A numerical approach to the double real radiation part of $e^+e^- \to 3$ jets at NNLO

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Motivation

experiments at high energy colliders in past years:

precision measurements

led to stringent tests of the Standard Model and bounds on New Physics

only possible in combination with

"precision calculations"

what to expect from "Old Physics" must be well under control $\Rightarrow$ precise knowledge of parameters like α_s, $\sin^2\theta_W$ important

- at the LHC : n-jet cross section $\sim \alpha_s^n$
- future International Linear Collider will reach precision at the per mille level

Jet production in e^+e^- annihilation

measurements of $e^+e^- \rightarrow$ jet rates & shape observables

- allow precision tests of the SM over wide range of energies
- offer possibility for determination of strong coupling constant α_s with unseen precision

α_s world average:

$\alpha_s(M_Z) = 0.1182 \pm 0.0027$ (stat. and sys.) [S. Bethke 04]

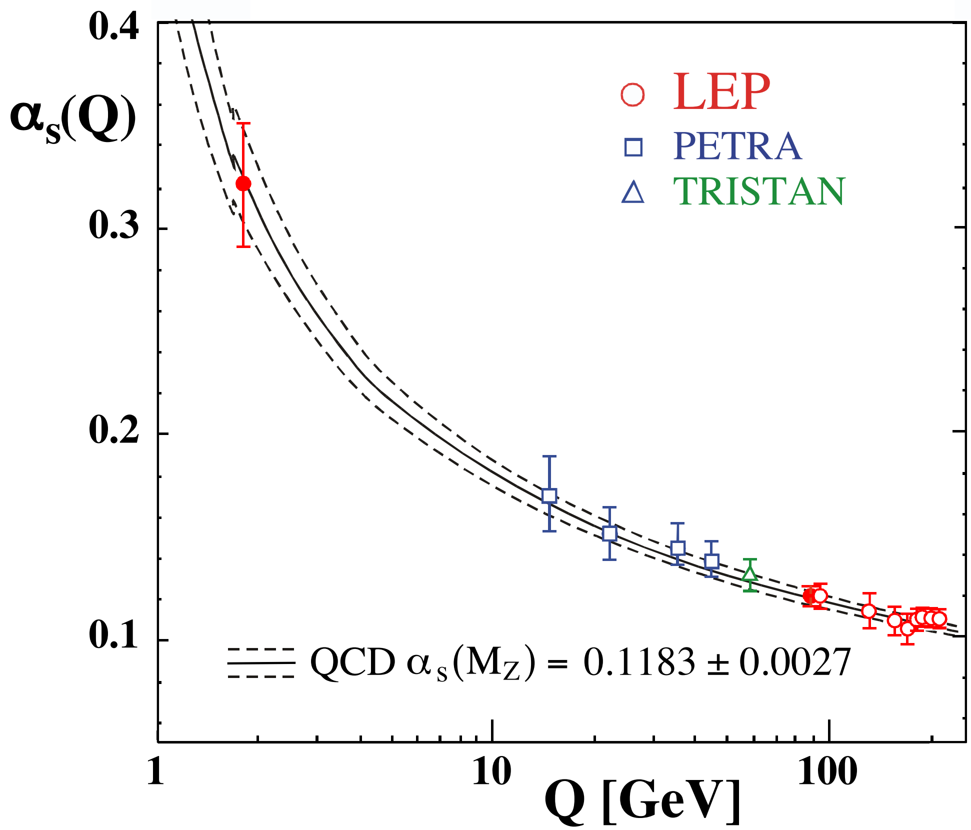
determination based only on data where

NNLO QCD theory predictions exist !

(DIS, $\Gamma(Z \rightarrow \mathrm{had})$, $\tau, \Upsilon -$ decays)

α_s determination

LEP data for jets and shape observables very precise,
but not used for world average : only NLO theory available!



[S. Bethke '04]

α_s determination

error on $\alpha_s(M_Z)$ from e^+e^- jet rates and event shape observables (NLO + NLL resum) :

theoretical: $\mathcal{O}(5\%)$, dominated by scale dependence !

experimental: $\mathcal{O}(1.3\%)$ (statistical, hadronisation, $\ldots$)

[R. Jones, Ford, Salam, Stenzel, Wicke '04]

$\Rightarrow$ NNLO will improve the error considerably !

mandatory to match experimental precision at the ILC

NLO calculation:

Ellis, Ross, Terrano '81, Fabricius, Schmitt, Kramer, Schierholz '81, Kramer, Lampe '89 , Kunszt, Nason, Marchesini, Webber '89

resummation:

