Holographic Currents and Chern-Simons Terms

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Introduction

Type IIB string theory on an AdS_5 x Y_5 background, where Y_5 is a Sasaki-Einstein manifold, for large radii and with a stack of N D3-branes at the tip of the conical singularity is approximated by a $\mathcal{N}$=2 conformal supergravity theory in five dimensions.

This supergravity theory in turn is dual to a $\mathcal{N}$=1 superconformal field theory on the Minkowski space-time M_4 boundary of the AdS_5 space. (Maldacena, Witten, Gubser et al.)

The values of the central charges of the superconformal field theory (SCFT), a and c, designate different boundary field theories dual to the bulk AdS supergravity. (Anselmi et al.)

The energy-momentum tensor trace and the R-symmetry current anomalies calculated holographically in the supergravity theory and obtained in terms of the central charges in the boundary field theory have been shown to match in leading order. (Henningson and Skenderis)

Imposing subleading order matching provides a measure of the string loop corrections to the supergravity action through the value of the coefficient governing the higher derivative action terms. (Nojiri and Odintsov, Blau et al, Anselmi and Kehagias, Fukuma et al, Hanaki et al, Cremonini et al.)

The purpose of this work is to construct the energy-momentum tensor and R-symmetry current in the case that a subleading mixed Chern-Simons term is added to the leading order supergravity action.

This truncated four derivative action provides the leading gravitational contributions to each anomaly (leading N^2 contributions with c=a in the trace anomaly case and subleading N^0 contributions with non-vanishing c-a in the R-anomaly case).

The pure $U(1)_R$ gauge field contributions are leading order N^2 terms for each anomaly.

The near boundary field equations are solved and the abbreviated action is holographically renormalized through the addition of boundary counter-terms. (de Haro et al, Bianchi et al, Skenderis)

The Brown-York energy-momentum tensor (Brown and York)and R-symmetry current are constructed via the boundary source variational principle for the action. (Henningson, Balasubramanian)

The covariant divergence of the energy-momentum tensor has the Brown-York form and yields the diffeomorphism invariance of the renormalized action Ward identity (Imbimbo et al), thus providing its interpretation as a boundary energy-momentum tensor.

Improvements to both currents can be constructed consistent with the Ward identities and trace and R-symmetry anomalies.

These improvements follow from the variation of finite boundary action counter terms.

Conformal field theory

The trace anomaly of the boundary N=1 SCFT in the presence of background gravitational and $U(1)_R$ gauge fields and in the absence of sources related to any other global symmetries has the general form: (Anselmi et al.)

$$\theta^{\mu}{}_{\mu} = \frac{1}{16\pi^2}\left[2(2a-c)R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}(c-3a)R^2 + (c-a)R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right] + \frac{c}{6\pi^2}F_{\mu\nu}F^{\mu\nu}.$$

For an R-current defined so that it lies in the same N=1 SUSY supercurrent multiplet as the energy-momentum tensor and with the trace and R-current anomalies also appearing in their own SUSY anomaly multiplet, then the R-symmetry current anomaly is given in terms of the same a and c coefficients and has the form:

$$\partial_\mu(\sqrt{-g}R^\mu) = \frac{1}{24\pi^2}(c-a)\epsilon^{\mu\nu\rho\sigma}R^{\xi\zeta}{}_{\mu\nu}R_{\xi\zeta\rho\sigma} + \frac{1}{18\pi^2}(5a-3c)\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}.$$

In order for these currents, along with the SUSY current, to lie in the same N=1 supermultiplet, they must have appropriate normalization and improvements.

The field theory determination of these anomaly coefficients is: (Anselmi et al.)

$$a = \frac{3}{32}\left(3\mathrm{Tr}R^3 - \mathrm{Tr}R\right) \qquad , \qquad c = \frac{1}{32}\left(9\mathrm{Tr}R^3 - 5\mathrm{Tr}R\right),$$

The leading N^2 contribution to the anomaly coefficients implies that c=a for the class of models considered here.

