The massive two-loop master integrals for Bhabha scattering

Janusz Gluza *

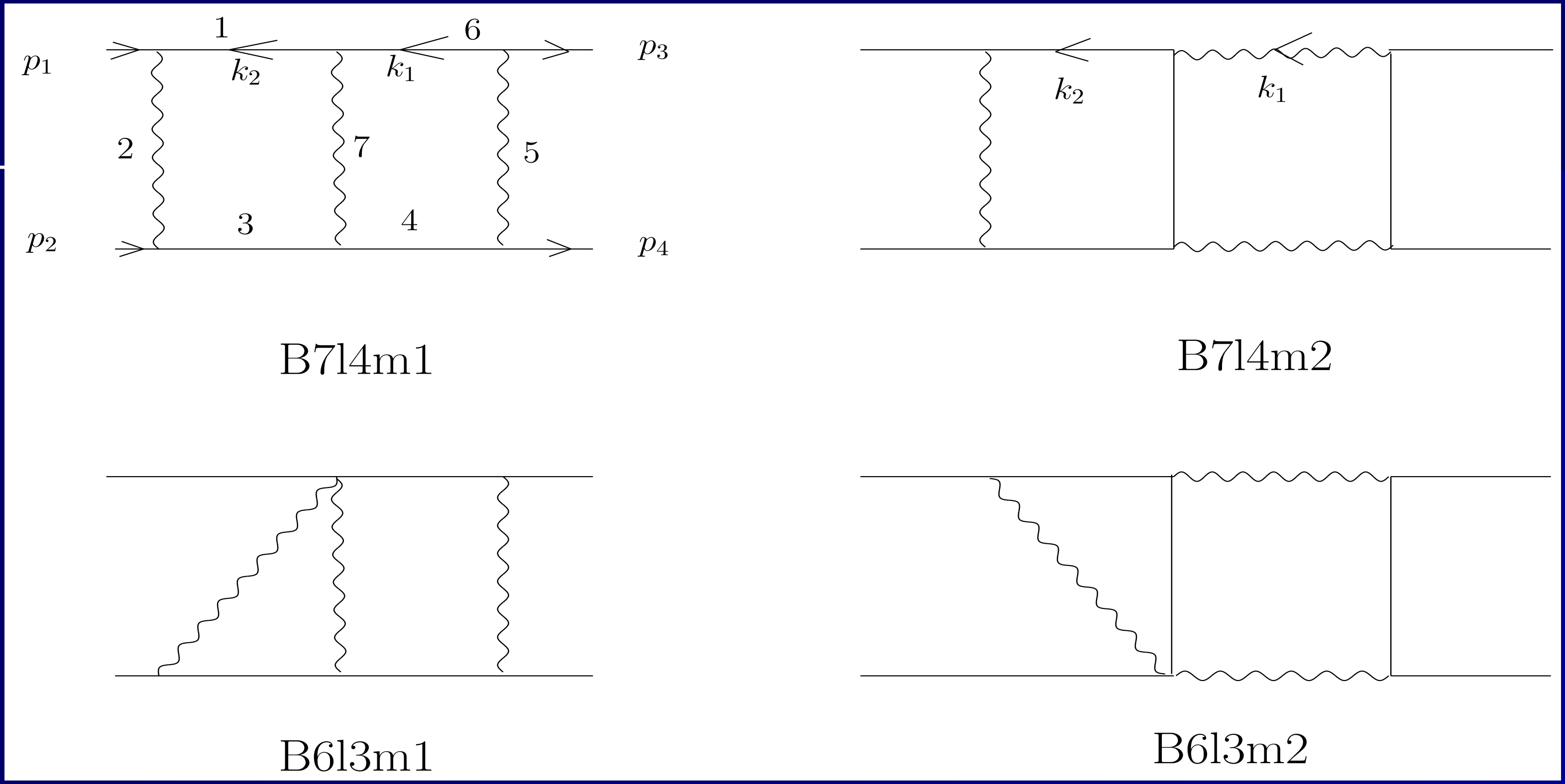
U. Silesia, Katowice

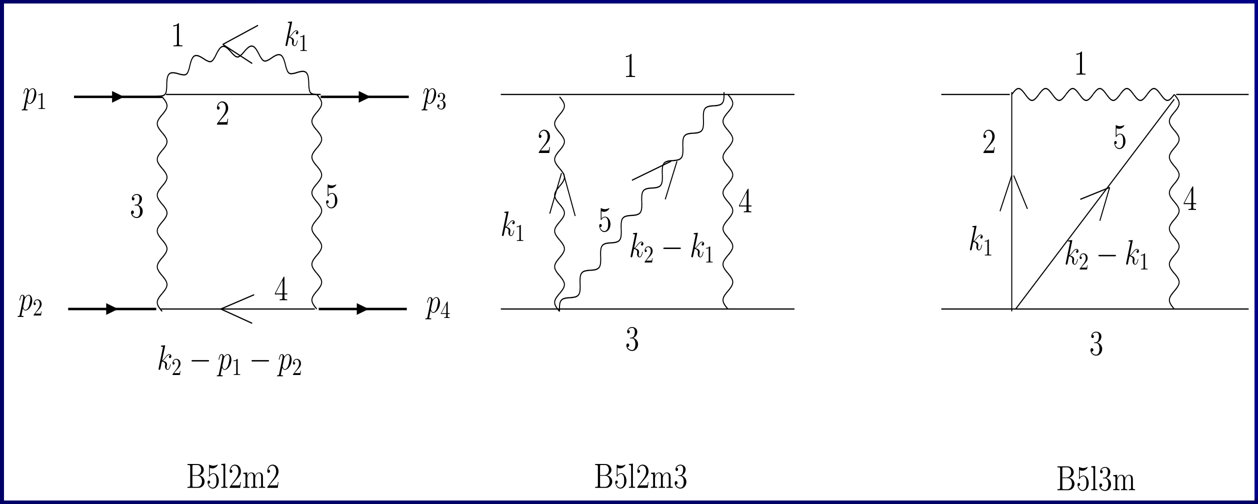
* *Together with: Stefano Actis, Michał Czakon, Tord Riemann*

The plan

- 4-point master integrals: present situation
- Mellin-Barnes representation for MIs, analytic continuation in ϵ
- Expansion of MB MIs in small fermion mass(es): The results
- Near future: Non-planar MIs, MIs with $N_f > 1$,
- Conclusions and outline

MI	B1	B2	B3	B4	B5	B6	solved:
B7l4m1	+	–	–	–	–	–	Smirnov:2001,our:2006
B7l4m1N	+	–	–	–	–	–	Smirnov,Heinrich:2004,our:2006
