Status of jet cross sections to NNLO

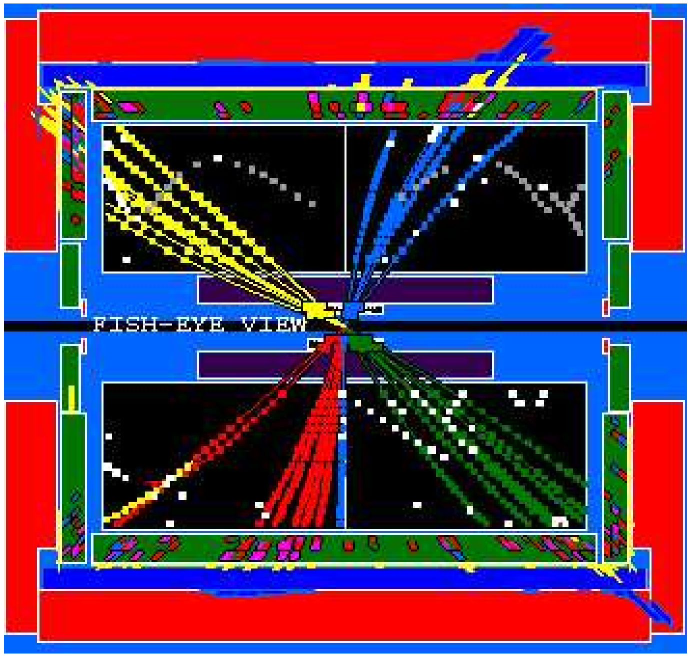
Stefan Weinzierl

Universität Mainz

Introduction:	**Jet physics**
I.:	**Two-loop amplitudes**
II:	**Multiple polylogarithms**
III:	**Infrared divergences**
IV.:	**Phase space generation**
V.:	**Numerics**

Jet physics

Jets: A bunch of particles moving in the same direction



Fully differential NNLO programs for processes like

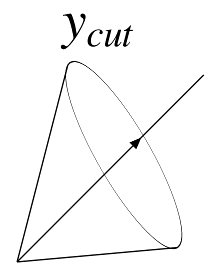
- Bhabha scattering
- $pp \longrightarrow 2$ jets
- $e^+e^- \longrightarrow 3$ jets

which allow the calculation of any infrared safe observable.

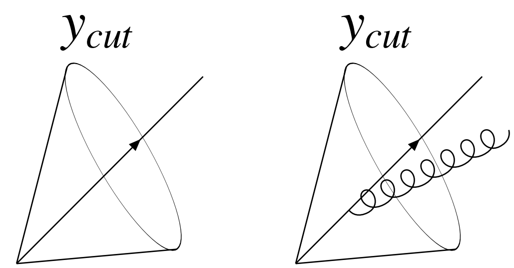
Modeling of jets:

In a perturbative calculation jets are modeled by only a few partons. This improves with the order to which the calculation is done.

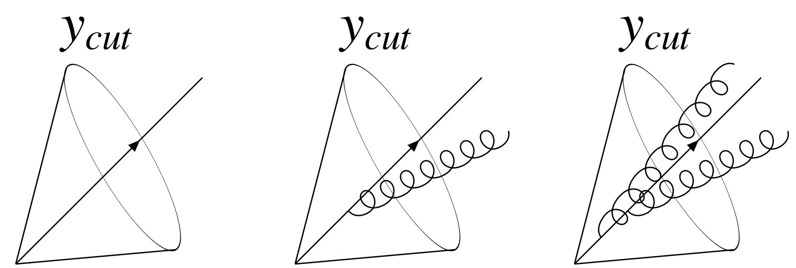
At leading order:



At next-to-leading order:



At next-to-next-to-leading order:



The master formula for the calculation of observables

$$\langle O \rangle = \underbrace{\frac{1}{2K(s)}}_{\text{flux factor}} \underbrace{\frac{1}{(2J_1+1)}\frac{1}{(2J_2+1)}}_{\text{average over initial spins}} \sum_n \underbrace{\int d\phi_{n-2}}_{\text{integral over phase space}} O(p_1,\ldots,p_n) \sum_{\text{helicity}} \underbrace{|\mathcal{A}_n|^2}_{\text{amplitude}}$$

