Three-loop corrections to electroweak observables in the large Higgs mass limit

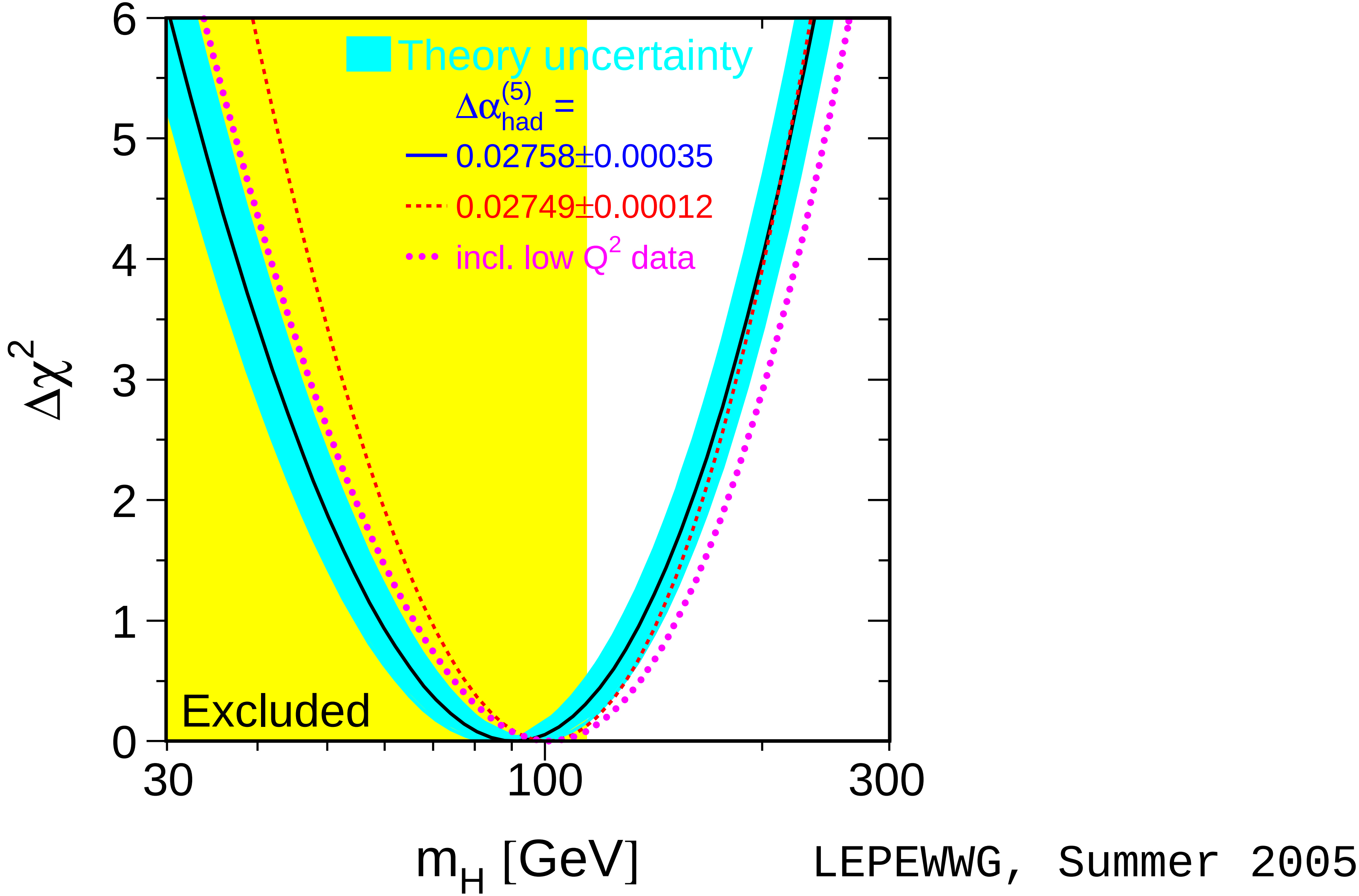
J.B. Tausk
University of Freiburg, Germany

