ADJUSTMENT WITH LEAST SQUARES METHOD TWO SOFTWARE PACKAGES – TWO RESULTS

Petra Radomi, Markus Schlösser, DESY, Hamburg

Abstract

Several software packages use the "least squares method" for the adjustment of observations to estimate coordinates. The mathematics behind this method minimizes the weighted sum of squared errors, $v^T Pv \to \min$. Consequently, it is possible to evaluate this sum and compare the results of different programs and their algorithms. The software packages PANDA and Spatial Analyzer are tested against various artificial networks using this minimum criterion.

INTRODUCTION

Several methods exist to compare results of different least squares software packages. Most of them compare the coordinates and their standard deviations. While the coordinates are normally the result the user is interested in, the algorithm does not imply any restrictions based on coordinate values. The algorithm is based on the requirement that the weighted sum of squared errors is minimal, $v^T Pv \to \min$. Thus comparisons of this sum are the only way to check the result of the algorithm in a well-defined mathematical context. Unfortunately it cannot be proven that the result of a software package (seen only as a "black box") is the minimal possible solution. On the other hand, comparing the $v^T Pv$ of different packages can give interesting insights into variations of the algorithms which were utilized.

At DESY, two software packages are available for the adjustment of networks, PANDA and Spatial Analyzer (SA). These two packages are tested with artificial data sets, the results are compared.

Based on a simulated network the error-free coordinates are known.

$$\tilde{X}^T = [x_1 y_1 z_1 \ldots x_n y_n z_n]$$

Errror-free observations can then be calculated from these coordinates.

$$\tilde{L}^T = [r_1 ... r_n z_1 ... z_n d_1 ... d_n] = \varphi\left(\tilde{X}^T\right)$$

The standard deviation of the azimuth and zenith angles is set to $\sigma_{r/z} = 0.3 mgon$ and the standard deviation of distance is set to $\sigma_d = 0.05mm + 0\,ppm$, which should approximately represent the accuracy of a laser tracker.

The observation vector L consists of the error-free observations $\tilde{L}$ and a random error ε, which is scaled with the standard deviation of each observation.

$$L = \tilde{L} + \varepsilon$$

$$\varepsilon_{r/z/d} = t \cdot \sigma_{r/z/d}$$

$$\varepsilon^T = [\varepsilon_{r1} ... \varepsilon_{rn}, \varepsilon_{z1} ... \varepsilon_{zn}, \varepsilon_{d1} ... \varepsilon_{dn}]$$

The Gaussian variables t were produced with the Box-Muller-Method, since most programming languages cannot produce Gaussian variables, but only uniform $([0,1])$ pseudo-random numbers. If two uniform random numbers $([0,1])$, u_1 and u_2 are available, a Gaussian variable t can be computed, using

$$t = \cos(2\pi u_1)\sqrt{(-2 \ln u_2)}$$

Now the artificial observations L can be adjusted using the least squares method $v^T Pv \to \min$.

The adjusted observations $\hat{L}$ are the sum of the observations L and the estimated correction vector v, $\hat{L} = L + v$.

DESIGN OF A SIMULATED 3-D-NETWORK

A 3-D network with 10 network points and three instruments was simulated, covering an area of 30*30 m² with elevation differences of up to 4m. A plot of the network is shown in figure 1. Since SA is unable to treat an instrument stand as a measured point, the instruments have the same theoretical coordinates as the identical target points, although with a different name. For all network simulations the observation data is related to gravity, meaning that the vertical axes of all instruments are truly vertical. This approach has been chosen for simplicity and is normally not the case when adjusting laser tracker networks.

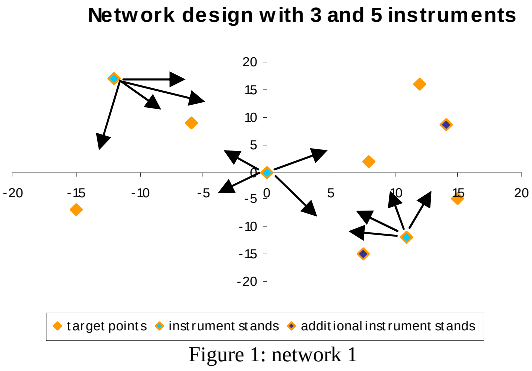

Figure 1: network 1

The 'true' observations $\tilde{L}$ for the instruments were calculated and distorted by the described method. The distorted observations L are imported into PANDA and Spatial Analyzer. The error-free coordinates of network points were also imported and used as coarse coordinates to locate the instruments in SA and as initial values for the vector of unknowns X_0 in PANDA.