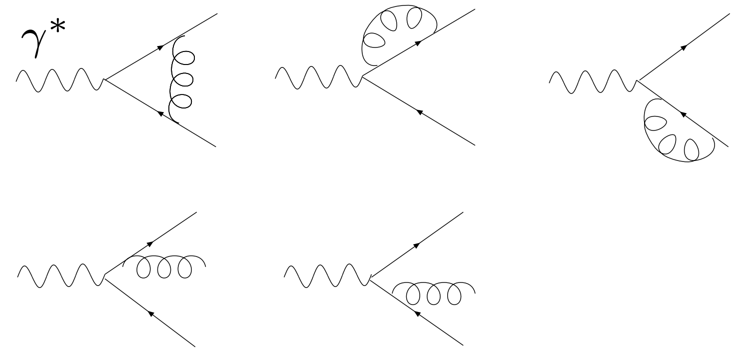
Catani et al. '92-'98, Dokshitzer et al. '98, Banfi et al. '02, Gardi et al. '01-'03

NNLO calculations

problems in NNLO calculations:

- enormous complexity of expressions
- analytic integrations very difficult
- direct numerical evaluation hampered by singularities
- massless particles $\Rightarrow$ soft and collinear (IR) singularities ($1/\epsilon^n$ poles in dim.reg.)
- IR singularities are entangled in a complicated way
  $\Rightarrow$ need to be isolated and subtracted
  - from loop integrals ($\rightarrow$ analytic integration)
  - from integrals over soft/collinear phase space regions ($\rightarrow$ subtraction terms)

subtraction of infrared poles: NLO



virtual: $d\sigma^V = P_2/\epsilon^2 + P_1/\epsilon + P_0$

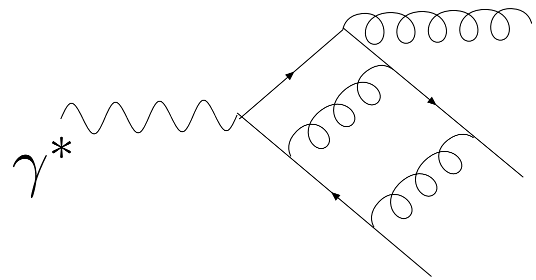
real: integration of subtraction terms
$d\sigma^S$ over singular regions of phase space

$\Rightarrow \int_{\text{sing}} d\sigma^S = -P_2/\epsilon^2 - P_1/\epsilon + Q_0$

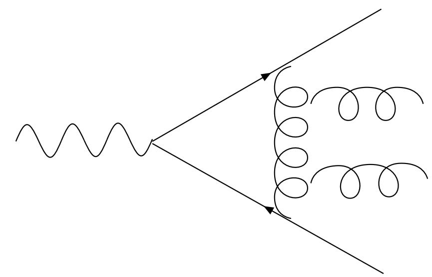
$$\sigma^{NLO} = \underbrace{\int_{m+1} \left[d\sigma^R - d\sigma^S \right]_{\epsilon=0}}_{\text{numerically}} + \underbrace{\int_m \left[\underbrace{d\sigma^V}_{\text{analytically}} + \underbrace{\int_1 d\sigma^S}_{\text{analytically}} \right]_{\epsilon=0}}_{\text{numerically}}$$

partonic ingredients for $e^+e^- \rightarrow 3$ jets at NNLO

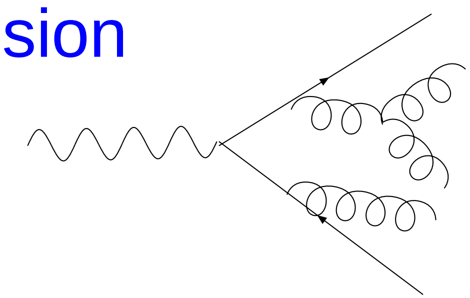
- 2-loop virtual



- one-loop plus single unresolved real emission



- double unresolved real emission



bottleneck: 5 partons in final state, up to 2 soft or collinear

difficulty: isolation of infrared poles from phase space integrals

subtraction of infrared poles: NNLO

m-jet production schematically:

$$\mathrm{d}\sigma_{NNLO} = \underbrace{\int_{\mathrm{d}\Phi_{m+2}} \left(\mathrm{d}\sigma^{R}_{NNLO} - \mathrm{d}\sigma^{S}_{NNLO}\right)}_{\text{finite}} + \int_{\mathrm{d}\Phi_{m+2}} \mathrm{d}\sigma^{S}_{NNLO}$$

$$+ \underbrace{\int_{\mathrm{d}\Phi_{m+1}} \left(\mathrm{d}\sigma^{V,1}_{NNLO} - \mathrm{d}\sigma^{VS,1}_{NNLO}\right)}_{\text{finite}} + \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\sigma^{VS,1}_{NNLO}$$

$$+ \int_{\mathrm{d}\Phi_{m}} \mathrm{d}\sigma^{V,2}_{NNLO}$$

$$\int_{\mathrm{d}\Phi_{m+2}} \mathrm{d}\sigma^{S}_{NNLO} + \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\sigma^{VS,1}_{NNLO} + \int_{\mathrm{d}\Phi_{m}} \mathrm{d}\sigma^{V,2}_{NNLO} = \text{finite}$$

subtraction of infrared poles: NNLO

two conceptually different approaches:

- "conventional" approach:

  manual construction of a subtraction scheme and analytic integration over subtraction terms in $D = 4 - 2\epsilon$ dimensions

  [A. Gehrmann-De Ridder, T. Gehrmann, N. Glover], [Kilgore], [Weinzierl], [Del Duca, Somogyi, Trocsanyi], [Frixione, Grazzini]

- sector decomposition: automated isolation of IR poles in parameter space and numerical integration of pole coefficients

  [Binoth, GH], [Anastasiou, Melnikov, Petriello]

sector decomposition

- no manual construction of subtraction terms
- isolation of poles in $1/\epsilon$ by automated procedure acting in parameter space
- pole coefficients will be complicated functions
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- application to real radiation [GH L&L '02; Anastasiou, Melnikov, Petriello, Binoth, GH '03/04]
- development as a powerful tool for differential NNLO calculations [Anastasiou, Melnikov, Petriello: $e^+e^- \to 2$ jets, H production, μ decay, W production '04-06]

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- development as a powerful tool for differential NNLO calculations [Anastasiou, Melnikov, Petriello: $e^+e^- \to 2$ jets, H production, μ decay, W production '04-06]
- $e^+e^- \to 3$ jets at NNLO: very complicated singularity structure due to high number of massless particles

basics of sector decomposition

integral of matrix element squared with some measurement function $\mathcal{J}$:

typically contains "overlapping" structures like

$$\int \mathrm{d}\Phi^{(D)}\,|\mathrm{ME}|^2\,\mathcal{J} \;\sim\; \int \mathrm{d}s_{13}\,\mathrm{d}s_{23}\; s_{13}^{-1-\epsilon}\,\frac{\mathcal{J}(s_{13},s_{23})}{s_{13}+s_{23}}$$

$$\sim\; \int_0^1 dx\,dy\,x^{-1-\epsilon}\,\frac{\mathcal{J}(x,y)}{x+y}$$

singularities for $x, y \to 0$ need to be factorised

sector decomposition is an algorithmic way to factorise this type of entangled singularities

sector decomposition

$$I = \int_0^1 dx\, dy\, x^{-1-\epsilon}\,(x+y)^{-1}\,\mathcal{J}(x,y)\,[\underbrace{\Theta(x-y)}_{(1)} + \underbrace{\Theta(y-x)}_{(2)}]$$

subst. $(1)\; y = x\,t \qquad (2)\; x = y\,t$

$$I = \int_0^1 dx\, x^{-1-\epsilon} \int_0^1 dt\,(1+t)^{-1}\mathcal{J}(x, x\,t)$$
$$+ \int_0^1 dy\, y^{-1-\epsilon} \int_0^1 dt\, t^{-1-\epsilon}\,(1+t)^{-1}\mathcal{J}(y\,t, y)$$

$\Rightarrow$ singularities factorised, number of integrals doubled

general algorithm

- map parameter integrals to unit hypercube (manual)
- map singularities to be located at the origin of parameter space (manual/automated)

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result:

$$I = \int_0^1 dx_1\, x_1{}^{b_1-\kappa_1\epsilon} \ldots \int_0^1 dx_n\; x_n{}^{b_n-\kappa_n\epsilon}\; F(x_1,\ldots,x_n)$$

$$\lim_{x_i\to 0} F(x_1,\ldots,x_n) = \text{const}$$

general algorithm

- perform subtractions and expansion in ϵ (fully automated)

use identities like

$$\int_0^1 dx \int_0^1 dy\, x^{-1-\kappa\epsilon} f(x,y) =$$

$$-\frac{1}{\kappa\epsilon} \int_0^1 dx\, \delta(x) \int_0^1 dy\, f(x,y) + \int_0^1 dx \int_0^1 dy\, x^{-\kappa\epsilon} \frac{f(x,y) - f(0,y)}{x}$$

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note: if $f(x,y) = \Theta(x-a)\, g(y) \Rightarrow$ pole term zero

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- result after ϵ-expansion: Laurent series in ϵ

$$I = \sum_{k=\mathrm{maxpole}}^{-n} C_k(y_i)/\epsilon^k \; + \; \mathcal{O}(\epsilon^{n+1})$$

  (y_i scaled invariants s_{ij})

- poles are isolated $\Rightarrow$ integrate finite coefficient functions $C_k(y_i)$ numerically over the phase space

application to $1 \to 5$ parton phase space

use scaled invariants $y_1 = s_{12}/q^2, \ldots, y_{10} = s_{45}/q^2$

$$\int d\Phi^D_{1\to 5} = C_\Gamma^{(5)} \int \prod_{j=1}^{10} dy_j \, \delta(1 - \sum_{i=1}^{10} y_i) \, (-\Delta_5)^{\frac{D}{2}-3} \Theta(-\Delta_5)$$