Leading action terms

The truncated bosonic part of the bulk AdS_5, N=2 conformal supergravity action including the single four derivative mixed Chern-Simons term is given by:

$$\Gamma = \int d^5x \det E \left[\frac{1}{2\kappa} R + \Lambda - \frac{Z_A}{4} F_{MN} F^{MN} \right.$$
$$\left. + C\, E^{MNRST} F_{MN} F_{RS} A_T - D\, E^{MNRST} R^X{}_{YMN} R^Y{}_{XRS} A_T \right]$$
$$+ \frac{1}{\kappa} \int_{\rho=\epsilon} d^4x \sqrt{-\gamma} K(\gamma) + \Gamma_{\mathrm{Counter-terms}},$$

The Fefferman-Graham metric for the (asymptotic) AdS_5 space of radius R_5 is:

$$ds^2 = \frac{R_5^2}{\rho} \left[g_{\mu\nu}(x,\rho) dx^\mu dx^\nu - \frac{d\rho^2}{4\rho} \right]$$

and γ is the induced metric on the fixed $\rho=\varepsilon$ hypersurface homeomorphic to the M_4 boundary.

The extrinsic curvature $K_{\mu\nu}$ on the $\rho=\varepsilon$ surface is given by the gradient of the normal vector there. The boundary term is required to have a well defined variational principle for the action.

The counter-term boundary action is determined through holographic renormalization and normalization conditions.

Equations of motion

Varying the bulk action with fixed boundary conditions for the fields yields the field equations. The Einstein equation has the form:

$$R_{MN} - \frac{1}{2} g_{MN} R = g_{MN} \kappa \Lambda - \kappa T_{MN},$$

The bulk energy-momentum tensor T_{MN} is obtained as

$$T_{MN} = T_{MN}^{\mathrm{Maxwell}} + \nabla_X \Theta^X_{MN},$$

with the R-symmetry gauge field's contribution to the energy-momentum tensor given by the Maxwell symmetric form

$$T_{MN}^{\mathrm{Maxwell}} = Z_A F_{MR} F^R{}_N - g_{MN} \left(-\frac{Z_A}{4} F_{RS} F^{RS} \right)$$

while the mixed gauge-gravity Chern-Simons contribution to the energy-momentum tensor is

$$\Theta^X_{MN} = 2D \, E^{QRSTU} F_{TU} \left(g_{QM} R^X{}_{NRS} + g_{QN} R^X{}_{MRS} \right).$$

The Maxwell equation is generalized to include the Chern-Simons terms so that

$$Z_A \nabla_M F^{ML} + 3C \, E^{MNRSL} F_{MN} F_{RS} = D \, E^{MNRSL} R^X{}_{YMN} R^Y{}_{XRS}.$$

Solutions to equations of motion

The boundary currents have an expectation value in the presence of the background gravitational and $U(1)_R$ gauge fields given in terms of the asymptotic behavior of the bulk fields at the M_4 boundary of the AdS_5 space. Consequently the field equations need only be solved close to the boundary at ρ =0 in order to determine these one point functions.

The $U(1)_R$ gauge field $A_\mu(x,\rho)$ has the asymptotic form in ρ close to the boundary given by

$$A_\mu(x,\rho) = A_\mu^{(0)}(x) + \rho A_\mu^{(2)}(x) + \rho \ln\left(\rho/R_5^2\right) \;\; B_\mu(x).$$

Likewise, near the boundary the gravitational field has the behavior

$$g_{\mu\nu}(x,\rho) = g_{\mu\nu}^{(0)}(x) + \rho g_{\mu\nu}^{(2)}(x) + \rho^2 g_{\mu\nu}^{(4)}(x) + \rho^2 \ln\left(\rho/R_5^2\right) \;\; h_{\mu\nu}(x).$$

Substituting these expansions into the Maxwell equations yields the field B_μ and the covariant divergence of $A_\mu^{(2)}$ in terms of $A_\mu^{(0)}$.

The transverse part of $A_\mu^{(2)}$ is undetermined. It corresponds to the other linearly independent solution to the second order Dirichlet problem A second boundary condition deeper into the bulk would be needed for its specification.

In order to find the asymptotic solution to the Einstein equations the bulk energy-momentum tensor must also be expanded in terms of ρ.