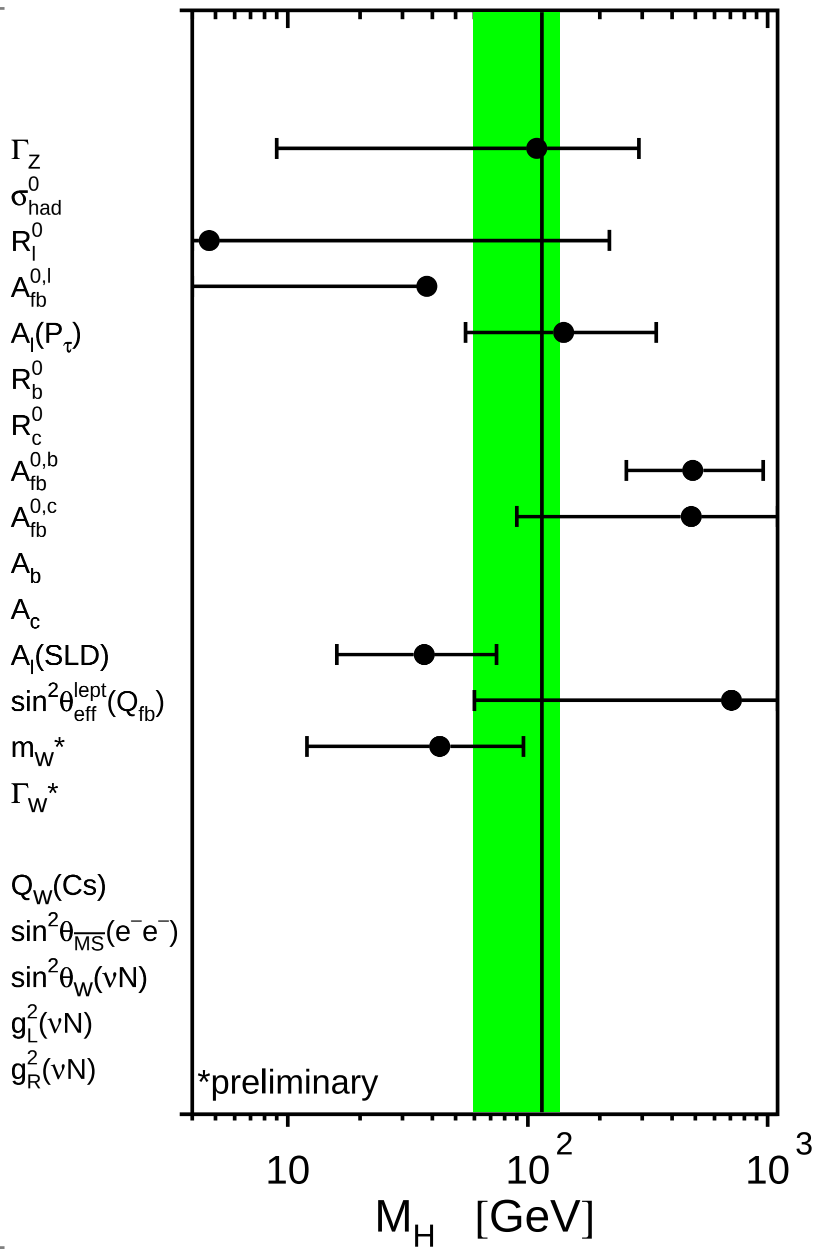
Loops and Legs, Eisenach, 28 April 2006

R. Boughezal, J. B. T., J. J. van der Bij, Nucl. Phys. B713 (2005) 278 [hep-ph/0410216] ;
Nucl. Phys. B725 (2005) 3 [hep-ph/0504092].