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$$\begin{aligned}
\Delta_5 \quad = \quad & y_{10}^2 y_1 y_2 y_3 + y_9^2 y_1 y_4 y_5 + y_8^2 y_2 y_4 y_6 + y_7^2 y_3 y_5 y_6 \\
& + y_6^2 y_1 y_7 y_8 + y_5^2 y_2 y_7 y_9 + y_4^2 y_3 y_8 y_9 + y_3^2 y_4 y_7 y_{10} \\
& + y_2^2 y_5 y_8 y_{10} + y_1^2 y_6 y_9 y_{10} \\
& + y_{10} \left[y_2 y_3 y_5 y_7 + y_1 y_3 y_6 y_7 + y_2 y_3 y_4 y_8 \right. \\
& \left. + y_1 y_2 y_6 y_8 + y_1 y_3 y_4 y_9 + y_1 y_2 y_5 y_9 \right] \\
& + y_9 \left[y_4 y_5 (y_3 y_7 + y_2 y_8) + y_1 y_6 (y_5 y_7 + y_4 y_8) \right] \\
& + y_6 y_7 y_8 (y_3 y_4 + y_2 y_5)
\end{aligned}$$

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  can be done once and for all, but costly in number of iterations for certain denominators

  better:

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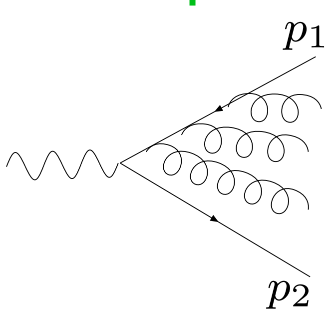
  better:

- use different parametrisations, optimized for certain denominator structures ("topologies")

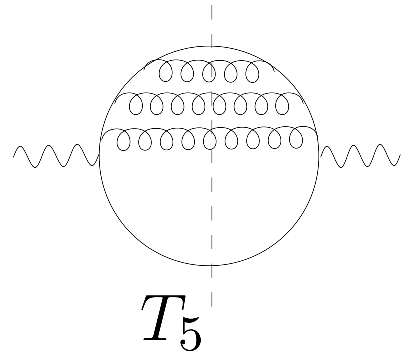
  (minimise square-root terms in denominators, maximize natural factorisations)

example from $e^+e^- \to 3$ jets

example:



$\Longrightarrow |\mathcal{M}|^2 \sim$



$+ \ldots$

convenient phase space parametrisation:

$s_{1345}/q^2 = t_1\,,\ s_{134}/q^2 = t_1\,t_2\,,\ s_{13}/q^2 = t_1\,t_2\,t_3\,,\ \ldots$

$s_{24}/q^2 = y_5^- + (y_5^+ - y_5^-)\,t_5\,,\ y_5^{\pm}$ solution of $\Delta_5 = 0$,

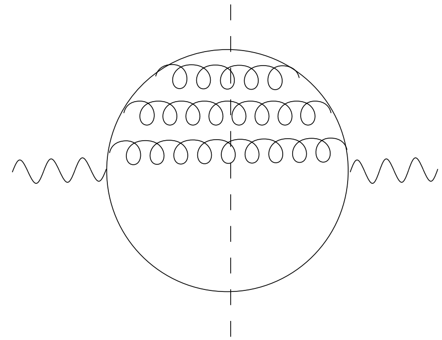
s_{25}, s_{45} also contain square-roots

$$\int d\Phi^D_{1\to 5} \;=\; \mathcal{K}_\Gamma^{(5)}(q^2)^{2D-5}\int_0^1 \prod_{j=1}^{9} dt_j\, t_1^{2-3\epsilon}\,[t_5(1-t_5)]^{-1-\epsilon}[t_8(1-t_8)t_9(1-t_9)]^{-\frac{1}{2}-\epsilon}$$

$$[(1-t_1)\,t_2(1-t_2)\,(1-t_3)]^{1-2\epsilon}[t_3\,t_4(1-t_4)t_6(1-t_6)t_7(1-t_7)]^{-\epsilon}$$

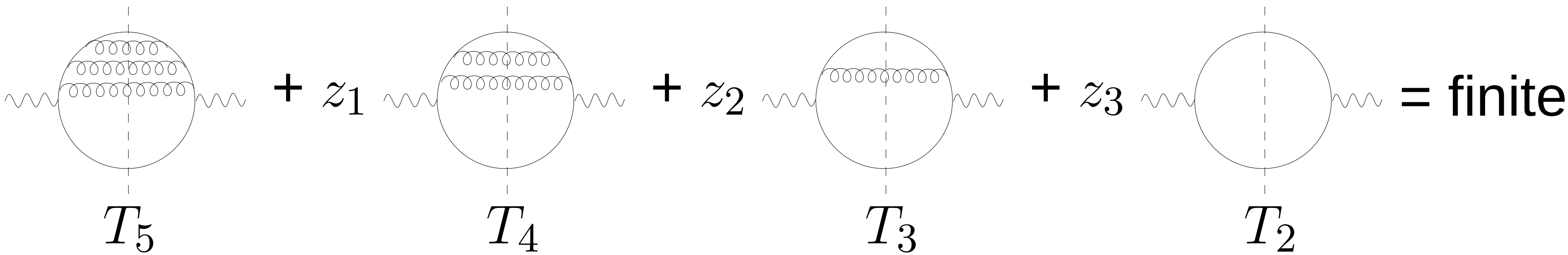
example from $e^+e^- \to 3$ jets

result for example graph T_5:



$$T_5 = C_F^3 \left(\frac{\alpha_s}{4\pi}\right)^3 T_2 \left\{ -\frac{0.166618 \pm 0.000086}{\epsilon^3} \right.$$
$$- \frac{1}{\epsilon^2} \left[(1.49928 \pm 0.00148) - (0.499853 \pm 0.000258) \log\left(\frac{Q^2}{\mu^2}\right) \right]$$
$$- \frac{1}{\epsilon} \left[(5.59588 \pm 0.0064) - (4.49783 \pm 0.0044) \log\left(\frac{Q^2}{\mu^2}\right) \right.$$
$$\left. \left. + (0.749779 \pm 0.00039) \log^2\left(\frac{Q^2}{\mu^2}\right) \right] + \text{finite} \right\}$$

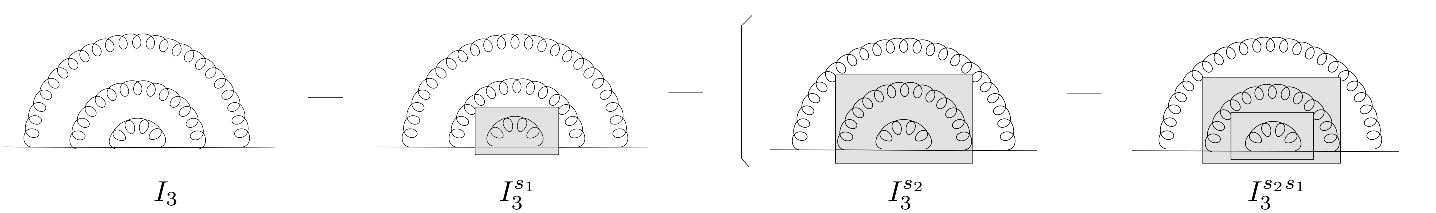
check by Kinoshita-Lee-Nauenberg theorem



renormalisation constants:

$$z_1 = C_F \frac{\alpha_s}{4\pi} \frac{1}{\epsilon} \;,\; z_2 = C_F^2 \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{1}{2\epsilon^2} - \frac{1}{4\epsilon}\right) \;,\; z_3 = C_F^3 \left(\frac{\alpha_s}{4\pi}\right)^3 \left(\frac{1}{6\epsilon^3} - \frac{1}{4\epsilon^2} + \frac{1}{6\epsilon}\right)$$

e.g. z_3 from



$-$ [overall divergence of expressions above]

check pole cancellations

$$z_1 T_4 + z_2 T_3 + z_3 T_2 =$$

$$C_F^3 \left(\frac{\alpha_s}{4\pi}\right)^3 T_2 \left\{ \frac{1}{6\epsilon^3} + \frac{1}{2\epsilon^2} \left[3 - \log\left(\frac{Q^2}{\mu^2}\right) \right] \right.$$

$$\left. + \frac{1}{\epsilon} \left[5.60813 - \frac{9}{2} \log\left(\frac{Q^2}{\mu^2}\right) + \frac{3}{4} \log^2\left(\frac{Q^2}{\mu^2}\right) \right] + \text{finite} \right\}$$

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differential results

- in order to compare to experimental data, we need to produce distributions which are differential in certain observables

differential results

- in order to compare to experimental data, we need to produce distributions which are differential in certain observables
- no problem: before numerical integration, pole coefficients and finite parts are functions of invariants $s_{12}, \ldots, s_{45}$

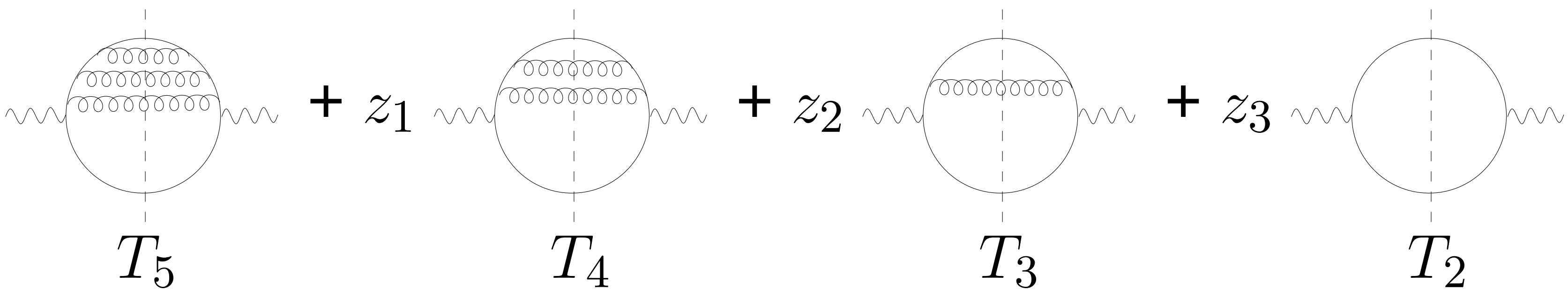
  $\Rightarrow$ 4-momenta of all final state particles can be reconstructed