B7l4m2	–	+	–	–	–	–	Smirnov,Heinrich:2004[†],our:2006
B7l4m2[d1--d3]	–	+	–	–	–	–	our:2006
B7l4m3	–	–	+	–	–	–	NP:Smirnov,Heinrich:2004[†]
B7l4m3[d1--d2]	–	–	+	–	–	–	NP
B6l3m1	+	–	+	–	–	–	our:2006
B6l3m1d	+	–	+	–	–	–	our:2006
B6l3m2	–	+	–	+	–	–	our:2006
B6l3m2d	–	+	–	+	–	–	our:2006
B6l3m3	–	–	+	–	–	–	NP
B6l3m3[d1--d5]	–	–	+	–	–	–	NP
B5l2m1	+	–	+	–	–	–	our:2004
B5l2m2	–	+	–	+	–	+	our:2006
B5l2m2[d1--d2]	–	+	–	+	–	+	our:2006
B5l2m3	+	–	+	–	–	–	our:2006
B5l2m3[d1--d3]	+	–	+	–	–	–	our:2006
B5l3m	–	+	+	+	–	–	our:2006
B5l3m[d1--d3]	–	+	+	+	–	–	our:2006
B5l4m	–	+	+	+	+	–	Bonciani, Mastrolia, Remiddi:2002
B5l4md	–	+	+	+	+	–	our:2004