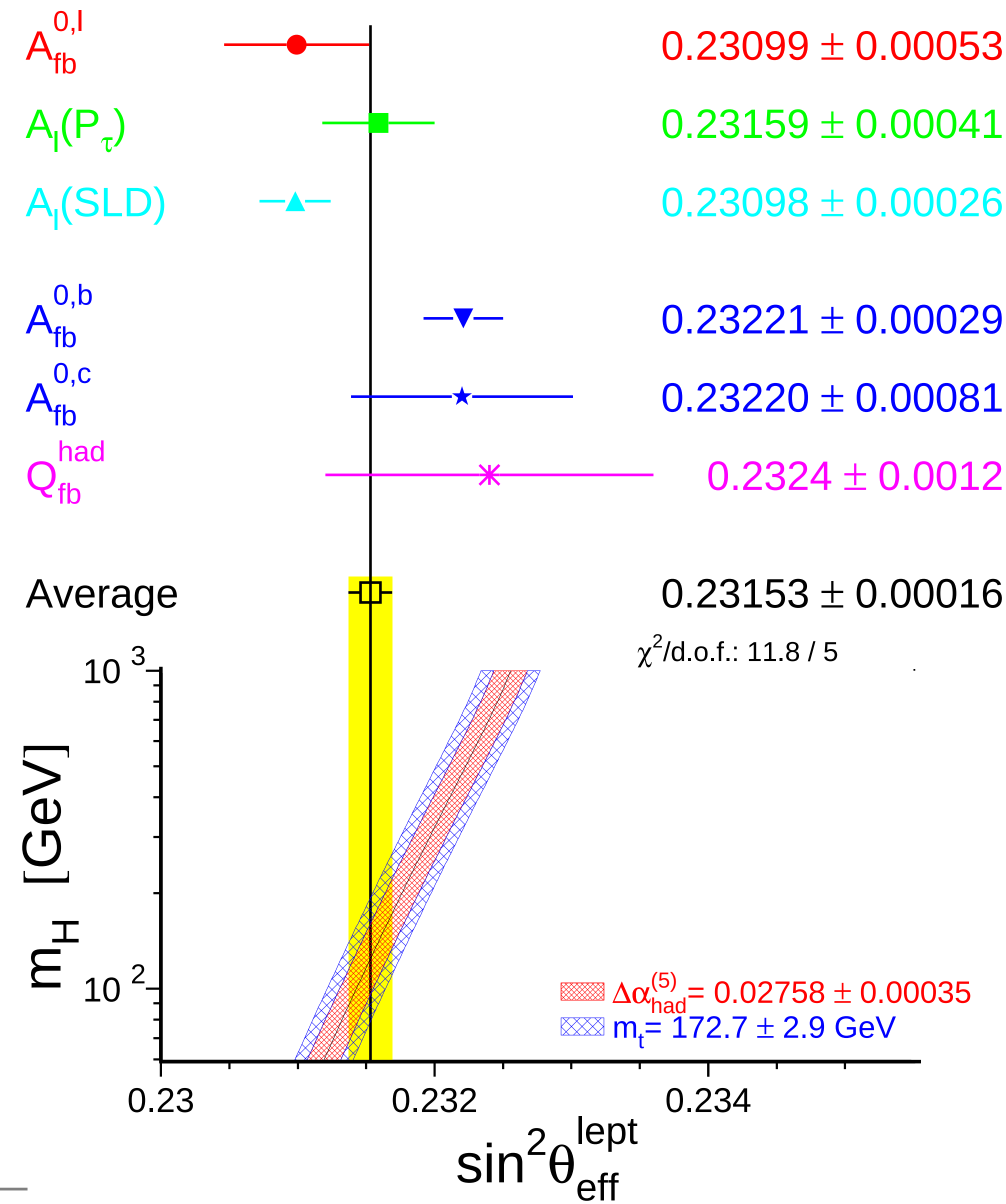
Introduction

- Indirect determinations of the Higgs boson mass are based on comparison of electroweak precision observables to theoretical predictions which depend on m_H through radiative corrections
- Higgs self-interaction $\lambda = g^2 m_H^2/(4M_W^2)$ grows quadratically with m_H
- Leading terms in heavy Higgs limit:
  - 1 loop: $g^2 \log(m_H^2/m_W^2)$
  - 2 loops: $g^4 m_H^2/m_W^2$
  - 3 loops: $g^6 m_H^4/m_W^4$


