.gap-system.org

"GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory."

```
group := SmallGroup(36,11);;
Display(StructureDescription(group));
Display(GeneratorsOfGroup(group));
for el in GeneratorsOfGroup(group) do
  Display(Factorization(group, el));
od;
for el in Elements(group) do
  Display(Factorization(group, el));
od;
chartab := Irr(group);;
Display(chartab);
SizesConjugacyClasses(CharacterTable(group));
LoadPackage("repsn");;
for i in [1..Size(chartab)] do
  Display(IrreducibleAffordingRepresentation(chartab[i]));
od;
```

➢ The matrices for the representations

The Character Table of $\mathbf{A_4} \times \mathbf{C_3}$

	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}	K_{11}	K_{12}
$\mathbf{1}$	1	1	1	1	1	1	1	1	1	1	1	1
$\mathbf{1'}$	1	1	ω^2	1	1	ω^2	ω	ω^2	ω^2	ω	ω	ω
$\mathbf{1''}$	1	1	ω	1	1	ω	ω^2	ω	ω	ω^2	ω^2	ω^2
$\mathbf{1'''}$	1	ω^2	1	1	ω	ω^2	1	1	ω	ω^2	1	ω
$\mathbf{1^{(4)}}$	1	ω	1	1	ω^2	ω	1	1	ω^2	ω	1	ω^2
$\mathbf{1^{(5)}}$	1	ω^2	ω^2	1	ω	ω	ω	ω^2	1	1	ω	ω^2
$\mathbf{1^{(6)}}$	1	ω	ω	1	ω^2	ω^2	ω^2	ω	1	1	ω^2	ω
$\mathbf{1^{(7)}}$	1	ω^2	ω	1	ω	1	ω^2	ω	ω^2	ω	ω^2	1
$\mathbf{1^{(8)}}$	1	ω	ω^2	1	ω^2	1	ω	ω^2	ω	ω^2	ω	1
$\mathbf{3}$	3	0	3	-1	0	0	3	-1	0	0	-1	0
$\mathbf{3'}$	3	0	3ω	-1	0	0	$3\omega^2$	ω	0	0	$1+\omega$	0
$\mathbf{3''}$	3	0	$3\omega^2$	-1	0	0	3ω	$1+\omega$	0	0	ω	0

$\omega = e^{2\pi i/3}$ is the primitive third root of unity

Match Irreps to Altarelli-Feruglio's Notation

Compare matrix representations of irreps for A_3, C_3 and $A_3 \times C_3$:

Field	$\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$	$\mathrm{U}(1)_R$	A_4	C_3	$A_4 \times C_3$
L	$(\mathbf{2}, \text{-}1)$	1	$\mathbf{3}$	ω	$\mathbf{3'}$
e	$(\mathbf{1}, 2)$	1	$\mathbf{1}$	ω^2	$\mathbf{1'}$
μ	$(\mathbf{1}, 2)$	1	$\mathbf{1''}$	ω^2	$\mathbf{1^{(8)}}$
τ	$(\mathbf{1}, 2)$	1	$\mathbf{1'}$	ω^2	$\mathbf{1^{(5)}}$
h_u	$(\mathbf{2}, 1)$	0	$\mathbf{1}$	1	$\mathbf{1}$
h_d	$(\mathbf{2}, \text{-}1)$	0	$\mathbf{1}$	1	$\mathbf{1}$
φ_T	$(\mathbf{1}, 0)$	0	$\mathbf{3}$	1	$\mathbf{3}$
φ_S	$(\mathbf{1}, 0)$	0	$\mathbf{3}$	ω	$\mathbf{3'}$
ξ	$(\mathbf{1}, 0)$	0	$\mathbf{1}$	ω	$\mathbf{1''}$

Decomposition of Tensor Products

From the character table and the dimensions of the conjugacy classes:

$\mathbf{1} \otimes \mathbf{1} = \mathbf{1}$
$\mathbf{1} \otimes \mathbf{1}^{(5)} = \mathbf{1}^{(5)}$
$\mathbf{1} \otimes \mathbf{3}' = \mathbf{3}'$
$\mathbf{1}' \otimes \mathbf{1}^{(4)} = \mathbf{1}^{(8)}$
$\mathbf{1}' \otimes \mathbf{3} = \mathbf{3}''$
$\mathbf{1}'' \otimes \mathbf{1}^{(4)} = \mathbf{1}^{(6)}$
$\mathbf{1}'' \otimes \mathbf{3} = \mathbf{3}'$
$\mathbf{1}''' \otimes \mathbf{1}^{(5)} = \mathbf{1}^{(8)}$
$\mathbf{1}''' \otimes \mathbf{3}' = \mathbf{3}'$
$\mathbf{1}^{(4)} \otimes \mathbf{1}^{(7)} = \mathbf{1}''$
$\mathbf{1}^{(5)} \otimes \mathbf{1}^{(5)} = \mathbf{1}^{(6)}$
$\mathbf{1}^{(5)} \otimes \mathbf{3}' = \mathbf{3}$
$\mathbf{1}^{(6)} \otimes \mathbf{3} = \mathbf{3}'$
$\mathbf{1}^{(7)} \otimes \mathbf{3} = \mathbf{3}'$
$\mathbf{1}^{(8)} \otimes \mathbf{3}' = \mathbf{3}$

$\mathbf{1} \otimes \mathbf{1}' = \mathbf{1}'$
$\mathbf{1} \otimes \mathbf{1}^{(6)} = \mathbf{1}^{(6)}$
$\mathbf{1} \otimes \mathbf{3}'' = \mathbf{3}''$
$\mathbf{1}' \otimes \mathbf{1}^{(5)} = \mathbf{1}^{(7)}$
$\mathbf{1}' \otimes \mathbf{3}' = \mathbf{3}$
$\mathbf{1}'' \otimes \mathbf{1}^{(5)} = \mathbf{1}'''$
$\mathbf{1}'' \otimes \mathbf{3}' = \mathbf{3}''$
$\mathbf{1}''' \otimes \mathbf{1}^{(6)} = \mathbf{1}''$
$\mathbf{1}''' \otimes \mathbf{3}'' = \mathbf{3}''$
$\mathbf{1}^{(4)} \otimes \mathbf{1}^{(8)} = \mathbf{1}^{(5)}$
$\mathbf{1}^{(5)} \otimes \mathbf{1}^{(6)} = \mathbf{1}$
$\mathbf{1}^{(5)} \otimes \mathbf{3}'' = \mathbf{3}'$
$\mathbf{1}^{(6)} \otimes \mathbf{3}' = \mathbf{3}''$
$\mathbf{1}^{(7)} \otimes \mathbf{3}' = \mathbf{3}''$
$\mathbf{1}^{(8)} \otimes \mathbf{3}'' = \mathbf{3}'$