  $\Rightarrow$ program has architecture of partonic event generator

  $\Rightarrow$ flexibility to include experimental cuts, jet algorithms, definition of shape observables, etc. at the Monte Carlo level

example

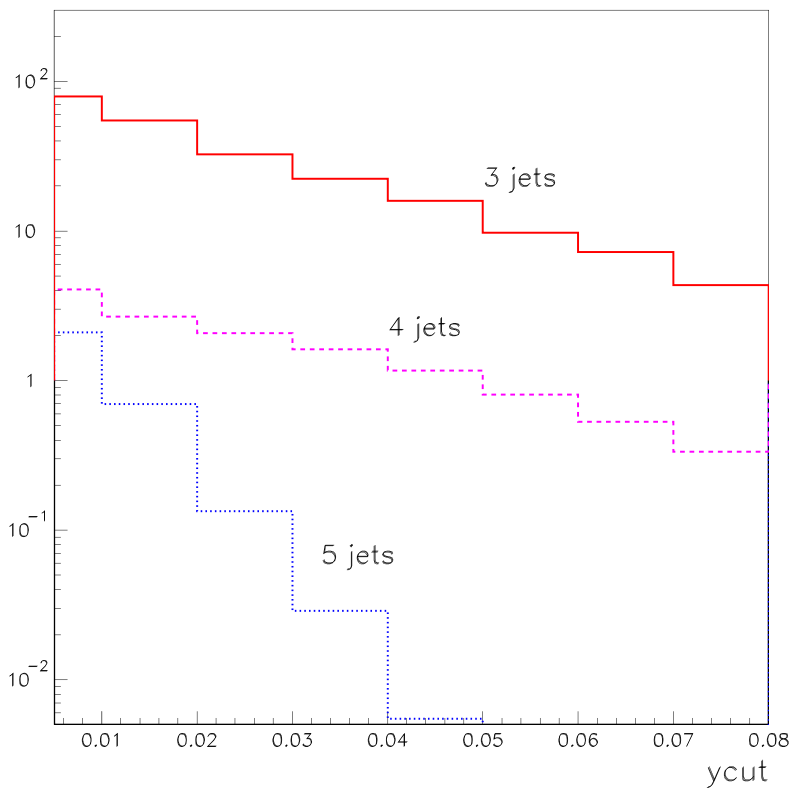
3,4,5-jet rates for "ladder" contribution



$$T_5 + z_1\, T_4 + z_2\, T_3 + z_3\, T_2$$

- virtual contributions calculated analytically
  (only renormalisation factors for ladder graph)
- phase space integrations for 3,4,5-parton final states by Monte Carlo program based on output of sector decomposition for T_4 and T_5

example: jet rates for ladder graph



y^{cut}: miminal separation between jets

in terms of invariant mass (JADE jet algorithm)

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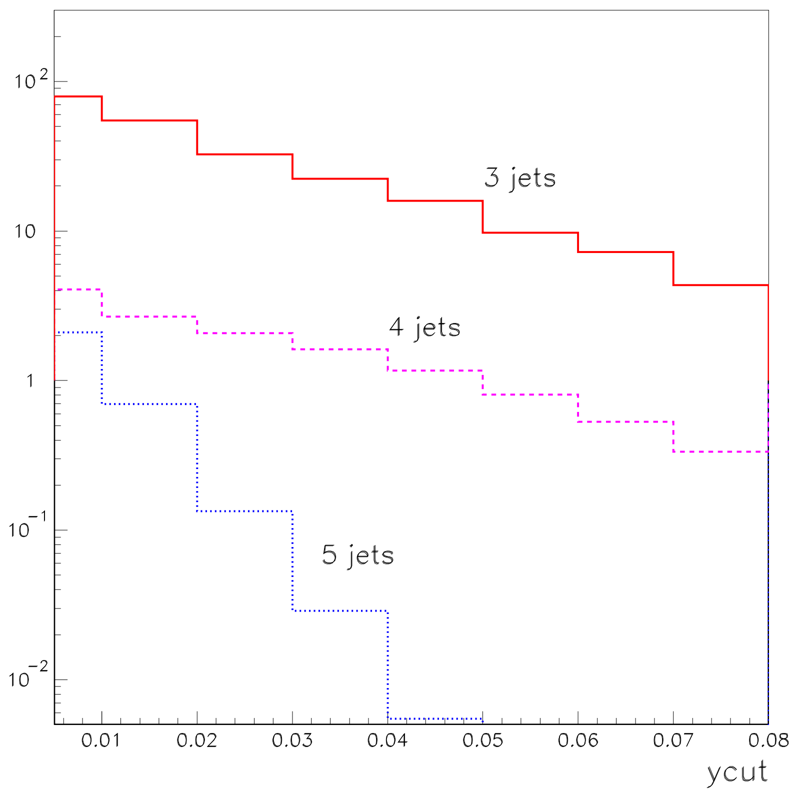