Infrared safe at NNLO:

$$O_{n+1}(p_1,\ldots,p_{n+1}) \to O_n(p'_1,\ldots,p'_n), \quad \text{(Single unresolved)}$$
$$O_{n+2}(p_1,\ldots,p_{n+2}) \to O_n(p'_1,\ldots,p'_n). \quad \text{(Double unresolved)}$$

Perturbative expansion of the amplitude (LO, NLO, NNLO):

$$|\mathcal{A}_n|^2 = \mathcal{A}_n^{(0)\,*}\mathcal{A}_n^{(0)} + \left(\mathcal{A}_n^{(0)\,*}\mathcal{A}_n^{(1)} + \mathcal{A}_n^{(1)\,*}\mathcal{A}_n^{(0)}\right) + \left(\mathcal{A}_n^{(0)\,*}\mathcal{A}_n^{(2)} + \mathcal{A}_n^{(2)\,*}\mathcal{A}_n^{(0)} + \mathcal{A}_n^{(1)\,*}\mathcal{A}_n^{(1)}\right),$$

$$|\mathcal{A}_{n+1}|^2 = \mathcal{A}_{n+1}^{(0)\,*}\mathcal{A}_{n+1}^{(0)} + \left(\mathcal{A}_{n+1}^{(0)\,*}\mathcal{A}_{n+1}^{(1)} + \mathcal{A}_{n+1}^{(1)\,*}\mathcal{A}_{n+1}^{(0)}\right),$$

$$|\mathcal{A}_{n+2}|^2 = \mathcal{A}_{n+2}^{(0)\,*}\mathcal{A}_{n+2}^{(0)}.$$

Necessary ingredients for a NNLO calculation

- Calculation of the (two-loop) amplitudes.

  Requires: Two-loop integrals and tensor reduction.

- Cancellation of IR divergences has to be done before any Monte Carlo integration.

  Requires: Extension of the subtraction or slicing method to NNLO.

- The final numerical computer program.

  Requires: Stable and efficient numerical methods.

The amplitudes for $e^+e^- \longrightarrow 3$ jets at NNLO

A NNLO calculation of $e^+e^- \longrightarrow 3$ jets requires the following amplitudes:

- Born amplitudes for $e^+e^- \longrightarrow 5$ jets:

  F. Berends, W. Giele and H. Kuijf.

- One-loop amplitudes for $e^+e^- \longrightarrow 4$ jets:

  Z. Bern, L. Dixon, D.A. Kosower and S.W.;

  J. Campbell, N. Glover and D. Miller.

- Two-loop amplitudes for $e^+e^- \longrightarrow 3$ jets:

  L. Garland, T. Gehrmann, N. Glover, A. Koukoutsakis and E. Remiddi;

  S. Moch, P. Uwer and S.W.

The calculation of two-loop integrals

- Techniques to calculate two-loop integrals
  - Mellin-Barnes transformation, Smirnov ’99, Tausk ’99, Bierenbaum and S.W. ’03.
  - Differential equations, Gehrmann, Remiddi ‘00.
  - Nested sums, Moch, Uwer, S.W. ‘01.
  - Sector decomposition (numerical), Binoth, Heinrich, ’00.