6
5
4
3
2
1
0
Theory uncertainty
$\Delta\alpha^{(5)}_{had}$ =
0.02758±0.00035
0.02749±0.00012
incl. low Q^2 data
Excluded
$\Delta\chi^2$
30
100
300
m_H [GeV]
LEPEWWG, Summer 2005



LEPEWWG, Summer 2005



LEPEWWG, Summer 2005

The ρ parameter

$$\boxed{\mu\text{-decay}} \quad : \quad \frac{G_F}{\sqrt{2}} \left[\bar{\nu}_\mu \gamma^\mu \left(1+\gamma_5\right) \mu\right] \left[\bar{e} \gamma_\mu \left(1+\gamma_5\right) \nu_e\right]$$

$$\boxed{\nu - e \text{ scattering}} \quad : \quad \frac{\rho\, G_F}{2\sqrt{2}} \left[\bar{\nu}_\mu \gamma^\mu \left(1+\gamma_5\right) \nu_\mu\right] \left[\bar{e} \gamma_\mu \left(1 - 4 s_W^2 + \gamma_5\right) e\right]$$

In the standard model (tree level):

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8\, M_W^2}\,, \qquad \frac{\rho\, G_F}{2\sqrt{2}} = \frac{g^2}{16\, c_W^2\, M_Z^2} \qquad \Rightarrow \qquad \rho = \frac{M_W^2}{c_W^2\, M_Z^2} = 1$$

At higher orders,

$$\rho = \frac{1}{1 - \Delta\rho}$$

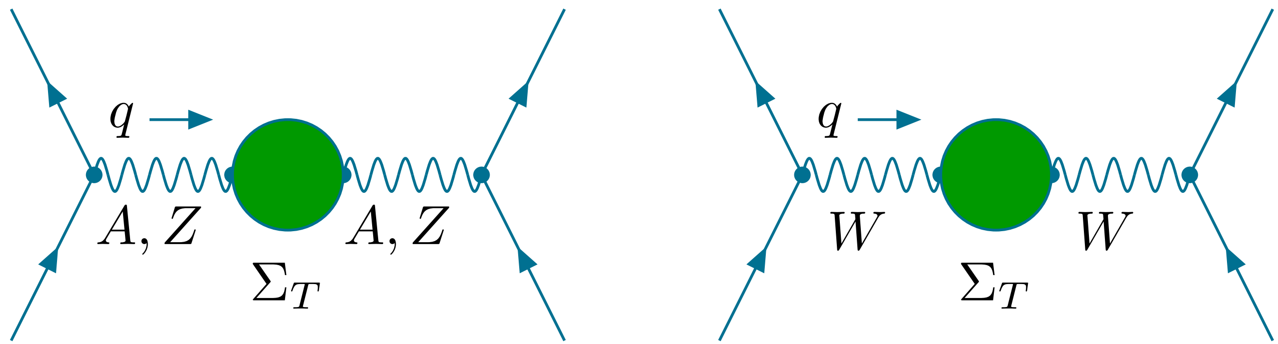
with

$$\Delta\rho = \frac{\Sigma_T^{ZZ}(0)}{M_Z^2} - \frac{\Sigma_T^{WW}(0)}{M_W^2}\,.$$

Oblique corrections to four-fermion processes

Neglect fermion masses $\Rightarrow$ no contributions from H or ϕ exchange diagrams