The standard deviations have been set to 0.3mgon for horizontal and vertical angles and 0.05mm+0ppm for distances.

PANDA runs a so-called 'free adjustment', where all used points together define the datum of the network, but where none of the points is fixed. Mathematically speaking this is a minimization of the entire trace of the normal matrix N, whereas the geometric meaning is that the corrections $\hat{x}$ of the coarse parameter vector X_0 are minimal, $\|\hat{x}\| = \min$. The output file of PANDA includes the adjusted coordinates and observations, as well as the corrections of the observations and the sum of squares of the corrections, referred to as "PANDA" in table 1.

Within SA the so-called 'Unified Spatial Metrology Network'-function is used to estimate the network coordinates. As previously stated, all observations are related to gravity in this example, therefore rotations around x and y are disabled for all instruments. The adjusted coordinates are saved in a new point group, the instruments are automatically moved to their adjusted positions. The adjusted coordinates of the instruments are not directly accessible in the output of the USMN, but can be accessed manually through the instrument properties dialog. SA does not report the corrections of the observations directly, so the adjusted coordinates were exported and the correction values of the observations were estimated externally by using an Excel sheet. This result is referred to as "SA-USMN" in table 1.

The corrections of the observations v were also derived from the estimated coordinates for the PANDA solution, referred to as "PANDA external" below. While this is not necessary, because PANDA already reports these values in its output files, it gives an idea about the numerical accuracy.

Additionally a 'Unified Spatial Metrology Network with point group' was estimated using SA. With this solution the original coordinates were allowed to move, making it in principle a 'free floating network', similar to the 'free adjustment' of PANDA. The results are referred as "SA-USMN with PG" in table 1.

There are some small variances between USMN with and without point group in the individual sum of squares, but the value [vv] remains approximately the same for both of the SA solutions.

The differences between 'PANDA' and 'PANDA external' give an idea about the numerical accuracy of a solution where the corrections are calculated backwards from the estimated coordinates. It shows that differences below 1% are not significant.

The sums of squares for network 1 are shown in table 1, where $[vv]_d$, $[vv]_a$ and $[vv]_z$ are the squared sums of distance, azimuth and zenith angle corrections and where [vv] is the total squared sum of all corrections.

While $[vv]_d$ and $[vv]_z$ are more or less the same for all solutions, there is a larger difference in $[vv]_a$ (and thus in [vv]) between PANDA and SA.

Table 1: [vv] as given from PANDA and SA for network 1 using 3 instruments

	$[vv]_d$	$[vv]_a$	$[vv]_z$	[vv]
PANDA	6.25	18,11	27,57	51,92
PANDA external	6.30	17,94	27,62	51,85
SA-USMN	6.21	29,15	27,35	62,71
SA-USMN with PG	7.11	26,74	28,88	62,73

To reinforce this result, the network was modified by introducing two additional stations, marked as "additional instrument stands" in figure 1. The number of observations rises by 54 to a total number of 135. Again the observations were distorted by random numbers and adjusted afterwards. The results are shown in table 2.

Increasing the number of observations has of course an effect on the sum of squares [vv]. Due to the geometry of the network the sum of squares rises substantially for the distance $[vv]_d$ and the vertical angle $[vv]_z$, but not for the azimuth $[vv]_a$.

Table 2: [vv] as given from PANDA and SA for network 1 using 5 instruments

	$[vv]_d$	$[vv]_a$	$[vv]_z$	[vv]
PANDA	27.03	20.36	33.87	81.26
SA-USMN	28.01	28.95	33.61	90.57

Again, the values obtained for $[vv]_d$ and $[vv]_z$ are similar, but $[vv]_a$ is more than 42% larger for SA than for PANDA, resulting in a larger total sum [vv] for SA.

DESIGN OF A SIMULATED 3-D LINEAR ACCELERATOR-NETWORK

Additionally, a second network was simulated as a linear accelerator network, which is similar to the ones used at DESY.