Czakon, Gluza, Riemann, hep-ph/0604101

DEqs and Mellin-Barnes representation

- *M. Czakon, J.G.,K. Kajda, T. Riemann,*
  "Differential equations and massive two-loop Bhabha scattering: The B5l2m3 case" [Musashi], hep-ph/0602102 (Radcor, Japan 2005)
- *M. Czakon, J.G., T. Riemann, Acta Phys.Polon.B36 (2005),3319 (Ustron 2005)*

Boos, Davydychev 1991, Smirnov 1999, Tausk 1999, Smirnov book 2004

$$\frac{1}{(A+B)^{\nu}} = \frac{B^{-\nu}}{2\pi i \Gamma(\nu)} \int\limits_{-i\infty}^{i\infty} d\sigma\, A^{\sigma}\, B^{-\sigma}\, \Gamma(-\sigma)\Gamma(\nu+\sigma)$$

B7l4m2, V.A. Smirnov, 6dim MB-integral

$$B_{\mathrm{pl},2} \quad = \quad \frac{\mathrm{const}}{(2\pi i)^6} \int_{-i\infty}^{+i\infty} \left[\frac{m^2}{-s}\right]^{z_5+z_6} \left[\frac{t}{s}\right]^{z_1} \prod_{j=1}^{6} [dz_j \Gamma(-z_j)] \frac{\prod_{k=7}^{18} \Gamma_k(\{z_i\})}{\prod_{l=19}^{24} \Gamma_l(\{z_i\})}$$

with $a = a_1 + \ldots + a_7$ *and* $d = 4 - 2\epsilon$

$$\mathrm{const} = \frac{(\mathrm{i}\pi^{\mathrm{d}/2})^2(-1)^{\mathrm{a}}(-\mathrm{s})^{\mathrm{d}-\mathrm{a}}}{\Gamma(\mathrm{a}_2)\Gamma(\mathrm{a}_4)\Gamma(\mathrm{a}_5)\Gamma(\mathrm{a}_6)\Gamma(\mathrm{a}_7)\Gamma(\mathrm{d}-\mathrm{a}_{4567})}$$

The integrand includes e.g.:

$$\Gamma_2 \quad = \quad \Gamma(-z_2),\ \Gamma_4 = \Gamma(-z_4),\ \ldots,$$

$$\Gamma_7 \quad = \quad \Gamma(a_4 + z_2 + z_4),$$

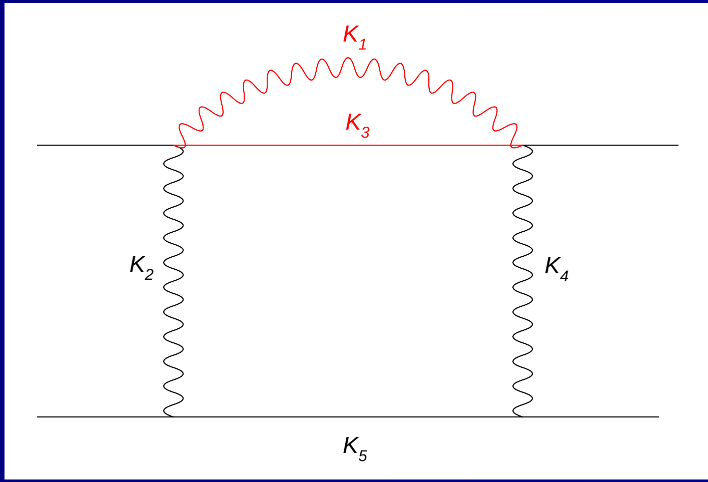
$$\Gamma_8 \quad = \quad \Gamma(d - a_{445667} - z_2 - z_3 - 2z_4), \ldots$$

$$a_1 \to 0,\ a_2 \to 2\ a_3 \to 0 \quad \longrightarrow \quad B_{\mathrm{pl},2} \to B5l2m2d2$$

2-loop Mellin-Barnes representation: construction

B5l2m2:

$$
\begin{aligned}
K_1 &= k_1^2 \\
K_2 &= (k_2 - p_1)^2 \\
K_3 &= (k_1 + k_2)^2 - m^2 \\
K_4 &= (k_2 - p_3)^2 \\
K_5 &= (k_2 - p_3 - p_4)^2 - m^2 \\
K_{B5l2m2} &= \int d^D k_2 \frac{1}{K_1^{\nu_1}} \frac{1}{K_4^{\nu_4}} \frac{1}{K_5^{\nu_5}} K_{up} \\
K_{up} &= \int d^D k_1 \frac{1}{K_1^{\nu_1}} \frac{1}{K_3^{\nu_3}}
\end{aligned}
$$