- Methods to reduce the work-load:
  - Integration-by-parts, Chetyrkin, Tkachov ‘81.
  - Reduction algorithms, Tarasov ‘96, Laporta ’01.
  - Cut technique Bern, Dixon, Kosower, ’00

The calculation of two-loop amplitudes

- Calculation of two-loop amplitudes
  - Bhabha, Bern, Dixon, Ghinculov ‘01.
  - $pp \longrightarrow$ 2 jets, Anastasiou, Glover, Oleari, Tejeda-Yeomans ’01;
    Bern, De Freitas, Dixon, Ghinculov, Wong ’01.
  - $e^+e^- \longrightarrow$ 3 jets, L. Garland, T. Gehrmann, N. Glover, A. Koukoutsakis and E. Remiddi ’02;
    S. Moch, P. Uwer and S.W. ’02
  - Higgs production, Harlander, Kilgore; Catani, de Florian, Grazzini; Anastasiou, Melnikov;
  - Drell-Yan, Anastasiou, Dixon, Melnikov, Petriello; Ravindran, Smith, van Neerven

- Calculation of three-loop splitting functions S. Moch, J. Vermaseren and A. Vogt ’04;

Results for the two-loop amplitude for $e^+e^- \longrightarrow 3$ jets

The finite part of the coefficient of a spinor structure:

$$c_{12}^{(2),\mathrm{fin}}(x_1,x_2) = N_f N\left(3\frac{\ln(x_1)}{(x_1+x_2)^2} + \frac{1}{4}\frac{\ln(x_2)^2-2\mathrm{Li}_2(1-x_2)}{x_1(1-x_2)} + \frac{1}{12}\frac{\zeta(2)}{(1-x_2)x_1} - \frac{1}{18}\frac{13{x_1}^2+36x_1-10x_1x_2-18x_2+31{x_2}^2}{(x_1+x_2)^2x_1(1-x_2)}\ln(x_2)\right.$$
$$+\frac{{x_1}^2-{x_2}^2-2x_1+4x_2}{(x_1+x_2)^4}\mathrm{R}_1(x_1,x_2) - \frac{1}{12}\frac{\mathrm{R}(x_1,x_2)}{x_1(x_1+x_2)^2}\left[5x_2+42x_1+5-\frac{(1+x_1)^2}{1-x_2}-4\frac{1-3x_1+3{x_1}^2}{1-x_1-x_2}-72\frac{{x_1}^2}{x_1+x_2}\right]+\left[\frac{1}{12}\frac{1}{x_1(1-x_2)}+\frac{6}{(x_1+x_2)^3}\right.$$
$$\left.\left.-\frac{1+2x_1}{x_1(x_1+x_2)^2}\right](\mathrm{Li}_2(1-x_2)-\mathrm{Li}_2(1-x_1)) - \frac{1}{(x_1+x_2)x_1}\right) - \frac{1}{2}I\pi N_f N\frac{\ln(x_2)}{x_1(1-x_2)}.$$

where $\mathrm{R}(x_1,x_2)$ and $\mathrm{R}_1(x_1,x_2)$ are defined by

$$\mathrm{R}(x_1,x_2) = \left(\frac{1}{2}\ln(x_1)\ln(x_2) - \ln(x_1)\ln(1-x_1) + \frac{1}{2}\zeta(2) - \mathrm{Li}_2(x_1)\right) + (x_1 \leftrightarrow x_2).$$

$$\mathrm{R}_1(x_1,x_2) = \left(\ln(x_1)\mathrm{Li}_{1,1}\left(\frac{x_1}{x_1+x_2},x_1+x_2\right) - \frac{1}{2}\zeta(2)\ln(1-x_1-x_2) + \mathrm{Li}_3(x_1+x_2) - \ln(x_1)\mathrm{Li}_2(x_1+x_2) - \frac{1}{2}\ln(x_1)\ln(x_2)\ln(1-x_1-x_2)\right.$$
$$\left.-\mathrm{Li}_{1,2}\left(\frac{x_1}{x_1+x_2},x_1+x_2\right) - \mathrm{Li}_{2,1}\left(\frac{x_1}{x_1+x_2},x_1+x_2\right)\right) + (x_1 \leftrightarrow x_2).$$

Multiple polylogarithms

$$\mathrm{Li}_{m_1,\ldots,m_k}(x_1,\ldots,x_k) = \sum_{i_1>i_2>\ldots>i_k>0} \frac{x_1^{i_1}}{i_1^{m_1}} \frac{x_2^{i_2}}{i_2^{m_2}} \cdots \frac{x_k^{i_k}}{i_k^{m_k}}.$$

- Special subsets: Harmonic polylogs, Nielsen polylogs, classical polylogs
  (Remiddi and Vermaseren, Gehrmann and Remiddi).
- Have also an integral representation.
- Fulfill two Hopf algebras (Moch, Uwer, S.W.).
- Can be evaluated numerically for all complex values of the arguments
  (Gehrmann and Remiddi, Vollinga and S.W.).