For this network 4 reference points are installed every 10m through the whole length of the tunnel. These 4 points are called a “ring”, they are installed in each corner of a tunnel profile (upper/lower, left/right). Four points per ring were chosen to keep the model simple for this simulation.

The network has a length of 80m and contains 36 reference points grouped in 9 rings. The instrument is placed in the middle between each two subsequent rings. Every network point of two rings back and forth is then measured, resulting in 48 observations (16 azimuths, vertical angles and distances) per instrument stand (figure 2). The number of observations varies for the leftmost instrument stand, because there is only one ring to its left, giving a total of 36 observations. On the right end of the network the last ring is measured without redundancy.

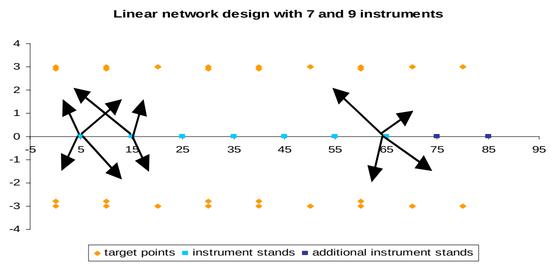

Figure 2 : network 2

For the seven instruments and 36 measured points 384 simulated observations (distance, azimuth, zenith angle) were computed, distorted and imported in PANDA and Spatial Analyzer.

PANDA adjusted the 384 observations and showed no corrections for the 12 observations to the four points of the rightmost ring, as they are only observed once. Since the observations to these points are not redundant, the Gauss-Markov-Model does not estimate corrections or à posteriori accuracy of observations.

SA adjusted only 372 observations, excluding all observations without redundancy. Surprisingly the observations without redundancy also received a correction during the adjustment process.

Again the individual sum of squares $[vv]_d$ and $[vv]_z$ of PANDA and SA are nearly identical, but $[vv]_a$ is significantly larger for SA than for PANDA as shown in table 3.

Table 3: [vv] as given from PANDA and SA for network 2 using 7 instruments

	$[vv]_d$	$[vv]_a$	$[vv]_z$	[vv]
PANDA	98.69	76.25	85.48	260.41
SA-USMN	97.07	122.50	85.45	305.02

To get more information about the different results of PANDA and SA the correction of the individual coordinates is visualized for the xy-plane. Figure 3 shows the vector difference between adjusted and coarse coordinates

$$\hat{x} = \hat{X} - X_0$$

The vectors are magnified by a factor of 100000, the results from PANDA are shown in black and the results from SA are shown in red.

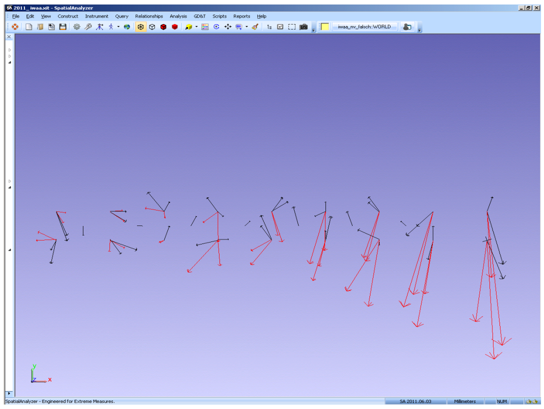

Figure 3: difference between adjusted and coarse coordinates for network 2 using 7 instruments, black=PANDA, red=SA

The distribution of coordinate corrections seems to be random for the PANDA solution. Since PANDA does a free adjustment of the network with $\|\hat{x}\| = \min$, this is the expected behavior.

For SA, however, there is a large lateral movement of the coordinates at the right end of the network. While there is no explanation for this behavior of SA, it is consistent with the result shown in table 3, $[vv]_a$ being much larger for SA than for PANDA.

To answer the question of whether these large coordinate corrections at the end of the network can be explained with the unused observations in SA, network 2 was extended by two additional stations to the right end of the network, so that all measurements are now redundant. The additional stations are marked as “additional instrument stands” in figure 2.

The network now has 468 observations with the same number of 36 target points. The result of this adjustment is shown in table 4.