B5l2m2

First step:

$$K_{up} = (-1)^{\nu_{13}} \frac{\Gamma(\nu_{13} - D/2)}{\Gamma(\nu_1)\Gamma(\nu_3)} \int_0^1 \prod_{j=1,3} dx_j x_j^{\nu_j - 1} \delta(1 - x_1 - x_3) \frac{U_{up}^{\nu_{13} - D}}{F_{up}^{\nu_{13} - D/2}}$$

F polynomial (now part of MB.m, M. Czakon, **hep-ph/0511200**)

$$F_{up} = (-k_2^2 + m^2) x_1 x_3 + m^2 x_3^2.$$

Second step:

1. use M-B relation for $F_{up}^{-(\nu_{13} - D/2)}$
2. integrate over x

B5l2m2

Final representation is:

$$\texttt{B5l2m2} = \frac{m^{4\epsilon}(-1)^{a_{12345}}e^{2\epsilon\gamma_E}}{\prod_{j=1}^{5}\Gamma[a_i]\Gamma[4-2\epsilon-a_{13}](2\pi i)^3}\int_{-i\infty}^{+i\infty}d\alpha\int_{-i\infty}^{+i\infty}d\beta\int_{-i\infty}^{+i\infty}d\gamma$$
$$(-s)^{2-\epsilon-a_{245}-\gamma-\alpha+\beta}(-t)^{\alpha}$$
$$\Gamma[-2+\epsilon+a_{13}+\beta]\Gamma[-\gamma]\Gamma[2-\epsilon-a_{245}-\gamma-\alpha]\Gamma[-\alpha]$$
$$\Gamma[a_2+\alpha]\Gamma[a_4+\alpha]\Gamma[4-2\epsilon-a_{113}-\beta]$$
$$\Gamma[-2+\epsilon+a_{245}+\gamma+\alpha-\beta]\Gamma[a_1+\beta]$$
$$\frac{\Gamma[4-2\epsilon-a_{2245}-2\alpha+\beta]\Gamma[2-\epsilon-a_{24}-\gamma-\alpha+\beta]}{\Gamma[4-2\epsilon-a_{245}+\beta]\Gamma[4-2\epsilon-a_{22445}-2\gamma-2\alpha+\beta]}$$

Typical situation

- `B5l2m2`:
  - *Directly from 7-line B7l4m1: After expansion in ϵ we are left with 11 integrals (one 4-dim.)*
  - *From our MB-representation: 4 integrals (one 3-dim)*
- `B5l2m2d2`:
  - *From B7m4m1 with help of MB.m: 102 integrals, including 4-dim (!)*
  - *From our MB-representation: only one, 3-dim integral,*

B5l2m2d2 case: fixed indices $a_1 = a_3 = a_4 = a_5 = 1$, $a_2 = 2$

General Tasks

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General Tasks

- *Find a region where integral is regular in the n-fold MB-integral* (`FindInstance`)

$$\Re(\alpha) = 361/384, \Re(\beta) = -117/128, \Re(\gamma) = -19/32, \Re(\epsilon) = 1/32$$

- *Go to the physical region where* $\epsilon \to 0$ *by distorting the integration path step by step (adding each crossed residuum – per residue this means one integral less (automatized in MB.m, M.Czakon,* **hep-ph/0511200***)*

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- *If we want to get exact analytical result: Take integrals by sums over residua, i.e. introduce infinite sums*
- *Sum these infinite multiple series into some known functions of a given class, e.g. Nielsen polylogs, Harmonic polylogs or whatever is appropriate (read: managable)*