$\mathbf{1} \otimes \mathbf{1}'' = \mathbf{1}''$
$\mathbf{1} \otimes \mathbf{1}^{(7)} = \mathbf{1}^{(7)}$
$\mathbf{1}' \otimes \mathbf{1}' = \mathbf{1}''$
$\mathbf{1}' \otimes \mathbf{1}^{(6)} = \mathbf{1}^{(4)}$
$\mathbf{1}' \otimes \mathbf{3}'' = \mathbf{3}'$
$\mathbf{1}'' \otimes \mathbf{1}^{(6)} = \mathbf{1}^{(8)}$
$\mathbf{1}'' \otimes \mathbf{3}'' = \mathbf{3}$
$\mathbf{1}''' \otimes \mathbf{1}^{(7)} = \mathbf{1}^{(6)}$
$\mathbf{1}^{(4)} \otimes \mathbf{1}^{(4)} = \mathbf{1}'''$
$\mathbf{1}^{(4)} \otimes \mathbf{3} = \mathbf{3}$
$\mathbf{1}^{(5)} \otimes \mathbf{1}^{(7)} = \mathbf{1}^{(4)}$
$\mathbf{1}^{(6)} \otimes \mathbf{1}^{(6)} = \mathbf{1}^{(5)}$
$\mathbf{1}^{(6)} \otimes \mathbf{3}'' = \mathbf{3}$
$\mathbf{1}^{(7)} \otimes \mathbf{3}'' = \mathbf{3}$

$\mathbf{1} \otimes \mathbf{1}''' = \mathbf{1}'''$
$\mathbf{1} \otimes \mathbf{1}^{(8)} = \mathbf{1}^{(8)}$
$\mathbf{1}' \otimes \mathbf{1}'' = \mathbf{1}$
$\mathbf{1}' \otimes \mathbf{1}^{(7)} = \mathbf{1}'''$
$\mathbf{1}'' \otimes \mathbf{1}'' = \mathbf{1}'$
$\mathbf{1}'' \otimes \mathbf{1}^{(7)} = \mathbf{1}^{(5)}$
$\mathbf{1}''' \otimes \mathbf{1}''' = \mathbf{1}^{(4)}$
$\mathbf{1}''' \otimes \mathbf{1}^{(8)} = \mathbf{1}'$
$\mathbf{1}^{(4)} \otimes \mathbf{1}^{(5)} = \mathbf{1}'$
$\mathbf{1}^{(4)} \otimes \mathbf{3}' = \mathbf{3}'$
$\mathbf{1}^{(5)} \otimes \mathbf{1}^{(8)} = \mathbf{1}''$
$\mathbf{1}^{(6)} \otimes \mathbf{1}^{(7)} = \mathbf{1}'$
$\mathbf{1}^{(7)} \otimes \mathbf{1}^{(7)} = \mathbf{1}^{(8)}$
$\mathbf{1}^{(8)} \otimes \mathbf{1}^{(8)} = \mathbf{1}^{(7)}$

$\mathbf{1} \otimes \mathbf{1}^{(4)} = \mathbf{1}^{(4)}$
$\mathbf{1} \otimes \mathbf{3} = \mathbf{3}$
$\mathbf{1}' \otimes \mathbf{1}''' = \mathbf{1}^{(5)}$
$\mathbf{1}' \otimes \mathbf{1}^{(8)} = \mathbf{1}^{(6)}$
$\mathbf{1}'' \otimes \mathbf{1}''' = \mathbf{1}^{(7)}$
$\mathbf{1}'' \otimes \mathbf{1}^{(8)} = \mathbf{1}^{(4)}$
$\mathbf{1}''' \otimes \mathbf{1}^{(4)} = \mathbf{1}$
$\mathbf{1}''' \otimes \mathbf{3} = \mathbf{3}$
$\mathbf{1}^{(4)} \otimes \mathbf{1}^{(6)} = \mathbf{1}^{(7)}$
$\mathbf{1}^{(4)} \otimes \mathbf{3}'' = \mathbf{3}''$
$\mathbf{1}^{(5)} \otimes \mathbf{3} = \mathbf{3}''$
$\mathbf{1}^{(6)} \otimes \mathbf{1}^{(8)} = \mathbf{1}'''$
$\mathbf{1}^{(7)} \otimes \mathbf{1}^{(8)} = \mathbf{1}$
$\mathbf{1}^{(8)} \otimes \mathbf{3} = \mathbf{3}''$

$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} + \mathbf{1}''' + \mathbf{1}^{(4)} + 2 \otimes \mathbf{3}$
$\mathbf{3} \otimes \mathbf{3}'' = \mathbf{1}' + \mathbf{1}^{(5)} + \mathbf{1}^{(8)} + 2 \otimes \mathbf{3}''$
$\mathbf{3}' \otimes \mathbf{3}'' = \mathbf{1} + \mathbf{1}''' + \mathbf{1}^{(4)} + 2 \otimes \mathbf{3}$

$\mathbf{3} \otimes \mathbf{3}' = \mathbf{1}'' + \mathbf{1}^{(6)} + \mathbf{1}^{(7)} + 2 \otimes \mathbf{3}'$
$\mathbf{3}' \otimes \mathbf{3}' = \mathbf{1}' + \mathbf{1}^{(5)} + \mathbf{1}^{(8)} + 2 \otimes \mathbf{3}''$
$\mathbf{3}'' \otimes \mathbf{3}'' = \mathbf{1}'' + \mathbf{1}^{(6)} + \mathbf{1}^{(7)} + 2 \otimes \mathbf{3}'$

Decomposition of Tensor Products

From the character table and the dimensions of the conjugacy classes:

$\mathbf{1} \otimes \mathbf{1} = \mathbf{1}$
$\mathbf{1} \otimes \mathbf{1}^{(5)} = \mathbf{1}^{(5)}$
$\mathbf{1} \otimes \mathbf{3}' = \mathbf{3}'$
$\mathbf{1}' \otimes \mathbf{1}^{(4)} = \mathbf{1}^{(8)}$
$\mathbf{1}' \otimes \mathbf{3} = \mathbf{3}''$
$\mathbf{1}'' \otimes \mathbf{1}^{(4)} = \mathbf{1}^{(6)}$
$\mathbf{1}'' \otimes \mathbf{3} = \mathbf{3}'$
$\mathbf{1}''' \otimes \mathbf{1}^{(5)} = \mathbf{1}^{(8)}$
$\mathbf{1}''' \otimes \mathbf{3}' = \mathbf{3}'$
$\mathbf{1}^{(4)} \otimes \mathbf{1}^{(7)} = \mathbf{1}''$
$\mathbf{1}^{(5)} \otimes \mathbf{1}^{(5)} = \mathbf{1}^{(6)}$
$\mathbf{1}^{(5)} \otimes \mathbf{3}' = \mathbf{3}$
$\mathbf{1}^{(6)} \otimes \mathbf{3} = \mathbf{3}'$
$\mathbf{1}^{(7)} \otimes \mathbf{3} = \mathbf{3}'$
$\mathbf{1}^{(8)} \otimes \mathbf{3}' = \mathbf{3}$