Table 4: [vv] as given from PANDA and SA for network 2 using 9 instruments

	$[vv]_d$	$[vv]_a$	$[vv]_z$	[vv]
PANDA	117.83	94.58	104.69	317.10
SA-USMN	134.64	122.82	122.49	379.95

Interestingly, the quantity $[vv]_a$ of the SA solution doesn't change substantially between the two different versions of the network as shown in table 3 and 4, although there are now major differences between PANDA and SA in $[vv]_d$ and $[vv]_z$ which did not exist before. The distribution of coordinate corrections shows the same lateral movement at the right end of the network (figure 4). Since the movements have the same systematic behavior shown in figure 3 and 4 they cannot be caused by bad redundancy at the end of the network.

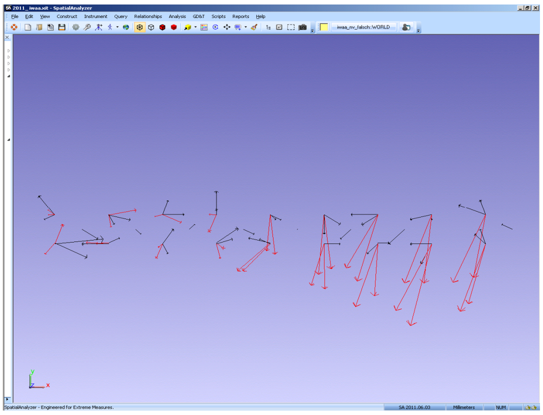

Figure 4: difference between adjusted and coarse coordinates for network 2 using 9 instruments, black=PANDA, red=SA

To find the reason behind this inexplicable lateral movement of coordinates, the instruments were copied to a different collection in reverse order within SA. The adjustment with the USMN function and instruments introduced in reverse order gives the same result as shown in figure 4.

Importing the measurements into SA in reverse order and then adjusting with USMN gives, however, a mirrored result as shown in figure 5.

This shows that the result of SA's adjustment is dependent on the timestamp of the measurements. It is believed that this is caused by a sequential adjustment of stations within SA's USMN.

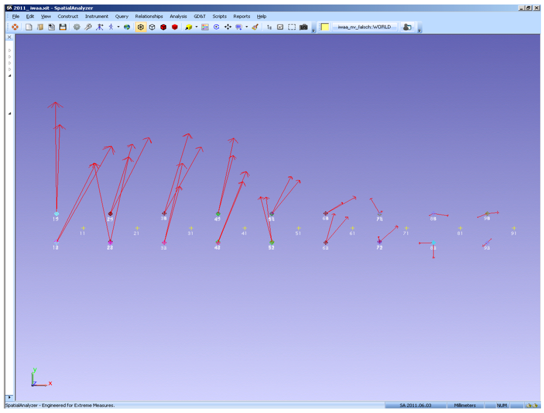

Figure 5: difference between adjusted and coarse coordinates for network 2 using 9 instruments (SA only), instruments imported in reverse order

CONCLUSION

A method for the comparison of different network adjustment packages has been shown, based only on the requirement that the weighted sum of squared errors is minimal, $v^T P v \rightarrow \min$. So far only PANDA and SA have been evaluated and only with a limited set of simulated networks.

While both packages claim to use the least squares method developed by Gauss, there are significant differences in the results. With the comparison of the sum of squares the correctness of a solution cannot be proven, because the minimum of a specific problem is not known analytically.

So at the moment the solution with the minimal sum of squares has to be considered the right one. In all estimated networks the weighted sum of squares of the residuals was larger for SA-USMN than for PANDA. The adjustment of SA is not optimal in terms of a geodetic Gauss-Markov model.

Because accelerators are most sensitive to lateral and height errors, the arbitrary lateral movement of points from start to end in the SA solution is especially important for the accelerator community. As has been demonstrated, the measurement timestamps form a critical element during the adjustment of Spatial Analyzer.

REFERENCES

[1] Niemeier, Wolfgang: *Ausgleichsrechnung. Statistische Auswertemethoden.* 2. Auflage. Berlin: de Gruyter 2008.

[2] w/o name.: *PANDA für WINDOWS*. Version 3.20. Laatzen 2006.

[3] w/o name: *SpatialAnalyzer User Manual*. Williamsburg, VA, 2009.