However,...., and we need approximations

e.g. (one initial 3-dim integral for B5l2m2d2)

$$
\begin{aligned}
B5l2m2d2 &= m^{-4-2\epsilon}\left(-m^2/s\right)^{\epsilon+\alpha}(-x)^{\beta} \\
&\times \Gamma[1-\epsilon-\alpha-\beta]\Gamma[-\beta]\Gamma[1+\beta] \\
&\times \Gamma[1+\epsilon+\beta]\Gamma[\epsilon+\alpha+\beta]\Gamma[-2\epsilon-\gamma]\Gamma[1-\alpha-\gamma].......
\end{aligned}
$$

$$\Re(\alpha) = 361/384, \Re(\beta) = -117/128, \Re(\gamma) = -19/32$$

$m^2/s << 1$, *so contour closed on right, this is an iterative procedure:*

$$
\begin{aligned}
Residue_{|\alpha=1-\gamma} &= \left(-m^2/s\right)^{1+\epsilon-\gamma}(-x)^{\beta} \\
&\times \Gamma[1+\gamma]\Gamma[1+\epsilon+\gamma]\Gamma[-ep-\beta+\gamma].....
\end{aligned}
$$

and again residues in $\gamma' s$

Final structure for approximated MIs

In all planar cases, after expansions, we arrived at 1-dimensional integrals of the kind (B5l2m2d2 example)

$$\left(-m^2/s\right)^{n=2}(-x)^{\beta}\Gamma[...\beta...]$$

or

$$\left(-m^2/s\right)^{...\beta...}(-x)^{...\beta...}\Gamma[...\beta...], \quad \text{or } ...\Gamma[...2\beta...]...,$$

which can be solved analytically using XSummer (Uwer, Moch), or PSLQ (Ferguson, Bailey) algorithms

Finding iteratively residues must be automatized (sometimes hundreds of terms must be considered).

Finally, we get

$$I \sim s^n F\left(\ln\left(-\frac{m^2}{s}\right), \frac{t}{s}\right) + O\left(m^2\right), \quad n = \frac{1}{2}\, \textit{dim}\, I + 2\epsilon,$$

This is requirement we impose for the structure of MIs

Approximations: chosen MIs for B5l2m2 system

$$\mathtt{B5l2m2} = - \frac{1}{\epsilon}\frac{1}{6t}\Big[-3\,\mathrm{L}^2 + 6\,\mathrm{L}\,\ln(x) - 3\,\ln^2(x) - 24\,\zeta_2\Big] - \frac{1}{6t}\Big\{const\Big\}$$

$$\begin{aligned}\mathtt{B5l2m2d2} = &+ \frac{1}{6st}\Big\{-12\,\mathrm{L}^3 + 9\,\mathrm{L}^2\,\ln(x) - 36\,\mathrm{L}\,\zeta_2 \\ &- 12\,\zeta_3 - 12\,\zeta_2\,\ln(x) - \ln^3(x) + 18\,\zeta_2\,\ln(1+x) \\ &+ 3\,\ln^2(x)\,\ln(1+x) + 6\,\ln(x)\,\mathrm{Li}_2(-x) - 6\,\mathrm{Li}_3(-x)\Big\}\end{aligned}$$

$$\begin{aligned}\mathtt{B5l2m2}(\mathtt{k}_2\cdot\mathtt{p}_3) &= +\frac{1}{\epsilon}\frac{s}{12t}\Big[3\,\mathrm{L}^2 + \mathrm{L}\,(12 + 6\,x - 6\,\ln(x)) \\ &+ 24\,\zeta_2 - 12\,\ln(x) - 6\,x\,\ln(x) + 3\,\ln^2(x)\Big] \\ &+ \frac{s}{12t}\Big\{const\Big\}\end{aligned}$$

It is not always so, B5l2m2d1

$$I \sim s^n F\left(\ln\left(-\frac{m^2}{s}\right), \frac{t}{s}\right) + O\left(m^2\right), \quad n = \frac{1}{2}\,\textit{dim}\; I + 2\epsilon,$$