Radiative corrections to gauge boson propagators only



Expand in q^2:

$$\Sigma_T(q^2) = \Sigma_T(0) + q^2\,\Sigma_T'(0)$$

Work to linear order in Σ $\Rightarrow$ radiative corrections to observables are linear combinations of **S,T,U**

S, T, and U

$$S = \frac{4s_W^2 c_W^2}{\alpha}\left(\Sigma_T'^{ZZ} - \frac{c_W^2 - s_W^2}{c_W s_W}\Sigma_T'^{AZ} - \Sigma_T'^{AA}\right) = \frac{-4s_W c_W}{\alpha}\Sigma_T'^{3B}$$

$$T = \frac{1}{\alpha M_W^2}\left(c_W^2 \Sigma_T^{ZZ} - \Sigma_T^{WW}\right) = \frac{1}{\alpha M_W^2}\left(\Sigma_T^{33} - \Sigma_T^{WW}\right)$$

$$U = \frac{4s_W^2}{\alpha}\left(\Sigma_T'^{WW} - c_W^2 \Sigma_T'^{ZZ} - 2c_W s_W \Sigma_T'^{AZ} - s_W^2 \Sigma_T'^{AA}\right)$$

$$= \frac{4s_W^2}{\alpha}\left(\Sigma_T'^{WW} - \Sigma_T'^{33}\right) .$$

- $$\Delta\rho = \alpha T$$

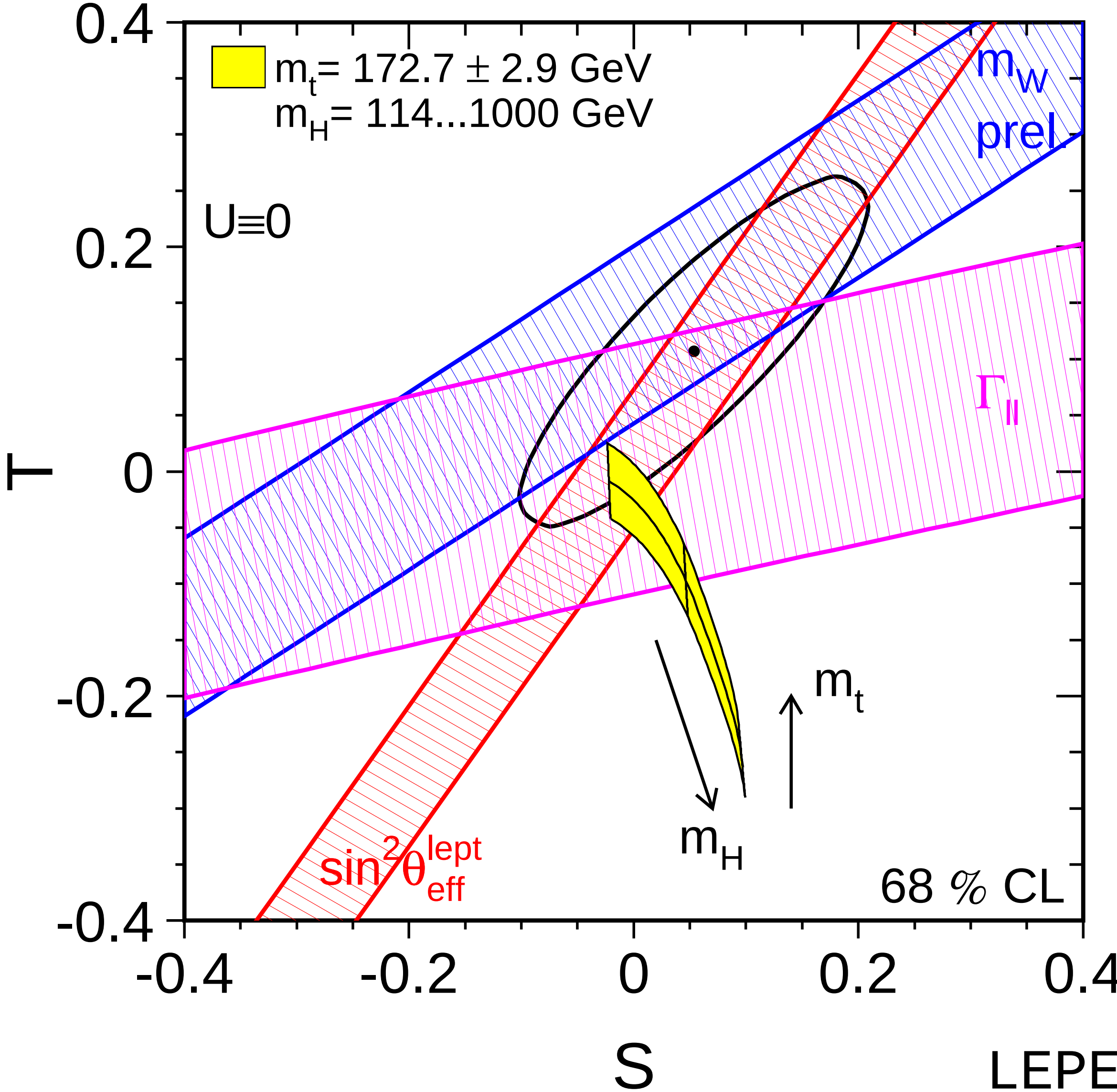
- The effective weak mixing angle is shifted relative to its tree level value, expressed in terms of α, G_F and M_Z, by