$\mathbf{1} \otimes \mathbf{1}' = \mathbf{1}'$
$\mathbf{1} \otimes \mathbf{1}^{(6)} = \mathbf{1}^{(6)}$
$\mathbf{1} \otimes \mathbf{3}'' = \mathbf{3}''$
$\mathbf{1}' \otimes \mathbf{1}^{(5)} = \mathbf{1}^{(7)}$
$\mathbf{1}' \otimes \mathbf{3}' = \mathbf{3}$
$\mathbf{1}'' \otimes \mathbf{1}^{(5)} = \mathbf{1}'''$
$\mathbf{1}'' \otimes \mathbf{3}' = \mathbf{3}''$
$\mathbf{1}''' \otimes \mathbf{1}^{(6)} = \mathbf{1}''$
$\mathbf{1}''' \otimes \mathbf{3}'' = \mathbf{3}''$
$\mathbf{1}^{(4)} \otimes \mathbf{1}^{(8)} = \mathbf{1}^{(5)}$
$\mathbf{1}^{(5)} \otimes \mathbf{1}^{(6)} = \mathbf{1}$
$\mathbf{1}^{(5)} \otimes \mathbf{3}'' = \mathbf{3}'$
$\mathbf{1}^{(6)} \otimes \mathbf{3}' = \mathbf{3}''$
$\mathbf{1}^{(7)} \otimes \mathbf{3}' = \mathbf{3}''$
$\mathbf{1}^{(8)} \otimes \mathbf{3}'' = \mathbf{3}'$

$\mathbf{1} \otimes \mathbf{1}'' = \mathbf{1}''$
$\mathbf{1} \otimes \mathbf{1}^{(7)} = \mathbf{1}^{(7)}$
$\mathbf{1}' \otimes \mathbf{1}' = \mathbf{1}''$
$\mathbf{1}' \otimes \mathbf{1}^{(6)} = \mathbf{1}^{(4)}$
$\mathbf{1}' \otimes \mathbf{3}'' = \mathbf{3}'$
$\mathbf{1}'' \otimes \mathbf{1}^{(6)} = \mathbf{1}^{(8)}$
$\mathbf{1}'' \otimes \mathbf{3}'' = \mathbf{3}$
$\mathbf{1}''' \otimes \mathbf{1}^{(7)} = \mathbf{1}^{(6)}$
$\mathbf{1}^{(4)} \otimes \mathbf{1}^{(4)} = \mathbf{1}'''$
$\mathbf{1}^{(4)} \otimes \mathbf{3} = \mathbf{3}$
$\mathbf{1}^{(5)} \otimes \mathbf{1}^{(7)} = \mathbf{1}^{(4)}$
$\mathbf{1}^{(6)} \otimes \mathbf{1}^{(6)} = \mathbf{1}^{(5)}$
$\mathbf{1}^{(6)} \otimes \mathbf{3}'' = \mathbf{3}$
$\mathbf{1}^{(7)} \otimes \mathbf{3}'' = \mathbf{3}$

$\mathbf{1} \otimes \mathbf{1}''' = \mathbf{1}'''$
$\mathbf{1} \otimes \mathbf{1}^{(8)} = \mathbf{1}^{(8)}$
$\mathbf{1}' \otimes \mathbf{1}'' = \mathbf{1}$
$\mathbf{1}' \otimes \mathbf{1}^{(7)} = \mathbf{1}'''$
$\mathbf{1}'' \otimes \mathbf{1}'' = \mathbf{1}'$
$\mathbf{1}'' \otimes \mathbf{1}^{(7)} = \mathbf{1}^{(5)}$
$\mathbf{1}''' \otimes \mathbf{1}''' = \mathbf{1}^{(4)}$
$\mathbf{1}''' \otimes \mathbf{1}^{(8)} = \mathbf{1}'$
$\mathbf{1}^{(4)} \otimes \mathbf{1}^{(5)} = \mathbf{1}'$
$\mathbf{1}^{(4)} \otimes \mathbf{3}' = \mathbf{3}'$
$\mathbf{1}^{(5)} \otimes \mathbf{1}^{(8)} = \mathbf{1}''$
$\mathbf{1}^{(6)} \otimes \mathbf{1}^{(7)} = \mathbf{1}'$
$\mathbf{1}^{(7)} \otimes \mathbf{1}^{(7)} = \mathbf{1}^{(8)}$
$\mathbf{1}^{(8)} \otimes \mathbf{1}^{(8)} = \mathbf{1}^{(7)}$

$\mathbf{1} \otimes \mathbf{1}^{(4)} = \mathbf{1}^{(4)}$
$\mathbf{1} \otimes \mathbf{3} = \mathbf{3}$
$\mathbf{1}' \otimes \mathbf{1}''' = \mathbf{1}^{(5)}$
$\mathbf{1}' \otimes \mathbf{1}^{(8)} = \mathbf{1}^{(6)}$
$\mathbf{1}'' \otimes \mathbf{1}''' = \mathbf{1}^{(7)}$
$\mathbf{1}'' \otimes \mathbf{1}^{(8)} = \mathbf{1}^{(4)}$
$\mathbf{1}''' \otimes \mathbf{1}^{(4)} = \mathbf{1}$
$\mathbf{1}''' \otimes \mathbf{3} = \mathbf{3}$
$\mathbf{1}^{(4)} \otimes \mathbf{1}^{(6)} = \mathbf{1}^{(7)}$
$\mathbf{1}^{(4)} \otimes \mathbf{3}'' = \mathbf{3}''$
$\mathbf{1}^{(5)} \otimes \mathbf{3} = \mathbf{3}''$
$\mathbf{1}^{(6)} \otimes \mathbf{1}^{(8)} = \mathbf{1}'''$
$\mathbf{1}^{(7)} \otimes \mathbf{1}^{(8)} = \mathbf{1}$
$\mathbf{1}^{(8)} \otimes \mathbf{3} = \mathbf{3}''$

$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} + \mathbf{1}''' + \mathbf{1}^{(4)} + 2 \otimes \mathbf{3}$
$\mathbf{3} \otimes \mathbf{3}'' = \mathbf{1}' + \mathbf{1}^{(5)} + \mathbf{1}^{(8)} + 2 \otimes \mathbf{3}''$
$\mathbf{3}' \otimes \mathbf{3}'' = \mathbf{1} + \mathbf{1}''' + \mathbf{1}^{(4)} + 2 \otimes \mathbf{3}$

$\mathbf{3} \otimes \mathbf{3}' = \mathbf{1}'' + \mathbf{1}^{(6)} + \mathbf{1}^{(7)} + 2 \otimes \mathbf{3}'$
$\mathbf{3}' \otimes \mathbf{3}' = \mathbf{1}' + \mathbf{1}^{(5)} + \mathbf{1}^{(8)} + 2 \otimes \mathbf{3}''$
$\mathbf{3}'' \otimes \mathbf{3}'' = \mathbf{1}'' + \mathbf{1}^{(6)} + \mathbf{1}^{(7)} + 2 \otimes \mathbf{3}'$