Choice of MIs: dot on a massive line (can) transform log:

$$\ln^n(-m^2/s) \to n\ln^{n-1}(-m^2/s)/m^2$$

e.g. $m_s \equiv -m^2/s,\; L \equiv \left(-m^2/s\right),\; x \equiv t/s$:

$$\begin{aligned} \mathtt{B5l2m2d1}(\mathtt{m_s}) = & - \frac{1}{\epsilon^2}\frac{1}{m^2 t} \\ & + \frac{1}{\epsilon}\frac{1}{m^2 t}\left[-2 - L + \ln(x)\right] \\ & + \frac{1}{m^2 t}\left\{-4 + 2\,\zeta_2 - 2\,L - 2\,L^2 + 2\,\ln(x) + L\,\ln(x)\right\}. \end{aligned}$$

Solutions: The planar box masters for $m^2/s << 1$

B7l4m1: planar I

$$
\begin{aligned}
&+ \quad \frac{1}{\epsilon^2}\,\frac{2\mathrm{L}^2}{s^2 t} \\
&- \quad \frac{1}{\epsilon}\,\frac{1}{3s^2 t}\left[-10\,\mathrm{L}^3 + 6\,\mathrm{L}\,\zeta_2 + 6\,\zeta_3 + 12\,\mathrm{L}^2\,\ln(x)\right] \\
&- \quad \frac{1}{6s^2 t}\Big\{-12\,\mathrm{L}^4 + 28\,\mathrm{L}^3\,\ln(x) - 4\,\mathrm{L}^2\left(-30\,\zeta_2 + 3\,\ln^2(x)\right) \\
&- \quad 4\,\mathrm{L}\left(9\,\zeta_3 + 30\,\zeta_2\,\ln(x) + \ln^3(x) - 18\,\zeta_2\,\ln(1+x)\right. \\
&- \quad \left. 3\,\ln^2(x)\,\ln(1+x) - 6\,\ln(x)\,\mathrm{Li}_2(-x) + 6\,\mathrm{Li}_3(-x)\right) \\
&- \quad 15\zeta_4 - 24\zeta_3 \ln(x)\Big\}
\end{aligned}
$$

B5l2m3

$$
\begin{aligned}
\texttt{B5l2m3} = \; & + \frac{1}{12u}\Big\{ -6\,\mathrm{L}^2\,(6\,\zeta_2 + \ln^2(x)) \\
& - 6\,\mathrm{L}\,(-4\,\zeta_3 + 4\,\zeta_2\,\ln(x) - 12\,\zeta_2\,\ln(1+x) \\
& - 2\,\ln^2(x)\,\ln(1+x) - 4\,\ln(x)\,\mathrm{Li}_2(-x) + 4\,\mathrm{Li}_3(-x)) + 312\,\zeta_4 \\
& + 72\,\zeta_3\,\ln(x) + 36\,\zeta_2\,\ln^2(x) + \ln^4(x) - 24\,\zeta_3\,\ln(1+x) \\
& + 24\,\zeta_2\,\ln(x)\,\ln(1+x) - 36\,\zeta_2\,\ln^2(1+x) \\
& - 6\,\ln^2(x)\,\ln^2(1+x) - 24\,\ln(x)\,\mathrm{S}_{1,2}(-x) + 12\,(8\,\zeta_2 + \ln^2(x) \\
& - 2\,\ln(x)\,\ln(1+x))\,\mathrm{Li}_2(-x) - 48\,\ln(x)\,\mathrm{Li}_3(-x) \\
& + 24\,\ln(1+x)\,\mathrm{Li}_3(-x) + 72\,\mathrm{Li}_4(-x) + 24\,\mathrm{S}_{2,2}(-x)\Big\}
\end{aligned}
$$

B5l2m3

$$
\begin{aligned}
\texttt{B5l2m3d2} = \quad & - \quad \frac{1}{\epsilon^2}\,\frac{1}{st}\mathrm{L} \\
& + \quad \frac{1}{\epsilon}\,\frac{1}{st}\left[-3\,\mathrm{L}^2 + 2\,\mathrm{L}\,\ln(x) - \zeta_2\right] \\
& + \quad \frac{1}{3st}\Big\{-10\,\mathrm{L}^3 + 12\,\mathrm{L}^2\,\ln(x) + 51\,\mathrm{L}\,\zeta_2 - 21\,\zeta_3 \\
& - \quad 42\,\zeta_2\,\ln(x) - 2\,\ln^3(x) + 36\,\zeta_2\,\ln(1+x) \\
& + \quad 6\,\ln^2(x)\,\ln(1+x) + 12\,\ln(x)\,\mathrm{Li}_2(-x) - 12\,\mathrm{Li}_3(-x)\Big\}
\end{aligned}
$$