$$\Delta \sin^2\theta^{\text{lept}}_{\text{eff}} = \frac{\alpha}{c_W^2 - s_W^2}\left(\frac{1}{4}S - s_W^2 c_W^2 T\right) .$$

- Similarly, the W-mass is shifted by

$$\Delta M_W = \frac{\alpha M_W}{2(c_W^2 - s_W^2)}\left(-\frac{1}{2}S + c_W^2 T + \frac{c_W^2 - s_W^2}{4 s_W^2}U\right) .$$

The S and T parameters



LEPEWWG, Summer 2005

Calculation

- Generate diagrams for gauge boson self-energies with QGRAF `(P. Nogueira)`

No. of Loops	AA	AZ	ZZ	WW
1	8	8	12	17
2	414	456	616	792
3	54652	63852	82985	104340

- Expand in external momentum p up to order $p^2 \Longrightarrow$ vacuum diagrams
- Large mass expansion
- Reduction to master integrals
- Renormalization
- Limit $d \to 4$

Large mass expansion

1 : Divide the integration domain into different regions, depending on whether the propagator momenta are large or small in those regions; in each region, expand the integrand into Taylor series w.r.t. the parameters that are considered small there

2 : Integrate each expanded integrand over the *whole integration domain* and sum the resulting contributions from all the regions

$$F_\Gamma \sim \sum_{regions} T_s F_\Gamma$$

There are 15 $regions$ to consider for a 3-loop vacuum diagram !