Invariant Lagrangian

➢ Terms that are invariant, have 2 leptons and mass dimension ≤ 5:

$$L\,L\,h_u\,h_u\,\varphi_S + L\,L\,h_u\,h_u\,\xi + L\,e\,h_d\,\varphi_T + L\,\mu\,h_d\,\varphi_T + L\,\tau\,h_d\,\varphi_T$$

Invariant Lagrangian

➢ Terms that are invariant, have 2 leptons and mass dimension ≤ 5:

$$L\,L\,h_u\,h_u\,\varphi_S + L\,L\,h_u\,h_u\,\xi + L\,e\,h_d\,\varphi_T + L\,\mu\,h_d\,\varphi_T + L\,\tau\,h_d\,\varphi_T$$

$$\mathbf{3'}\otimes\mathbf{3'}\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{3'}=\left(\mathbf{1'}+\mathbf{1^{(5)}}+\mathbf{1^{(8)}}+2\times\mathbf{3''}\right)\otimes\mathbf{3'}=2\times\mathbf{1}+2\times\mathbf{1'''}+2\times\mathbf{1^{(4)}}+7\times\mathbf{3}$$

Invariant Lagrangian

- Terms that are invariant, have 2 leptons and mass dimension ≤ 5:

$$L\,L\,h_u\,h_u\,\varphi_S + L\,L\,h_u\,h_u\,\xi + L\,e\,h_d\,\varphi_T + L\,\mu\,h_d\,\varphi_T + L\,\tau\,h_d\,\varphi_T$$

$$\mathbf{3'}\otimes\mathbf{3'}\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{3'}=\left(\mathbf{1'}+\mathbf{1^{(5)}}+\mathbf{1^{(8)}}+2\times\mathbf{3''}\right)\otimes\mathbf{3'}=2\times\mathbf{1}+2\times\mathbf{1'''}+2\times\mathbf{1^{(4)}}+7\times\mathbf{3}$$

- Contract family indices:

$$\tfrac{1}{\sqrt{3}}L_2\,L_3\,h_u\,h_u\,\varphi_{S,1}+\tfrac{1}{\sqrt{3}}L_3\,L_1\,h_u\,h_u\,\varphi_{S,2}+\tfrac{1}{\sqrt{3}}L_1\,L_2\,h_u\,h_u\,\varphi_{S,3}+\tfrac{1}{\sqrt{3}}L_1\,L_1\,h_u\,h_u\,\xi$$
$$+\tfrac{1}{\sqrt{3}}L_2\,L_2\,h_u\,h_u\,\xi+\tfrac{1}{\sqrt{3}}L_3\,L_3\,h_u\,h_u\,\xi$$

Invariant Lagrangian

➢ Terms that are invariant, have 2 leptons and mass dimension ≤ 5:

$$L\,L\,h_u\,h_u\,\varphi_S + L\,L\,h_u\,h_u\,\xi + L\,e\,h_d\,\varphi_T + L\,\mu\,h_d\,\varphi_T + L\,\tau\,h_d\,\varphi_T$$

$$\mathbf{3'}\otimes\mathbf{3'}\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{3'}=\left(\mathbf{1'}+\mathbf{1^{(5)}}+\mathbf{1^{(8)}}+2\times\mathbf{3''}\right)\otimes\mathbf{3'}=2\times\mathbf{1}+2\times\mathbf{1'''}+2\times\mathbf{1^{(4)}}+7\times\mathbf{3}$$

➢ Contract family indices:

$$\tfrac{1}{\sqrt{3}}L_2\,L_3\,h_u\,h_u\,\varphi_{S,1}+\tfrac{1}{\sqrt{3}}L_3\,L_1\,h_u\,h_u\,\varphi_{S,2}+\tfrac{1}{\sqrt{3}}L_1\,L_2\,h_u\,h_u\,\varphi_{S,3}+\tfrac{1}{\sqrt{3}}L_1\,L_1\,h_u\,h_u\,\xi$$
$$+\tfrac{1}{\sqrt{3}}L_2\,L_2\,h_u\,h_u\,\xi+\tfrac{1}{\sqrt{3}}L_3\,L_3\,h_u\,h_u\,\xi$$

➢ Contract $\mathrm{SU}(2)$ indices and substitute *vevs* $\langle\varphi_S\rangle = (v_S, 0, 0)$, etc:

$$\tfrac{1}{\sqrt{3}}L_2^{(1)}\,L_3^{(1)}\,v_u\,v_u\,v_S+\tfrac{1}{\sqrt{3}}L_1^{(1)}\,L_1^{(1)}\,v_u\,v_u\,\xi+\tfrac{1}{\sqrt{3}}L_2^{(1)}\,L_2^{(1)}\,v_u\,v_u\,\xi+\tfrac{1}{\sqrt{3}}L_3^{(1)}\,L_3^{(1)}\,v_u\,v_u\,\xi$$

Finally, the Mixing Angles

- Mass matrices

$$M_{\ell^+} = \begin{matrix} & e & \mu & \tau \\ L_1^{(2)} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ L_2^{(2)} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ L_3^{(2)} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \end{matrix}, \qquad M_\nu = \begin{matrix} & L_1^{(1)} & L_2^{(1)} & L_3^{(1)} \\ L_1^{(1)} & \frac{1}{\sqrt{3}} & 0 & 0 \\ L_2^{(1)} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ L_3^{(1)} & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \end{matrix}$$

Finally, the Mixing Angles

➢ Mass matrices

$$M_{\ell^+} = \begin{matrix} L_1^{(2)} \\ L_2^{(2)} \\ L_3^{(2)} \end{matrix} \overset{\begin{matrix} e & \mu & \tau \end{matrix}}{\begin{pmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \end{pmatrix}}, \qquad M_\nu = \begin{matrix} L_1^{(1)} \\ L_2^{(1)} \\ L_3^{(1)} \end{matrix} \overset{\begin{matrix} L_1^{(1)} & L_2^{(1)} & L_3^{(1)} \end{matrix}}{\begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}}$$

➢ Singular value decomposition: $\hat{M}_{\ell^+} = D_L M_{\ell^+} D_R^\dagger, \quad \hat{M}_\nu = U_L M_\nu U_R^\dagger$

$$D_\mathrm{L} = \begin{pmatrix} -0.5774+i\,0.0000 & -0.5774+i\,0.0000 & -0.5774+i\,0.0000 \\ 0.5738-i\,0.0636 & -0.2319+i\,0.5287 & -0.3420-i\,0.4652 \\ 0.5731-i\,0.0702 & -0.3474-i\,0.4612 & -0.2257+i\,0.5314 \end{pmatrix}, \quad U_\mathrm{L} = \begin{pmatrix} 0.0000 & -0.7071 & -0.7071 \\ -1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.7071 & -0.7071 \end{pmatrix}$$