B5l2m3

Higher orders in ϵ may be determined:

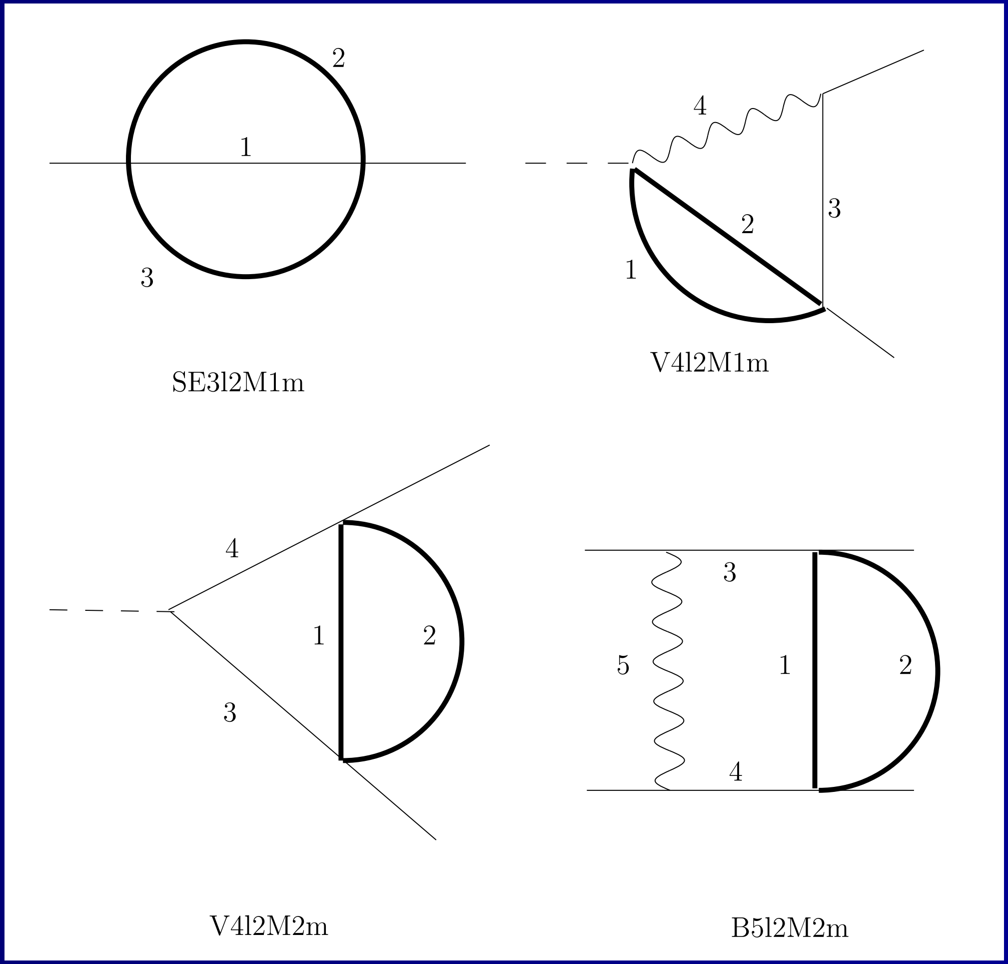
$$
\begin{aligned}
& \mathtt{B5l2m3d2} = \cdots \\
+ \quad & \epsilon \, \frac{1}{3st} \Bigg\{ \Bigg\{ -5 \, \mathrm{L}^4 + 111 \, \mathrm{L}^2 \, \zeta_2 + 10 \, \mathrm{L}^3 \, \ln(x) + \mathrm{L} \, (104 \, \zeta_3 \\
- \quad & 126 \, \zeta_2 \, \ln(x) - 6 \, \ln^3(x) + 108 \, \zeta_2 \, \ln(1+x) + 18 \, \ln^2(x) \, \ln(1+x) \\
+ \quad & 36 \, \ln(x) \, \mathrm{Li}_2(-x) - 36 \, \mathrm{Li}_3(-x)) - 372 \, \zeta_4 - 78 \, \zeta_3 \, \ln(x) + 30 \, \zeta_2 \, \ln^2(x) \\
+ \quad & \ldots \\
- \quad & 6 \, (4 \, \zeta_2 + 3 \, \ln^2(x) - 2 \, \ln(x) \, \ln(1+x)) \, \mathrm{Li}_2(-x) + 24 \, \ln(x) \, \mathrm{Li}_3(-x) \\
- \quad & 12 \, \ln(1+x) \, \mathrm{Li}_3(-x) - 12 \, \mathrm{Li}_4(-x) - 12 \, \mathrm{S}_{2,2}(-x) \Bigg\} \Bigg\}
\end{aligned}
$$