Iterated integrals

Define the functions G by

$$G(z_1, \ldots, z_k; y) = \int_0^y \frac{dt_1}{t_1 - z_1} \int_0^{t_1} \frac{dt_2}{t_2 - z_2} \ldots \int_0^{t_{k-1}} \frac{dt_k}{t_k - z_k}.$$

Short hand notation:

$$G_{m_1,\ldots,m_k}(z_1, \ldots, z_k; y) = G(\underbrace{0, \ldots, 0}_{m_1-1}, z_1, \ldots, z_{k-1}, \underbrace{0 \ldots, 0}_{m_k-1}, z_k; y)$$

Conversion to multiple polylogarithms:

$$\mathrm{Li}_{m_1,\ldots,m_k}(x_1, \ldots, x_k) = (-1)^k G_{m_1,\ldots,m_k}\left(\frac{1}{x_1}, \frac{1}{x_1 x_2}, \ldots, \frac{1}{x_1 \ldots x_k}; 1\right).$$

Numerical evaluations of multiple polylogarithms

Example: Numerical evaluation of the dilogarithm ('t Hooft, Veltman, Nucl. Phys. B153, (1979), 365)

$$\text{Li}_2(x) \quad = \quad -\int_0^x dt \frac{\ln(1-t)}{t} = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Map into region $-1 \leq \text{Re}(x) \leq 1/2$, using

$$\text{Li}_2(x) = -\text{Li}_2\left(\frac{1}{x}\right) - \frac{\pi^2}{6} - \frac{1}{2}(\ln(-x))^2, \qquad \text{Li}_2(x) = -\text{Li}_2(1-x) + \frac{\pi^2}{6} - \ln(x)\ln(1-x).$$

Accelaration using Bernoulli numbers:

$$\text{Li}_2(x) \quad = \quad \sum_{i=0}^{\infty} \frac{B_i}{(i+1)!} (-\ln(1-x))^{i+1},$$

Generalization to multiple polylogarithms, using arbitrary precision arithmetic in C++.

J. Vollinga, S.W., (2004)

Numerical evaluations of multiple polylogarithms

Use the integral representation

$$G_{m_1,\dots,m_k}(z_1,z_2,\dots,z_k;y) \quad = \quad \int\limits_0^y \left(\frac{dt}{t}\circ\right)^{m_1-1} \frac{dt}{t-z_1} \left(\frac{dt}{t}\circ\right)^{m_2-1} \frac{dt}{t-z_2} \dots \left(\frac{dt}{t}\circ\right)^{m_k-1} \frac{dt}{t-z_k}$$

to transform all arguments into a region, where we have a converging power series expansion:

$$G_{m_1,\dots,m_k}(z_1,\dots,z_k;y) =$$
$$\sum_{j_1=1}^{\infty} \dots \sum_{j_k=1}^{\infty} \frac{1}{(j_1+\dots+j_k)^{m_1}} \left(\frac{y}{z_1}\right)^{j_1} \frac{1}{(j_2+\dots+j_k)^{m_2}} \left(\frac{y}{z_2}\right)^{j_2} \dots \frac{1}{(j_k)^{m_k}} \left(\frac{y}{z_k}\right)^{j_k}.$$

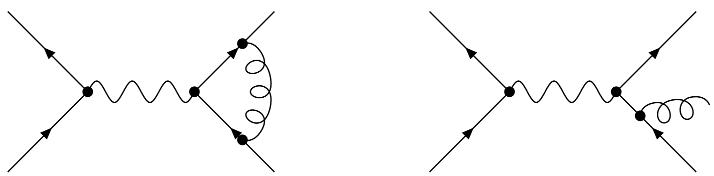
Use the Hölder convolution to accelerate the convergent series.