Finally, the Mixing Angles

- Mass matrices

$$M_{\ell^+} = \begin{array}{c} \\ L_1^{(2)} \\ L_2^{(2)} \\ L_3^{(2)} \end{array} \begin{array}{c} \begin{array}{ccc} e & \mu & \tau \end{array} \\ \begin{pmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \end{pmatrix} \end{array}, \qquad M_\nu = \begin{array}{c} \\ L_1^{(1)} \\ L_2^{(1)} \\ L_3^{(1)} \end{array} \begin{array}{c} \begin{array}{ccc} L_1^{(1)} & L_2^{(1)} & L_3^{(1)} \end{array} \\ \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \end{array}$$

- Singular value decomposition: $\hat{M}_{\ell^+} = D_L M_{\ell^+} D_R^\dagger, \quad \hat{M}_\nu = U_L M_\nu U_R^\dagger$

$$D_\mathrm{L} = \begin{pmatrix} -0.5774+i\,0.0000 & -0.5774+i\,0.0000 & -0.5774+i\,0.0000 \\ 0.5738-i\,0.0636 & -0.2319+i\,0.5287 & -0.3420-i\,0.4652 \\ 0.5731-i\,0.0702 & -0.3474-i\,0.4612 & -0.2257+i\,0.5314 \end{pmatrix}, \quad U_\mathrm{L} = \begin{pmatrix} 0.0000 & -0.7071 & -0.7071 \\ -1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.7071 & -0.7071 \end{pmatrix}$$

- Neutrino mixing matrix: $U_\mathrm{PMNS} = D_\mathrm{L} U_\mathrm{L}^\dagger$ (needs rephasing)

$$U_\mathrm{PMNS} = \begin{pmatrix} 0.8165+i\,0.0000 & 0.5774+i\,0.0000 & 0.0000+i\,0.0000 \\ 0.4058-i\,0.0449 & -0.5738+i\,0.0636 & 0.0778+i\,0.7028 \\ 0.4052-i\,0.0497 & -0.5731+i\,0.0702 & -0.0860-i\,0.7019 \end{pmatrix}$$

- Mixing angles: $\theta_{12} = 35.26, \quad \theta_{23} = 45.00, \quad \theta_{13} = 0.00$ Tribimaximal ✔

Finally, the Mixing Angles

- Mass matrices

$$M_{\ell^+} = \begin{array}{c} \\ L_1^{(2)} \\ L_2^{(2)} \\ L_3^{(2)} \end{array} \begin{array}{c} \begin{array}{ccc} e & \mu & \tau \end{array} \\ \begin{pmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \end{pmatrix} \end{array}, \qquad M_\nu = \begin{array}{c} \\ L_1^{(1)} \\ L_2^{(1)} \\ L_3^{(1)} \end{array} \begin{array}{c} \begin{array}{ccc} L_1^{(1)} & L_2^{(1)} & L_3^{(1)} \end{array} \\ \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \end{array}$$

- Singular value decomposition: $\hat{M}_{\ell^+} = D_L M_{\ell^+} D_R^\dagger$, $\quad \hat{M}_\nu = U_L M_\nu U_R^\dagger$

$$D_{\mathrm{L}} = \begin{pmatrix} -0.5774+i\,0.0000 & -0.5774+i\,0.0000 & -0.5774+i\,0.0000 \\ 0.5738-i\,0.0636 & -0.2319+i\,0.5287 & -0.3420-i\,0.4652 \\ 0.5731-i\,0.0702 & -0.3474-i\,0.4612 & -0.2257+i\,0.5314 \end{pmatrix}, \quad U_{\mathrm{L}} = \begin{pmatrix} 0.0000 & -0.7071 & -0.7071 \\ -1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.7071 & -0.7071 \end{pmatrix}$$

- Neutrino mixing matrix: $U_{\mathrm{PMNS}} = D_{\mathrm{L}} U_{\mathrm{L}}^\dagger$ (needs rephasing)

$$|U_{\mathrm{PMNS}}| = \begin{pmatrix} 0.8165 & 0.5774 & 0.0000 \\ 0.4082 & 0.5774 & 0.7071 \\ 0.4082 & 0.5774 & 0.7071 \end{pmatrix}$$

- Mixing angles: $\theta_{12} = 35.26$, $\theta_{23} = 45.00$, $\theta_{13} = 0.00$ Tribimaximal ✔

Finally, the Mixing Angles

➢ Mass matrices

$$M_{\ell^+} = \begin{matrix} & e & \mu & \tau \\ L_1^{(2)} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ L_2^{(2)} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ L_3^{(2)} & -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \end{matrix}, \qquad M_\nu = \begin{matrix} & L_1^{(1)} & L_2^{(1)} & L_3^{(1)} \\ L_1^{(1)} & \frac{1}{\sqrt{3}} & 0 & 0 \\ L_2^{(1)} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ L_3^{(1)} & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \end{matrix}$$

➢ Singular value decomposition: $\hat{M}_{\ell^+} = D_L M_{\ell^+} D_R^\dagger, \quad \hat{M}_\nu = U_L M_\nu U_R^\dagger$

$$D_\mathrm{L} = \begin{pmatrix} -0.5774+i\,0.0000 & -0.5774+i\,0.0000 & -0.5774+i\,0.0000 \\ 0.5738-i\,0.0636 & -0.2319+i\,0.5287 & -0.3420-i\,0.4652 \\ 0.5731-i\,0.0702 & -0.3474-i\,0.4612 & -0.2257+i\,0.5314 \end{pmatrix}, \quad U_\mathrm{L} = \begin{pmatrix} 0.0000 & -0.7071 & -0.7071 \\ -1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.7071 & -0.7071 \end{pmatrix}$$

➢ Neutrino mixing matrix: $U_\mathrm{PMNS} = D_\mathrm{L} U_\mathrm{L}^\dagger$ (needs rephasing)

$$|U_\mathrm{PMNS}| = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$$

➢ Mixing angles: $\theta_{12} = 35.26, \;\; \theta_{23} = 45.00, \;\; \theta_{13} = 0.00$ Tribimaximal ✔

What is the Bottom Line?

- We used the Altarelli-Feruglio model only as a paradigm
- The analysis is completely independent of the family symmetry
- `GAP` gives us all the relevant information about the group
- Complexity of group is hereby irrelevant
- We use `Python` to interact w/`GAP` and do symbolic manipulations
- From symmetry to lagrangian to mixing angles takes less than 5 ms/model

Where Do We Go From Here?