The metric coefficients $g^{(2)}$ and h can be determined from the field equation expansion in terms of the boundary metric $g^{(0)}$ and the boundary R-symmetry gauge field $A_\mu^{(0)}$ while only the trace and divergence of $g^{(4)}$ are determined in terms of $g^{(0)}$ and $A_\mu^{(0)}$ by the near boundary expansion.

$$B^\mu(x) \;=\; \frac{1}{4}\nabla^{(0)}_\nu F^{(0)\nu\mu}$$

$$2R_5 Z_A \nabla^{(0)}_\mu A^{(2)\mu}(x) \;=\; -3CE^{\mu\nu\rho\sigma}(g^{(0)})F^{(0)}_{\mu\nu}F^{(0)}_{\rho\sigma} + DE^{\mu\nu\rho\sigma}(g^{(0)})R^{(0)\kappa}{}_{\lambda\mu\nu}R^{(0)\lambda}{}_{\kappa\rho\sigma}.$$

$$g^{(2)}_{\mu\nu} = \frac{1}{2}\left[R^{(0)}_{\mu\nu} - \frac{1}{6}g^{(0)}_{\mu\nu}R^{(0)}\right],$$

$$h_{\mu\nu} \;=\; \frac{1}{8}\left[R^{(0)}_{\mu\rho}R^{(0)\rho}{}_\nu - \frac{1}{3}R^{(0)}R^{(0)}_{\mu\nu} - \frac{1}{36}g^{(0)}_{\mu\nu}R^{(0)2}\right] - \frac{1}{32}g^{(0)}_{\mu\nu}\left[R^{(0)}_{\mu\nu}R^{(0)\mu\nu} - \frac{2}{9}R^{(0)2}\right] + \ldots,$$

$$g^{(0)\mu\nu}g^{(4)}_{\mu\nu} \;=\; \frac{1}{16}\left[R^{(0)}_{\mu\nu}R^{(0)\mu\nu} - \frac{2}{9}R^{(0)2}\right] - \frac{\kappa Z_A}{24R_5^2}F^{(0)}_{\mu\nu}F^{(0)\mu\nu}.$$

$$\nabla^{(0)\rho}g^{(4)}_{\rho\mu} \;=\; -\frac{1}{2}\nabla^{(0)\rho}h_{\rho\mu} + \frac{1}{2}\nabla^{(0)\rho}\left[g^{(2)}_{\rho\sigma}g^{(0)\sigma\lambda}g^{(2)}_{\lambda\mu}\right]$$

Renormalization

The near-boundary analysis of the field equations allows the boundary divergences of the action, now regulated at the surface ρ=ε, to be determined. The regulated action is given by

$$\Gamma_{\text{Reg.}} = \int d^4x \int_{\rho=\epsilon} d\rho \det E \left[\frac{1}{2\kappa}R + \Lambda - \frac{Z_A}{4}F_{MN}F^{MN} + C\, E^{MNRST}F_{MN}F_{RS}A_T - D\, E^{MNRST}R^X{}_{YMN}R^Y{}_{XRS}A_T\right] + \frac{1}{\kappa}\int_{\rho=\epsilon} d^4x\sqrt{-\gamma}K(\gamma).$$

Applying the Fefferman-Graham form of the metric and employing the near-boundary solutions to the field equations the divergent terms in the action are isolated as

$$\Gamma_{\text{Reg.}} = \int d^4x\sqrt{-g^{(0)}}\left\{-\frac{1}{\epsilon^2}\frac{3R_5^3}{\kappa} + \ln(\epsilon/R_5^2)\frac{R_5^3}{16\kappa}\left[\left(R^{(0)}_{\mu\nu}R^{(0)\mu\nu} - \frac{1}{3}R^{(0)2}\right) + \frac{2\kappa Z_A}{R_5^2}F^{(0)}_{\mu\nu}F^{(0)\mu\nu}\right]\right\} + O(\epsilon^0).$$

The holographically renormalized action is defined by choosing the near-boundary counter-term action to cancel the divergent terms in the regulated action and to impose normalization conditions on the remaining finite terms.