(Borwein, Bradley, Broadhurst and Lisonek)

Infrared divergences and the Kinoshita-Lee-Nauenberg theorem

In addition to ultraviolet divergences, loop integrals can have infrared divergences.

For each IR divergence there is a corresponding divergence with the opposite sign in the real emission amplitude, when particles becomes soft or collinear (e.g. unresolved).



The Kinoshita-Lee-Nauenberg theorem: Any observable, summed over all states degenerate according to some resolution criteria, will be finite.

General methods at NLO

Fully differential NLO Monte Carlo programs need a general method to handle the cancelation of infrared divergencies.

- Phase space slicing
  - e^+e^-: W. Giele and N. Glover, (1992)
  - initial hadrons: W. Giele, N. Glover and D.A. Kosower, (1993)
  - massive partons, fragmentation: S. Keller and E. Laenen, (1999)

- Subtraction method
  - residue approach: S. Frixione, Z. Kunzst and A. Signer, (1995)
  - dipole formalism: S. Catani and M. Seymour, (1996)
  - massive partons: L. Phaf and S.W. (2001), S. Catani, S. Dittmaier, M. Seymour and Z. Trócsányi, (2002)

The dipole formalism

The dipole formalism is based on the subtraction method. The NLO cross section is rewritten as

$$\sigma^{NLO} = \int_{n+1} d\sigma^R + \int_n d\sigma^V$$

$$= \int_{n+1} \left(d\sigma^R - d\sigma^A\right) + \int_n \left(d\sigma^V + \int_1 d\sigma^A\right)$$

The approximation $d\sigma^A$ has to fulfill the following requirements:

- $d\sigma^A$ must be a proper approximation of $d\sigma^R$ such as to have the same pointwise singular behaviour in D dimensions as $d\sigma^R$ itself. Thus, $d\sigma^A$ acts as a local counterterm for $d\sigma^R$ and one can safely perform the limit $\varepsilon \to 0$.

- Analytic integrability in D dimensions over the one-parton subspace leading to soft and collinear divergences.

The subtraction method at NNLO

- Singular behaviour
  - Factorization of tree amplitudes in double unresolved limits, Berends, Giele, Cambell, Glover, Catani, Grazzini, Del Duca, Frizzo, Maltoni, Kosower '99
  - Factorization of one-loop amplitudes in single unresolved limits, Bern, Del Duca, Kilgore, Schmidt, Kosower, Uwer, Catani, Grazzini, '99
- Extension of the subtraction method to NNLO Kosower; S.W.; Anastasiou, Melnikov, Petriello; Kilgore; Gehrmann-De Ridder, Gehrmann, Glover, Heinrich; Frixione, Grazzini; Somogyi, Trócsányi and Del Duca;
- Applications:
  - $pp \to W$, Anastasiou, Dixon, Melnikov, Petriello '03,
  - $e^+e^- \to$ 2 jets, Anastasiou, Melnikov, Petriello '04,

The subtraction method at NNLO

Contributions at NNLO:

$$
\begin{aligned}
d\sigma_{n+2}^{(0)} &= \left({\mathcal{A}_{n+2}^{(0)}}^* \mathcal{A}_{n+2}^{(0)} \right) d\phi_{n+2}, \\
d\sigma_{n+1}^{(1)} &= \left({\mathcal{A}_{n+1}^{(0)}}^* \mathcal{A}_{n+1}^{(1)} + {\mathcal{A}_{n+1}^{(1)}}^* \mathcal{A}_{n+1}^{(0)} \right) d\phi_{n+1}, \\
d\sigma_{n}^{(2)} &= \left({\mathcal{A}_{n}^{(0)}}^* \mathcal{A}_{n}^{(2)} + {\mathcal{A}_{n}^{(2)}}^* \mathcal{A}_{n}^{(0)} + {\mathcal{A}_{n}^{(1)}}^* \mathcal{A}_{n}^{(1)} \right) d\phi_{n},
\end{aligned}
$$