1. Generalize family symmetry:

$$A_4 \times C_3 \longrightarrow 1048 \text{ groups of order } \leq 100$$

2. Particle content is the same:

$$L, \quad e, \quad \mu, \quad \tau, \quad h_u, \quad h_d, \quad \varphi_T, \quad \varphi_S, \quad \xi$$

3. Generalize family charge assignments:

$$\begin{aligned}(L, e, \mu, \tau, h_u, h_d, \varphi_T, \varphi_S, \xi) &\rightarrow (\mathbf{3'}, \mathbf{1'}, \mathbf{1^{(8)}}, \mathbf{1^{(5)}}, \mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{3'}, \mathbf{1''}) \\ &\rightarrow (*, \ *, \ *, \ *, \ *, \ *, \ *, \ *, \ *)\end{aligned}$$

4. Generalize symmetry breaking patterns:

$$\langle\varphi_T\rangle = (*, *, *), \quad \langle\varphi_S\rangle = (*, *, *), \quad \langle\xi\rangle = *$$

Small Groups

The first few of the 1048 groups of order ≤ 100
✔ $= \mathrm{U}(n)$ and ✔ $= \mathrm{SU}(n)$ for $n = 2, 3$

GAP ID	Group	**3**	$\mathrm{U}(3)$	$\mathrm{U}(2)$	$\mathrm{U}(2)\times\mathrm{U}(1)$	A_4
[1, 1]	1	✗	✗	✗	✗	✗
[2, 1]	C_2	✗	✗	✗	✗	✗
[3, 1]	C_3	✗	✗	✗	✗	✗
[4, 1]	C_4	✗	✗	✗	✗	✗
[4, 2]	$C_2 \times C_2$	✗	✗	✗	✗	✗
[5, 1]	C_5	✗	✗	✗	✗	✗
[6, 1]	S_3	✗	✔	✔	✔	✗
[6, 2]	C_6	✗	✗	✗	✗	✗
[7, 1]	C_7	✗	✗	✗	✗	✗
[8, 1]	C_8	✗	✗	✗	✗	✗

Small Groups

The first few of the 1048 groups of order ≤ 100

✔ (blue) $= \mathrm{U}(n)$ and ✔ (orange) $= \mathrm{SU}(n)$ for $n = 2, 3$

GAP ID	Group	**3**	U(3)	U(2)	U(2)×U(1)	A_4
[8, 2]	$C_4 \times C_2$	✗	✗	✗	✗	✗
[8, 3]	D_4	✗	✔ (orange)	✔ (blue)	✔ (blue)	✗
[8, 4]	Q_8	✗	✔ (orange)	✔ (orange)	✔ (orange)	✗
[8, 5]	$C_2 \times C_2 \times C_2$	✗	✗	✗	✗	✗
[9, 1]	C_9	✗	✗	✗	✗	✗
[9, 2]	$C_3 \times C_3$	✗	✗	✗	✗	✗
[10, 1]	D_5	✗	✔ (orange)	✔ (blue)	✔ (blue)	✗
[10, 2]	C_{10}	✗	✗	✗	✗	✗
[11, 1]	C_{11}	✗	✗	✗	✗	✗
[12, 1]	$C_3 \rtimes_\varphi C_4$	✗	✔ (orange)	✔ (orange)	✔ (orange)	✗

Small Groups

The first few of the 1048 groups of order ≤ 100
✔ $= \mathrm{U}(n)$ and ✔ $= \mathrm{SU}(n)$ for $n = 2, 3$

GAP ID	Group	**3**	$\mathrm{U}(3)$	$\mathrm{U}(2)$	$\mathrm{U}(2)\times\mathrm{U}(1)$	A_4
$[12, 2]$	C_{12}	✗	✗	✗	✗	✗
$[12, 3]$	A_4	✔	✔	✗	✗	✔
$[12, 4]$	D_6	✗	✔	✔	✔	✗
$[12, 5]$	$C_6 \times C_2$	✗	✗	✗	✗	✗
$[13, 1]$	C_{13}	✗	✗	✗	✗	✗
$[14, 1]$	D_7	✗	✔	✔	✔	✗
$[14, 2]$	C_{14}	✗	✗	✗	✗	✗
$[15, 1]$	C_{15}	✗	✗	✗	✗	✗
$[16, 1]$	C_{16}	✗	✗	✗	✗	✗
$[16, 2]$	$C_4 \times C_4$	✗	✗	✗	✗	✗

Small Groups

The first few of the 1048 groups of order ≤ 100

✔ (blue) $= \mathrm{U}(n)$ and ✔ (orange) $= \mathrm{SU}(n)$ for $n = 2, 3$

`GAP ID`	Group	**3**	$\mathrm{U}(3)$	$\mathrm{U}(2)$	$\mathrm{U}(2)\times\mathrm{U}(1)$	A_4
[16, 3]	$(C_4 \times C_2) \rtimes_\varphi C_2$	✗	✔ (orange)	✗	✔ (blue)	✗
[16, 4]	$C_4 \rtimes_\varphi C_4$	✗	✔ (orange)	✗	✔ (orange)	✗
[16, 5]	$C_8 \times C_2$	✗	✗	✗	✗	✗
[16, 6]	$C_8 \rtimes_\varphi C_2$	✗	✔ (orange)	✔ (blue)	✔ (blue)	✗
[16, 7]	D_8	✗	✔ (orange)	✔ (blue)	✔ (blue)	✗
[16, 8]	QD_8	✗	✔ (orange)	✔ (blue)	✔ (blue)	✗
[16, 9]	Q_{16}	✗	✔ (orange)	✔ (orange)	✔ (orange)	✗
[16, 10]	$C_4 \times C_2 \times C_2$	✗	✗	✗	✗	✗
[16, 11]	$C_2 \times D_4$	✗	✔ (orange)	✗	✔ (blue)	✗
[16, 12]	$C_2 \times Q_8$	✗	✔ (orange)	✗	✔ (orange)	✗

Small Groups

The first few of the 1048 groups of order ≤ 100
✔ (blue) $= \mathrm{U}(n)$ and ✔ (orange) $= \mathrm{SU}(n)$ for $n = 2, 3$

GAP ID	Group	**3**	U(3)	U(2)	U(2)×U(1)	A_4
[16, 13]	$(C_4 \times C_2) \rtimes_\varphi C_2$	✗	✔ (orange)	✔ (blue)	✔ (blue)	✗
[16, 14]	$C_2 \times C_2 \times C_2 \times C_2$	✗	✗	✗	✗	✗
[17, 1]	C_{17}	✗	✗	✗	✗	✗
[18, 1]	D_9	✗	✔ (orange)	✔ (blue)	✔ (blue)	✗
[18, 2]	C_{18}	✗	✗	✗	✗	✗
[18, 3]	$C_3 \times S_3$	✗	✔ (orange)	✔ (blue)	✔ (blue)	✗
[18, 4]	$(C_3 \times C_3) \rtimes_\varphi C_2$	✗	✔ (orange)	✗	✔ (blue)	✗
[18, 5]	$C_6 \times C_3$	✗	✗	✗	✗	✗
[19, 1]	C_{19}	✗	✗	✗	✗	✗
[20, 1]	$C_5 \rtimes_\varphi C_4$	✗	✔ (orange)	✔ (orange)	✔ (orange)	✗