Improvement terms

After inverting the near-boundary expansion of the fields to write the boundary quantities in terms of the tensors at the surface $\rho = \varepsilon$, the near-boundary counter-term action is found to be

$$\Gamma_{\mathrm{Ct.-terms}} = \frac{3}{\kappa R_5}\int_{\rho=\epsilon} d^4x\sqrt{-\gamma} - \frac{R_5}{4\kappa}\int_{\rho=\epsilon} d^4x\sqrt{-\gamma}R(\gamma)$$
$$- \ln\left(\epsilon/R_5^2\right)\frac{R_5^3}{16\kappa}\int_{\rho=\epsilon} d^4x\sqrt{-\gamma}\left[\left(R_{\mu\nu}(\gamma)R^{\mu\nu}(\gamma) - \frac{1}{3}R^2(\gamma)\right) + \frac{2\kappa Z_A}{R_5^2}F_{\mu\nu}(A)F^{\mu\nu}(A)\right]$$

The finite holographic normalization is chosen through the dimensionless ratio in the logarithmic counter-term as ε/R_5^2. (minimal subtraction) A different normalization such as $\varepsilon/\tau R_5^2$ will correspond to a finite boundary term in the action,

$$\Gamma_{\mathrm{Improve}} = \ln\tau\frac{R_5^3}{16\kappa}\int_{\rho=\epsilon} d^4x\sqrt{-\gamma}\left[\left(R_{\mu\nu}(\gamma)R^{\mu\nu}(\gamma) - \frac{1}{3}R^2(\gamma)\right) + \frac{2\kappa Z_A}{R_5^2}F_{\mu\nu}(A)F^{\mu\nu}(A)\right]$$

Such a finite boundary term leads to finite holographic improvements to the currents that do not alter the form of the scale and chiral R anomalies and are consistent with the current Ward identities.

Definition of currents

According to gravity/gauge duality, the expectation values of the boundary currents, the energy-momentum tensor $\theta_{\mu\nu}(x)$ and the R-symmetry current $R_\mu(x)$, in the presence of their respective external sources $g_{\mu\nu}^{(0)}(x)$ and $A_\mu^{(0)}(x)$ are found by varying the boundary sources in the on-shell renormalized action as

$$\delta\Gamma[g^{(0)}_{\mu\nu}, A^{(0)}_\mu] = \int d^4x \sqrt{-g^{(0)}} \left[\frac{1}{2}\theta_{\mu\nu}(x)\delta g^{(0)\mu\nu}(x) + R_\mu(x)\delta A^{(0)\mu}(x)\right].$$

The currents thus take the form

$$\begin{aligned} \theta_{\mu\nu} &\equiv \frac{2}{\sqrt{-g^{(0)}}}\frac{\delta\Gamma}{\delta g^{(0)\mu\nu}} \\ R^\mu &\equiv \frac{1}{\sqrt{-g^{(0)}}}\frac{\delta\Gamma}{\delta A^{(0)}_\mu}. \end{aligned}$$

The mixed gauge-gravity Chern-Simons term in the action has higher order derivatives and therefore must be treated separately and with special care. (For details, please see our paper.)

This calculations yields the currents:

$$
\begin{aligned}
\theta_{\mu\nu} &= \theta^{\mathrm{EMCS}}_{\mu\nu} + \theta^{\mathrm{CS}}_{\mu\nu} \\
&= \frac{R_5^3}{\kappa}\left[2g^{(4)}_{\mu\nu} + h_{\mu\nu} - g^{(2)}_{\mu\nu} g^{(0)\rho\sigma} g^{(2)}_{\rho\sigma} - g^{(0)}_{\mu\nu}\left(2g^{(0)\rho\sigma} g^{(4)}_{\rho\sigma} + g^{(2)\rho\sigma} g^{(2)}_{\rho\sigma}\right)\right] \\
&\quad -\frac{R_5^3}{4\kappa}\left[R^{(0)}_{\mu\lambda} R^{(0)\lambda}{}_{\nu} + R^{(0)}_{\lambda\mu\rho\nu} R^{(0)\lambda\rho} - \frac{1}{2}\nabla^{(0)2} R^{(0)}_{\mu\nu} + \frac{1}{6}\nabla^{(0)}_{\mu}\nabla^{(0)}_{\nu} R^{(0)} + \frac{1}{12} g^{(0)}_{\mu\nu}\nabla^{(0)2} R^{(0)2}\right. \\
&\quad \left. + \frac{1}{2} R^{(0)}_{\rho\sigma} R^{(0)\rho\sigma} g^{(0)}_{\mu\nu} - \frac{1}{2} R^{(0)} R^{(0)}_{\mu\nu}\right] \\
&\quad -4D E^{\xi\rho\sigma\lambda}(g^{(0)})\left\{\nabla^{(0)}_{\alpha}\left[A^{(0)}_{\lambda}\left(g^{(0)}_{\xi\mu} R^{(0)\alpha}{}_{\nu\rho\sigma} + g^{(0)}_{\xi\nu} R^{(0)\alpha}{}_{\mu\rho\sigma}\right)\right]\right. \\
&\quad \left. +2\nabla^{(0)}_{\rho}\left[A^{(0)}_{\lambda}\left(g^{(0)}_{\xi\mu} g^{(2)}_{\nu\sigma} + g^{(0)}_{\xi\nu} g^{(2)}_{\mu\sigma}\right)\right] - 2A^{(0)}_{\lambda}\left(g^{(0)}_{\xi\mu}\nabla^{(0)}_{\rho} g^{(2)}_{\nu\sigma} + g^{(0)}_{\xi\nu}\nabla^{(0)}_{\rho} g^{(2)}_{\mu\sigma}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
R^{\mu} &= R^{\mathrm{EMCS}\mu} \\
&= -2R_5 Z_A\left(A^{(2)\mu} + B^{\mu}\right) - 4C E^{\mu\nu\rho\sigma}(g^{(0)}) A^{(0)}_{\nu} F^{(0)}_{\rho\sigma}.
\end{aligned}
$$

Anomaly structure

Exploiting the near boundary solutions to the field equations, the anomalous divergence of the R-symmetry current is found to be

$$\nabla_\mu R^\mu = C E^{(0)\mu\nu\rho\sigma} F^{(0)}_{\mu\nu} F^{(0)}_{\rho\sigma} + D E^{(0)\mu\nu\rho\sigma} R^{(0)\xi\zeta}_{\mu\nu} R^{(0)}_{\xi\zeta\rho\sigma}.$$

The energy-momentum tensor has contributions from gravity and matter flowing into the boundary of the form

$$\begin{aligned} \nabla^\mu \theta^{\mathrm{EMCS}}_{\mu\nu} &= -2R_5^3 T^{(2)}_{\nu 4} = -2R_5^3 \left(T^{(2)\mathrm{Maxwell}}_{\nu 4} + T^{(2)\mathrm{D}}_{\nu 4} \right) \\ \nabla^\mu \theta^{\mathrm{CS}}_{\mu\nu} &= 2R_5^3 T^{(2)\mathrm{D}}_{\nu 4} + A^{(0)}_\nu \mathcal{A}^{\mathrm{D}}_R, \end{aligned}$$

with the mixed Chern-Simons contribution to the R-symmetry anomaly given by

$$\mathcal{A}^{\mathrm{D}}_R = D E^{\mu\nu\rho\sigma}(g^{(0)}) R^{(0)\xi\zeta}{}_{\mu\nu} R^{(0)}_{\rho\sigma\xi\zeta}.$$

From these follows the Ward identity relating the divergence of the energy-momentum tensor with that of the R-symmetry current as

$$\nabla^\mu \theta_{\mu\nu} = F^{(0)}_{\mu\nu} R^\mu + A^{(0)}_\nu \nabla_\mu R^\mu.$$

The trace of the energy-momentum tensor is found from the field equations to be

$$\theta^\mu{}_\mu = \frac{R_5^3}{\kappa} \left[\frac{1}{8} \left(R^{(0)}_{\mu\nu} R^{(0)\mu\nu} - \frac{1}{3} R^{(0)2} \right) + \frac{\kappa Z_A}{4R_5^2} F^{(0)}_{\mu\nu} F^{(0)\mu\nu} \right]$$

These results agree with the general diffeomorphism and R-symmetry transformations of the action.