B5l2m3

$$
\begin{aligned}
\texttt{B5l2m3N1} \quad = \quad & \frac{1}{4}\left(\frac{s}{u}\right)^2 \Big\{ \mathrm{L}^2 \left(6\, x\, \zeta_2 + 2\, x\, \ln(x) + 2\, x^2\, \ln(x) + x\, \ln^2(x)\right) \\
+ \quad & \mathrm{L}\, (16\, x\, \zeta_2 - 8\, x^2\, \zeta_2 - 4\, x\, \zeta_3 - 2\, \ln(x) + 2\, x^2\, \ln(x) \\
+ \quad & 4\, x\, \zeta_2\, \ln(x) + 2\, x\, \ln^2(x) - 2\, x^2\, \ln^2(x) - 12\, x\, \zeta_2\, \ln(1+x) \\
- \quad & 2\, x\, \ln^2(x)\, \ln(1+x) - 4\, x\, \ln(x)\, \mathrm{Li}_2(-x) + 4\, x\, \mathrm{Li}_3(-x)) \Big\} \\
+ \quad & \frac{1}{120}\left(\frac{s}{u}\right)^2 \Big\{ +120\, \zeta_2 + \ldots. \Big\}
\end{aligned}
$$

The $N_f > 1$ contributions (talk by S. Actis)

The 2-box-diagrams represent a three-scale problem: $s/m^2, t/m^2, M^2/m^2$



Results for the Finite Parts

$$
\begin{aligned}
[\mathrm{B5l2M2m}]_{\mathrm{fin}} &= \frac{1}{st}\Big\{-2\ln^2(m_s)\ln(M_s) + \zeta_2\ln(m_s) \\
&+ \left[2\ln^2(m_s) + 2\ln(m_s)\ln(M_s)\right]\ln\left(\frac{t}{s}\right) \\
&- 2\ln(m_s)\ln^2\left(\frac{t}{s}\right) \\
&+ \left[3\zeta_2 + \frac{1}{2}\ln^2\left(\frac{t}{s}\right)\right]\ln\left(1+\frac{t}{s}\right) + \ln\left(\frac{t}{s}\right)\mathit{Li}_2\left(-\frac{t}{s}\right) \\
&- \mathit{Li}_3\left(-\frac{t}{s}\right)\Big\}
\end{aligned}
$$

$$
m_s \equiv -\frac{m^2}{s} \qquad M_s \equiv -\frac{M^2}{s}
$$

Numerical checks

- Multidimensional MB-integrals checked with MB.m
- Comparison with sector decomposition (Binoth, Heinrich)
- Expansions (analytical form)

$$m^2/s = 1/1000, \; t/s = 1/4$$

$$\begin{aligned} \texttt{B5l2m2d2} &= 0.00252823 \\ \texttt{B5l2m2d2}_{expanded} &= 0.0025307 \end{aligned}$$

Conclusions

- Two-loop virtual corrections are known since A. Penin (2005), obtained by merging information from massless case (Bern, Dixon, Ghinculov) and massive vertices plus $N_f = 1$ (Bonciani, Mastrolia, Remiddi),
- we used another approach based on calculation of MIs
- fully analytic results seems to be not possible
- planar MIs are expanded (enough to get leading electron mass terms in the cross section)
- remain non-planar cases