Small Groups

The first few of the 1048 groups of order ≤ 100
✔ $= \mathrm{U}(n)$ and ✔ $= \mathrm{SU}(n)$ for $n = 2, 3$

`GAP ID`	Group	**3**	U(3)	U(2)	U(2)×U(1)	A_4
[20, 2]	C_{20}	✗	✗	✗	✗	✗
[20, 3]	$C_5 \rtimes_\varphi C_4$	✗	✗	✗	✗	✗
[20, 4]	D_{10}	✗	✔ (SU)	✔ (U)	✔ (U)	✗
[20, 5]	$C_{10} \times C_2$	✗	✗	✗	✗	✗
[21, 1]	$C_7 \rtimes_\varphi C_3$	✔ (U)	✔ (SU)	✗	✗	✗
[21, 2]	C_{21}	✗	✗	✗	✗	✗
[22, 1]	D_{11}	✗	✔ (SU)	✔ (U)	✔ (U)	✗
[22, 2]	C_{22}	✗	✗	✗	✗	✗
[23, 1]	C_{23}	✗	✗	✗	✗	✗
[24, 1]	$C_3 \rtimes_\varphi C_8$	✗	✔ (SU)	✔ (U)	✔ (SU)	✗

Small Groups

The first few of the 1048 groups of order ≤ 100
✔ (blue) $= \mathrm{U}(n)$ and ✔ (orange) $= \mathrm{SU}(n)$ for $n = 2, 3$

GAP ID	Group	**3**	$\mathrm{U}(3)$	$\mathrm{U}(2)$	$\mathrm{U}(2)\times\mathrm{U}(1)$	A_4
$[24, 2]$	C_{24}	✗	✗	✗	✗	✗
$[24, 3]$	$\mathrm{SL}(2, 3)$	✔ (blue)	✔ (orange)	✔ (orange)	✔ (orange)	✗
$[24, 4]$	$C_3 \rtimes_\varphi Q_8$	✗	✔ (orange)	✔ (orange)	✔ (orange)	✗
$[24, 5]$	$C_4 \times S_3$	✗	✔ (orange)	✔ (blue)	✔ (blue)	✗
$[24, 6]$	D_{12}	✗	✔ (orange)	✔ (blue)	✔ (blue)	✗
$[24, 7]$	$C_2 \times (C_3 \rtimes_\varphi C_4)$	✗	✔ (orange)	✗	✔ (orange)	✗
$[24, 8]$	$(C_6 \times C_2) \rtimes_\varphi C_2$	✗	✔ (orange)	✔ (blue)	✔ (blue)	✗
$[24, 9]$	$C_{12} \times C_2$	✗	✗	✗	✗	✗
$[24, 10]$	$C_3 \times D_4$	✗	✔ (orange)	✔ (blue)	✔ (blue)	✗
$[24, 11]$	$C_3 \times Q_8$	✗	✔ (orange)	✔ (blue)	✔ (orange)	✗

Small Groups

The first few of the 1048 groups of order ≤ 100

✔ $= \mathrm{U}(n)$ and ✔ $= \mathrm{SU}(n)$ for $n = 2, 3$

GAP ID	Group	**3**	$\mathrm{U}(3)$	$\mathrm{U}(2)$	$\mathrm{U}(2)\times\mathrm{U}(1)$	A_4
$[24, 12]$	S_4	✔	✔	✗	✔	✔
$[24, 13]$	$C_2 \times A_4$	✔	✔	✗	✗	✔
$[24, 14]$	$C_2 \times C_2 \times S_3$	✗	✔	✗	✔	✗
$[24, 15]$	$C_6 \times C_2 \times C_2$	✗	✗	✗	✗	✗
$[25, 1]$	C_{25}	✗	✗	✗	✗	✗
$[25, 2]$	$C_5 \times C_5$	✗	✗	✗	✗	✗
$[26, 1]$	D_{13}	✗	✔	✔	✔	✗
$[26, 2]$	C_{26}	✗	✗	✗	✗	✗
$[27, 1]$	C_{27}	✗	✗	✗	✗	✗
$[27, 2]$	$C_9 \times C_3$	✗	✗	✗	✗	✗

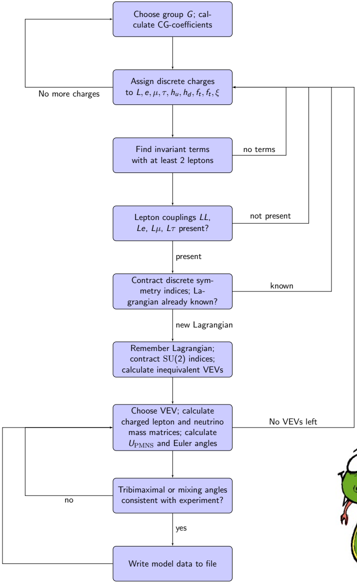
Particles and Symmetry Breaking

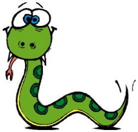
➢ Assignment of family symmetry charges

Field	$\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$	$\mathrm{U}(1)_R$	$A_4 \times C_3$	
L	$(\mathbf{2},\text{-}1)$	1	$\mathbf{3'}$	$\longrightarrow$ Any 3-dim irrep
e	$(\mathbf{1},\ 2)$	1	$\mathbf{1'}$	$\longrightarrow$ Any 1-dim irrep
μ	$(\mathbf{1},\ 2)$	1	$\mathbf{1^{(8)}}$	$\longrightarrow$ Any 1-dim irrep
τ	$(\mathbf{1},\ 2)$	1	$\mathbf{1^{(5)}}$	$\longrightarrow$ Any 1-dim irrep
h_u	$(\mathbf{2},\ 1)$	0	$\mathbf{1}$	$\longrightarrow$ Any 1-dim irrep
h_d	$(\mathbf{2},\text{-}1)$	0	$\mathbf{1}$	$\longrightarrow$ Any 1-dim irrep
φ_T	$(\mathbf{1},\ 0)$	0	$\mathbf{3}$	$\longrightarrow$ Any irrep
φ_S	$(\mathbf{1},\ 0)$	0	$\mathbf{3'}$	$\longrightarrow$ Any irrep
ξ	$(\mathbf{1},\ 0)$	0	$\mathbf{1''}$	$\longrightarrow$ Any irrep