Adding and subtracting:

$$
\begin{aligned}
\langle O \rangle_n^{NNLO} = & \int \left(O_{n+2}\, d\sigma_{n+2}^{(0)} - O_{n+1} \circ d\alpha_{n+1}^{(0,1)} - O_n \circ d\alpha_n^{(0,2)} \right) \\
& + \int \left(O_{n+1}\, d\sigma_{n+1}^{(1)} + O_{n+1} \circ d\alpha_{n+1}^{(0,1)} - O_n \circ d\alpha_n^{(1,1)} \right) \\
& + \int \left(O_n\, d\sigma_n^{(2)} + O_n \circ d\alpha_n^{(0,2)} + O_n \circ d\alpha_n^{(1,1)} \right).
\end{aligned}
$$

NNLO subtraction terms

The $(n+2)$-parton contribution:

$$\int \left(O_{n+2}\, d\sigma^{(0)}_{n+2} - O_{n+1} \circ d\alpha^{(0,1)}_{n+1} - O_n \circ d\alpha^{(0,2)}_n \right), \qquad d\alpha^{(0,2)}_n = d\alpha^{(0,2)}_{(0,0)n} - d\alpha^{(0,2)}_{(0,1)n}.$$

has to be integrable for all double and single unresolved limits.

The $(n+1)$-parton contribution:

$$\int \left(O_{n+1}\, d\sigma^{(1)}_{n+1} + O_{n+1} \circ d\alpha^{(0,1)}_{n+1} - O_n \circ d\alpha^{(1,1)}_n \right), \qquad d\alpha^{(1,1)}_n = d\alpha^{(1,1)}_{(1,0)\,n} + d\alpha^{(1,1)}_{(0,1)\,n}$$

has to be integrable over single unresolved limits.
In addition, explicit poles in ε have to cancel.

Example: $qgg\bar{q}$ final state for $e^+e^- \to$ 2 jets

NNLO subtraction terms for the $(n+2)$-parton configuration:

$$d\alpha^{(0,2)}_{(0,0)} = \frac{1}{2}\left\{\frac{N}{2}C_F\left[A_4^0(1,2,3,4)+A_4^0(1,3,2,4)\right] - \frac{1}{2N}C_F\left[A_{4,sc}^0(1,2,3,4)+A_{4,sc}^0(1,3,2,4)\right]\right\}\left|\mathcal{A}_2^{(0)}\right|^2$$

$$d\alpha^{(0,2)}_{(0,1)} = \frac{1}{2}\left\{\frac{N}{2}\left[D_3^0(1,2,3)+D_3^0(1,3,2)+D_3^0(4,2,3)+D_3^0(4,3,2)\right] - \frac{1}{2N}\left[A_3^0(1,2,4)+A_3^0(1,3,4)\right]\right\}C_F A_3^0(1',2',3')\left|\mathcal{A}_2^{(0)}\right|^2.$$

Spin-averaged antenna functions

Spin-averaged $qgg\bar{q}$ antenna function obtained from the matrix element $\gamma^* \to qgg\bar{q}$:

$$\mathcal{A}_4^{(0)}(q_1,g_2,g_3,\bar{q}_4) = eg^2\left[\left(T^2T^3\right)_{14}A_4^{(0)}(q_1,g_2,g_3,\bar{q}_4)+\left(T^3T^2\right)_{14}A_4^{(0)}(q_1,g_3,g_2,\bar{q}_4)\right]$$

$$\left|\mathcal{A}_4^{(0)}\right|^2 = e^2g^4\frac{N(N^2-1)}{4}\left(A_4^{(0)}(2,3),A_4^{(0)}(3,2)\right)\begin{pmatrix} 1-\frac{1}{N^2} & -\frac{1}{N^2} \\ -\frac{1}{N^2} & 1-\frac{1}{N^2}\end{pmatrix}\begin{pmatrix} A_4^{(0)}(2,3) \\ A_4^{(0)}(3,2)\end{pmatrix}$$