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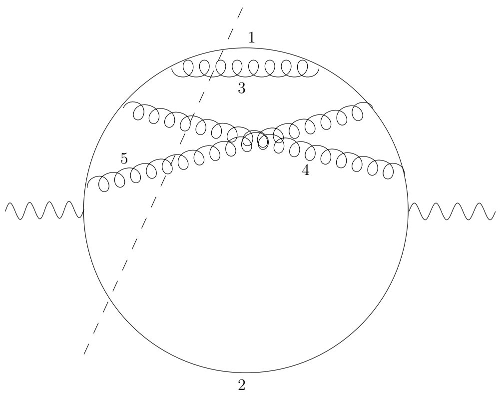
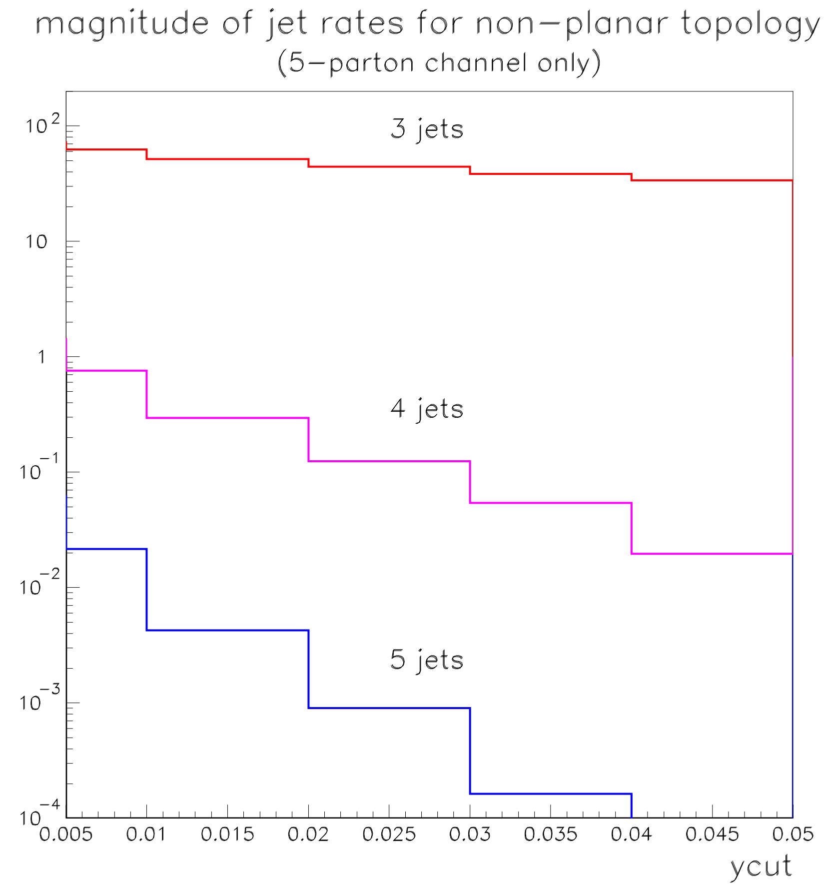
note:

- ladder graph only, not a physical observable, but architecture of program as a partonic event generator same for remaining parts
- CPU time $\mathcal{O}(10\,\text{min})$ for 1% precision (2.8 GHz Pentium IV)

a non-planar topology

more complex because square-root term in denominator unavoidable





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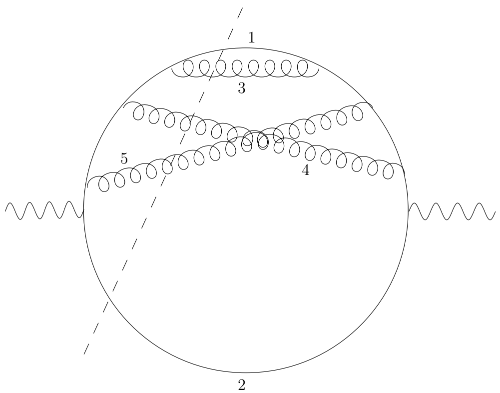