-- From the definition of the boundary currents it is found that the action is invariant under the diffeomorphism transformations

$$\begin{aligned} \delta g^{(0)\mu\nu} &= \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu \\ \delta A^{(0)\mu} &= -\nabla^\mu \xi_\nu A^{(0)\nu} - \xi_\nu \nabla^\nu A^{(0)\mu} \end{aligned}$$

so that

$$\begin{aligned} \delta\Gamma &= \int d^4x \sqrt{-g^{(0)}} \xi^\nu \left[-\nabla^\mu \theta_{\mu\nu} + F^{(0)}_{\mu\nu} R^\mu + A^{(0)}_\nu \nabla_\mu R^\mu \right] \\ &= 0. \end{aligned}$$

-- The anomalous R-symmetry variation of the action follows directly from the R-current divergence equation. For the R-symmetry transformations

$$\delta g^{(0)\mu\nu} = 0 \quad , \quad \delta A^{(0)\mu} = \partial^\mu \omega,$$

it follows that

$$\begin{aligned} \delta\Gamma &= -\int d^4x \sqrt{-g^{(0)}} \omega \nabla^{(0)\mu} R_\mu \\ &= -\int d^4x \sqrt{-g^{(0)}} \omega \left[C E^{(0)\mu\nu\rho\sigma} F^{(0)}_{\mu\nu} F^{(0)}_{\rho\sigma} + D E^{(0)\mu\nu\rho\sigma} R^{(0)\xi\zeta}_{\mu\nu} R^{(0)}_{\xi\zeta\rho\sigma} \right]. \end{aligned}$$

-- Finally, the energy-momentum trace anomaly equation implies that the Weyl scale transformation

$$\delta g^{(0)\mu\nu} = 2\sigma g^{(0)\mu\nu} \quad , \quad \delta A^{(0)\mu} = 0,$$

Is anomalous for the renormalized action,

$$\begin{aligned} \delta\Gamma &= \int d^4x \sqrt{-g^{(0)}}\sigma(x)\theta^{\mu}{}_{\mu}(x) \\ &= \int d^4x \sqrt{-g^{(0)}}\sigma(x)\frac{R_5^3}{\kappa}\left[\frac{1}{8}\left(R^{(0)}_{\mu\nu}R^{(0)\mu\nu} - \frac{1}{3}R^{(0)2}\right) + \frac{\kappa Z_A}{4R_5^2}F^{(0)}_{\mu\nu}F^{(0)\mu\nu}\right]. \end{aligned}$$

Summary and discussion

The subleading mixed gravitational field-$U(1)_R$ gauge field Chern-Simons term was added to the action as it gives rise to subleading gravitational contributions to the R-anomaly.

The modifications to the near boundary solutions to the field equations were then obtained along with the boundary counter-terms and normalization required by holographic renormalization.

Once the on-shell action was obtained, the Brown-York energy-momentum tensor and R-symmetry current were constructed.

The near boundary solutions were used to secure the Ward identity obeyed by the currents.

The Ward identities for diffeomorphism invariance of the action then followed. Likewise the R-symmetry transformation of the action was obtained and the Weyl scaling of the action was determined.

Since the Ward identities provide the interpretation of the holographic currents as the energy-momentum tensor and R-symmetry current, improvements to the currents can be constructed which leave the Ward identities unchanged. Such improvements are expressed as additional finite boundary terms in the action.

The fermionic gravitino sector of the conformal supergravity action can be included This work is in progress.

The boundary $\mathcal{N}$=1 SCFT includes the supersymmetry current in the supercurrent multiplet and the superconformal anomaly, given by the gamma trace of the supersymmetry current, as part of the anomaly multiplet.

The central charges a and c also describe the superconformal anomaly.

The holographic supersymmetry currents can be constructed and their divergence and trace determined.

The Ward identities will then include these fermionic currents as well, while anomaly matching will provide additional consistency checks for the AdS/gauge duality.