Leading-colour antenna function:

$$A_4^0(1,2,3,4) = \left|A_4^{(0)}(2,3)\right|^2 / \left|A_2^{(0)}\right|^2$$

Subleading-colour:

$$A_{4,sc}^0(1,2,3,4)+A_{4,sc}^0(1,3,2,4) = \left|A_4^{(0)}(2,3)+A_4^{(0)}(3,2)\right|^2 / \left|A_2^{(0)}\right|^2$$

Example: Leading colour $qgg\bar{q}$ antenna function

$A_4^0(1,2,3,4) =$

$$\frac{1}{4s_{1234}}\left(\frac{48\,s_{1234}}{s_{234}{}^2}+\frac{32\,s_{1234}}{s_{23}{}^2}+\frac{48\,s_{1234}}{s_{123}{}^2}+\frac{48\,s_{23}-48\,s_{123}+64\,s_{1234}}{s_{12}\,s_{234}}+\frac{-32\,s_{123}\,s_{1234}+16\,s_{123}{}^2-32\,s_{34}\,s_{1234}+16\,s_{34}{}^2+32\,s_{1234}{}^2}{s_{12}\,s_{23}\,s_{234}}\right.$$

$$+\frac{-48\,s_{12}-96\,s_{23}-48\,s_{34}-96\,s_{1234}}{s_{123}\,s_{234}}-\frac{16\,s_{1234}}{s_{34}\,s_{234}}+\frac{-32\,s_{123}\,s_{1234}+16\,s_{123}{}^2-32\,s_{1234}\,s_{234}+16\,s_{234}{}^2+32\,s_{1234}{}^2}{s_{12}\,s_{23}\,s_{34}}+\frac{96}{s_{123}}+\frac{32\,s_{1234}}{s_{12}\,s_{34}}$$

$$-\frac{16\,s_{1234}}{s_{12}\,s_{123}}+\frac{64\,s_{12}\,s_{34}\,s_{1234}}{s_{23}{}^2 s_{123}\,s_{234}}+\frac{64\,s_{12}\,s_{1234}-32\,s_{12}{}^2+64\,s_{34}\,s_{1234}-32\,s_{34}{}^2-128\,s_{1234}{}^2}{s_{23}\,s_{123}\,s_{234}}+\frac{16\,s_{23}\,s_{1234}}{s_{34}\,s_{234}{}^2}+\frac{48\,s_{23}-48\,s_{234}+64\,s_{1234}}{s_{123}\,s_{34}}$$

$$+\frac{48\,s_{12}-48\,s_{123}+32\,s_{1234}}{s_{23}\,s_{234}}+\frac{64\,s_{34}\,s_{1234}}{s_{23}\,s_{234}{}^2}+\frac{64\,s_{12}\,s_{1234}}{s_{23}\,s_{123}{}^2}-\frac{64\,s_{34}\,s_{1234}}{s_{23}{}^2 s_{234}}+\frac{48\,s_{34}-48\,s_{234}+32\,s_{1234}}{s_{23}\,s_{123}}+\frac{32\,s_{34}{}^2 s_{1234}}{s_{23}{}^2 s_{234}{}^2}-\frac{64\,s_{12}\,s_{1234}}{s_{23}{}^2 s_{123}}$$

$$+\frac{32\,s_{12}{}^2 s_{1234}}{s_{23}{}^2 s_{123}{}^2}+\frac{16\,s_{23}\,s_{1234}}{s_{12}\,s_{123}{}^2}+\frac{-32\,s_{12}\,s_{1234}+16\,s_{12}{}^2-32\,s_{1234}\,s_{234}+16\,s_{234}{}^2+32\,s_{1234}{}^2}{s_{23}\,s_{123}\,s_{34}}$$