➢ Choose *vevs* that break to all subgroups

Scanning for the Models

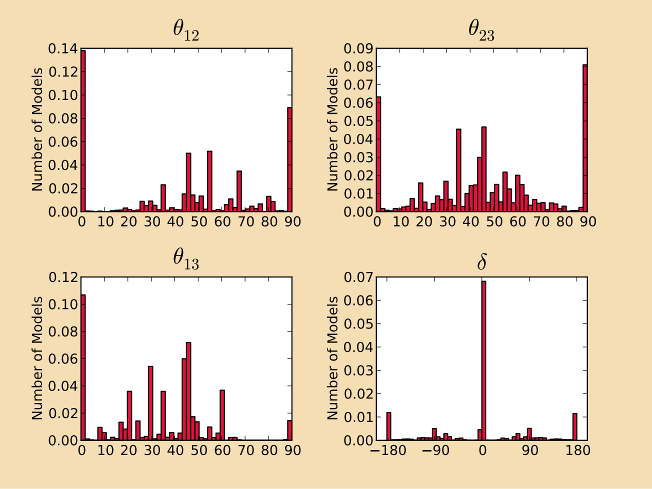




- Consider $A_4 \times C_3$
- 14,594,580 different family charge assignments (particles w/same gauge/R-symmetry charges are considered identical)
- Computer (2.6 GHz, 2 GB) takes 19.35 hours $\rightsquigarrow$ Distribute job to 10 machines and get results in 1.9 hours
- 41,086 different Lagrangians
- 8,526 cases of tribimaximal mixing, i.e. 20.75%
- Results for other groups on the way

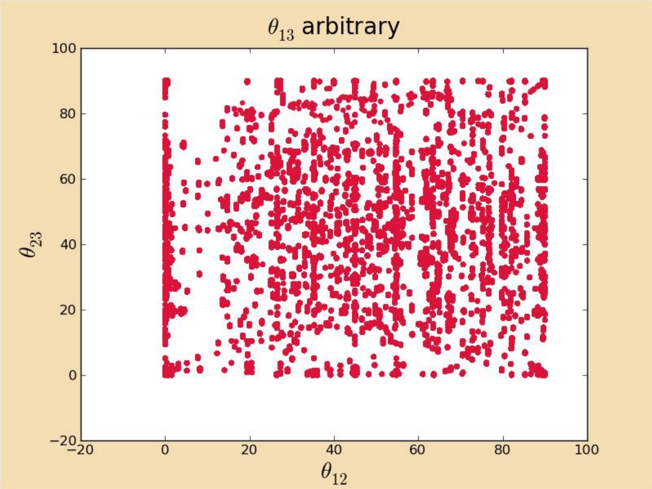
Distribution of Angles

➢ Frequency of angles and phase



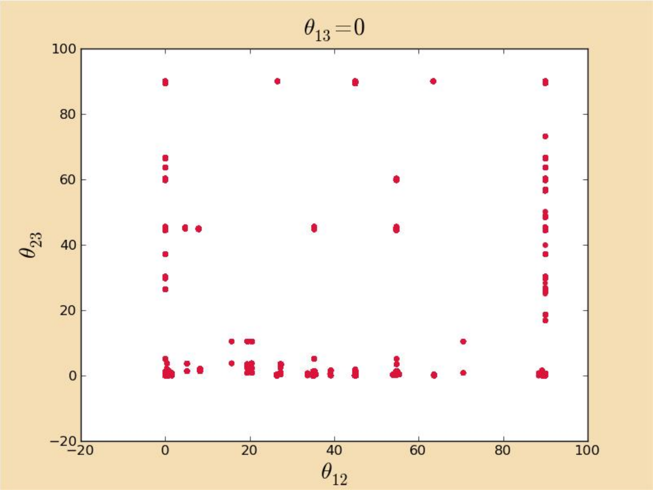
Distribution of Angles

- θ_{12} vs. θ_{23} for arbitrary θ_{13}



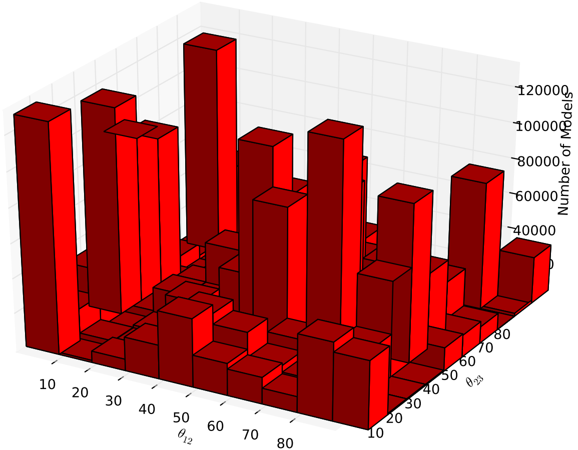
Distribution of Angles

- θ_{12} vs. θ_{23} for $\theta_{13} = 0$



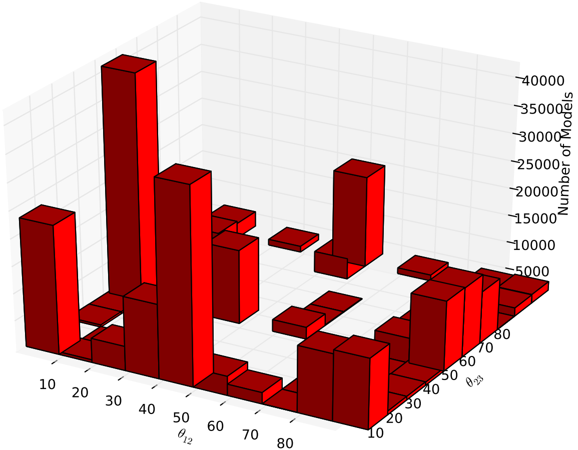
Distribution of Angles

➢ Frequency of models: θ_{12} vs. θ_{23} for arbitrary θ_{13}



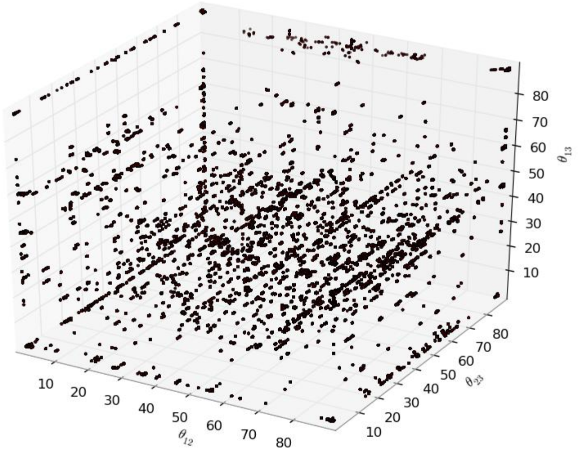
Distribution of Angles

- Frequency of models: θ_{12} vs. θ_{23} for $\theta_{13} = 0$



Distribution of Angles

- Scatter plot for θ_{12}, θ_{23}, θ_{13}



Conclusions

- Work in progress
- Disclaimer: I am not a landscaper
- Showing what one can do with data
- Our objective: Clarify connection between TBM and symmetry
- TBM occurs in 20.75% of all cases for $A_4 \times C_3$. Encouraging!
- TBM is not "privileged" or singled out, though
- Next: Extend analysis to other "small" groups