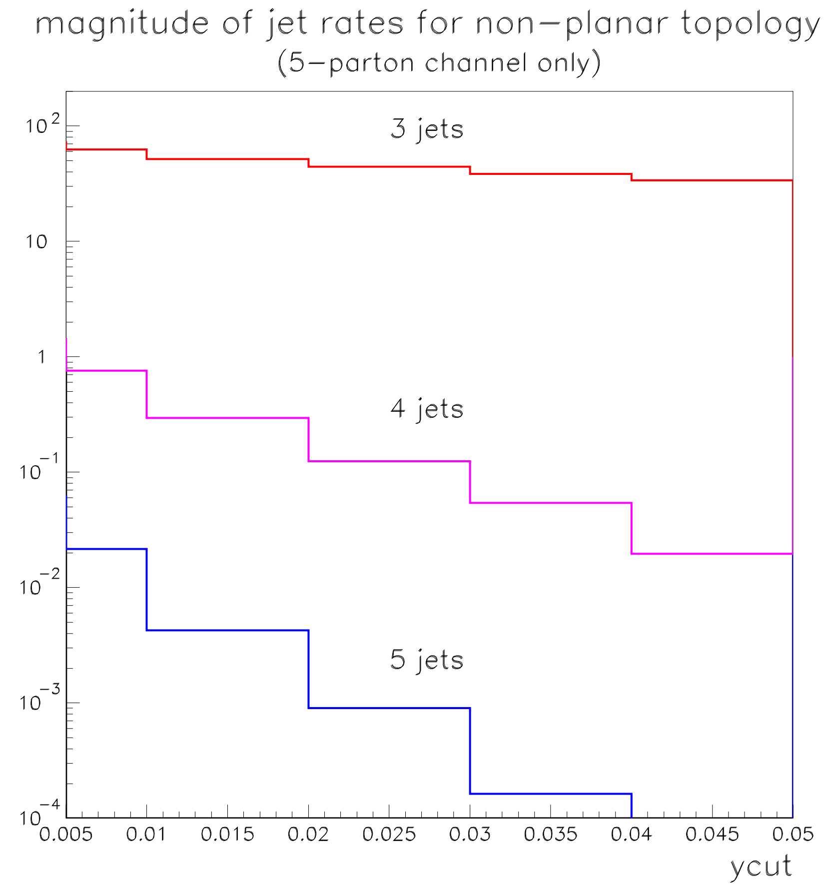

note:

- virtual corrections not yet included
- CPU time $\mathcal{O}$(2 h) for 1% precision
- other topologies can be calculated in parallel
- $\mathcal{O}(100)$ topologies
- $\mathcal{O}(10)$ PS parametrisations

roadmap

- include all topologies of $e^+e^- \to q\bar{q}ggg, q\bar{q}Q\bar{Q}g$ squared matrix elements

  limit size of expressions produced by iterated sector decomposition by using convenient phase space parametrisations and information on physical limits

- combine with one-loop + single real and two-loop virtual corrections
  - one-loop + single real: use a combination of analytic integration and sector decomposition
  - two-loop virtual: use analytic expressions to save CPU time

    [Garland, Gehrmann, Glover, Koukoutsakis, Remiddi '01],
    [Moch, Uwer, Weinzierl '02]

Summary: comparison of the two approaches

analytic subtraction

- ⊕ moderate number of subtraction terms
- ⊕ maximal (analytical) control over pole parts
- ⊕ insights into infrared structure of QCD
- ⊖ different for each colour structure
- ⊖ analytical integration of subtraction terms may become impossible for processes involving several mass scales

sector decomposition

- ⊖ produces large expressions
- ⊖ numerical cancellation of pole coefficients
- ⊕ high level of automatisation
- ⊕ basic algorithm same for all colour structures of a given process
- ⊕ subtraction terms are integrated numerically $\Rightarrow$ no need to have simple subtraction terms $\Rightarrow$ application to other processes easier

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the two methods are complementary

Summary: comparison of the two approaches

analytic subtraction	sector decomposition
⊕ moderate number of subtraction terms	⊖ produces large expressions
⊕ maximal (analytical) control over pole parts	⊖ numerical cancellation of pole coefficients
⊕ insights into infrared structure of QCD	⊕ high level of automatisation
⊖ different for each colour structure	⊕ basic algorithm same for all colour structures of a given process
⊖ analytical integration of subtraction terms may become impossible for processes involving several mass scales	⊕ subtraction terms are integrated numerically $\Rightarrow$ no need to have simple subtraction terms $\Rightarrow$ application to other processes easier

the sector decomposition approach has been/will be a useful tool for NNLO calculations where analytical methods reach their limits

backup

for Monte Carlo program: remap to energies and angles

$$E_1 = \frac{q^2 - s_{2345}}{2q}, \; E_2 = \frac{q^2 - s_{1345}}{2q},$$

$$E_3 = \frac{q^2 - s_{1245}}{2q}, \; E_4 = \frac{q^2 - s_{1235}}{2q}$$

$$\cos\theta_1 = -1 + 2\left(s_{1345}s_{2345} - s_{345}\right)/(1 - s_{1345})/(1 - s_{2345})$$

$$\cos\theta_2 = -1 + 2\left(s_{1245}s_{2345} - s_{245}\right)/(1 - s_{1245})/(1 - s_{2345})$$

$$\cos\theta_3 = -1 + 2\left(s_{1235}s_{2345} - s_{235}\right)/(1 - s_{1235})/(1 - s_{2345})$$

$$\vdots$$