$$+\frac{-32\,s_{23}\,s_{1234}-16\,s_{23}{}^2+48\,s_{34}\,s_{1234}-16\,s_{34}{}^2-64\,s_{1234}{}^2}{s_{12}\,s_{123}\,s_{234}}+\frac{48\,s_{12}\,s_{1234}-16\,s_{12}{}^2-32\,s_{23}\,s_{1234}-16\,s_{23}{}^2-64\,s_{1234}{}^2}{s_{123}\,s_{34}\,s_{234}}$$

$$\left.+\frac{32\,s_{23}\,s_{1234}{}^2+16\,s_{23}{}^2 s_{1234}+32\,s_{1234}{}^3}{s_{12}\,s_{123}\,s_{34}\,s_{234}}+\frac{-32\,s_{23}\,s_{1234}+16\,s_{123}\,s_{1234}-32\,s_{1234}{}^2}{s_{12}\,s_{34}\,s_{234}}+\frac{-32\,s_{23}\,s_{1234}+16\,s_{1234}\,s_{234}-32\,s_{1234}{}^2}{s_{12}\,s_{123}\,s_{34}}+\frac{96}{s_{234}}\right)$$

Correct singular limits:

$$\lim_{2,3\ soft} A_4^0(1,2,3,4) \quad = \quad 8\frac{(s_{12}s_{234}+s_{123}s_{34}-s_{123}s_{234})^2}{s_{23}^2 s_{123}^2 s_{234}^2}+\frac{8s_{1234}}{s_{23}}\left(\frac{1}{s_{12}s_{34}}+\frac{1}{s_{123}s_{34}}+\frac{1}{s_{12}s_{234}}+\frac{1}{s_{123}s_{234}}\right)+8\frac{s_{1234}^2}{s_{12}s_{34}s_{123}s_{234}}.$$

Spin correlations

In the collinear limit spin correlations remain:

$$A_\mu \frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2} A_\nu, \qquad \text{where } k_\perp = (1-z)p_i + zp_j - (1-2z)\frac{y}{1-y}p_k.$$

Let φ be the azimuthal angle of p_i around $p_i + p_j$. Then

$$A_\mu \frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2} A_\nu \quad \sim \quad C_0 + C_2 \cos(2\varphi + \alpha).$$

One can perform the average with two points:

$$\varphi, \qquad \varphi + \frac{\pi}{2},$$

while all other coordinates remain fixed.

Phase space generation

Dimension of phase space for n final state particles: $3n-4$.

Split the phase space into different channels, according to which invariants are the smallest.

For each channel, use a parameterization such that φ is along a coordinate axis:

$$
\begin{aligned}
d\phi_{n+1} &= d\phi_n \; d\phi_{dipole}, \\
& \quad d\phi_{dipole} = \frac{s_{ijk}}{32\pi^3}\,(1-y)\,dy\,dz\,d\varphi.
\end{aligned}
$$

Construct the momenta of the $(n+1)$ event from the ones of the n parton event and the values of y, z and φ.

Numerical results for $e^+e^- \to$ 2 jets

Durham jet algorithm with $y = 0.01$ and E-scheme, $\alpha_s = 0.118$, $\sqrt{s} = 91.187$ GeV.

Contributions at α_s^2:

2-parton final state	81.13 ± 0.06	pb
3-parton final state	-5050.2 ± 1.3	pb
4-parton final state	1536 ± 95	pb
2-jet NNLO (total)	-3433 ± 95	pb

For comparison:	LO	(α_s^0):	40807.4	pb
	NLO	(α_s^1):	-10476 ± 3	pb

Checks

total cross section at α_s^2		81.1 pb
4-jet LO	$-$	1214.6 ± 0.7 pb
3-jet NLO	$-$	2210.7 ± 5.9 pb
2-jet NNLO (indirect)		-3344 ± 6 pb

Summary and outlook

NNLO calculations:

- Matrix elements: Two-loop amplitudes.

- New functions: Multiple polylogarithms.

- Subtraction terms: Double unresolved and one-loop single unresolved.

- Phase space: Variance-reducing techniques.

Proof of concept for $e^+e^- \longrightarrow$ 2 jets.