The 15 regions

large momentum

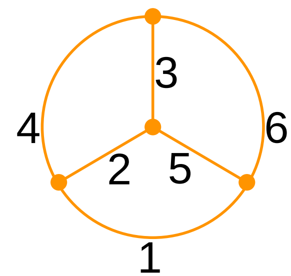
small momentum

$1 = k_1 \qquad 4 = k_1 + k_2$

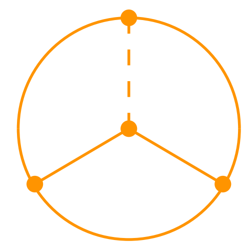
$2 = k_2 \qquad 5 = k_2 + k_3$

$3 = k_3 \qquad 6 = k_1 + k_2 + k_3$

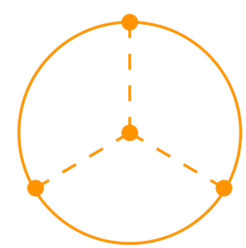
All momenta large

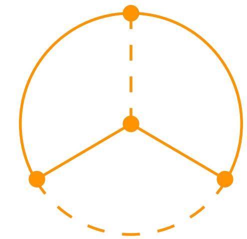


One momentum small, 6 regions

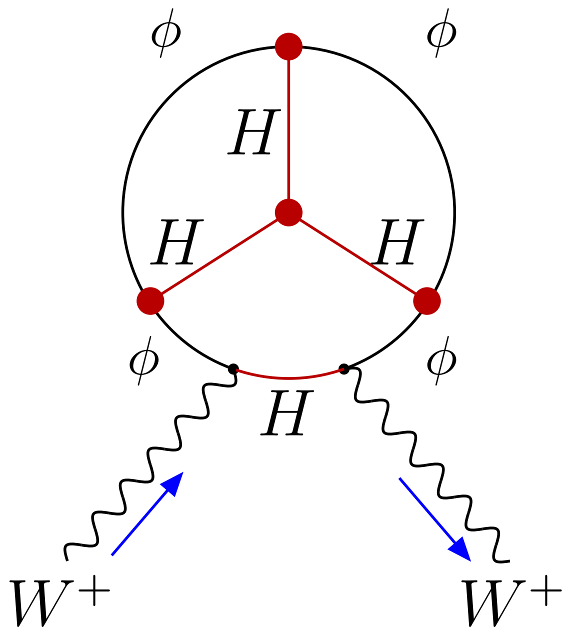


Two adjacent momenta small, 4 regions



Two non-adjacent momenta small, 3 regions	
All momenta small	

Three-loop example



$$\Gamma \sim g^6 \left(\frac{m_H^2}{M_W} \right)^4 \int \frac{d^d k_1 \, d^d k_2 \, d^d k_3 \quad k_1^2}{(k_1^2 + M_\phi^2)^2 \, (k_1^2 + m_H^2) \, (k_2^2 + m_H^2) \, (k_3^2 + m_H^2)}$$

$$\times \frac{1}{((k_1 + k_2)^2 + M_\phi^2) \, ((k_2 + k_3)^2 + m_H^2) \, ((k_1 + k_2 + k_3)^2 + M_\phi^2)}$$

Regions which give contributions that grow like m_H^4 or higher :

1. $k_1,\ k_2,\ k_3$ are of the same order as $m_H \Longrightarrow$ all momenta large

Expand propagators

$$\frac{1}{(k_1^2+M^2)^2},\ \frac{1}{((k_1+k_2)^2+M^2)},\ \frac{1}{((k_1+k_2+k_3)^2+M^2)}$$

in powers of M.

$$e.g. \qquad \mathcal{T}_M \frac{1}{(k_1^2+M^2)^2} \longrightarrow \frac{1}{(k_1^2)^2}\left(1-\frac{2M^2}{k_1^2}+\dots\right)$$

$\Longrightarrow$ three-loop *one-scale* (m_H) integral

2. k_1 is small, i.e $\ll m_H$

Expand propagators

$$\frac{1}{(k_1^2+m^2)}, \quad \frac{1}{((k_1+k_2)^2+M^2)}, \quad \frac{1}{((k_1+k_2+k_3)^2+M^2)}$$

in k_1 and M.

$$e.g. \qquad \mathcal{T}_{k_1}\frac{1}{(k_1^2+m^2)} \longrightarrow \frac{1}{m^2}\left(1-\frac{k_1^2}{m^2}+\ldots\right)$$

$\Longrightarrow$ The integral *factorises* into $1 \times\ 2-$loop integrals

3. k_1+k_2 is *small* $\Longrightarrow 1 \times\ 2-$loop integrals

4. $k_1+k_2+k_3$ is *small* $\Longrightarrow 1 \times\ 2-$loop integrals

Reduction to master integrals

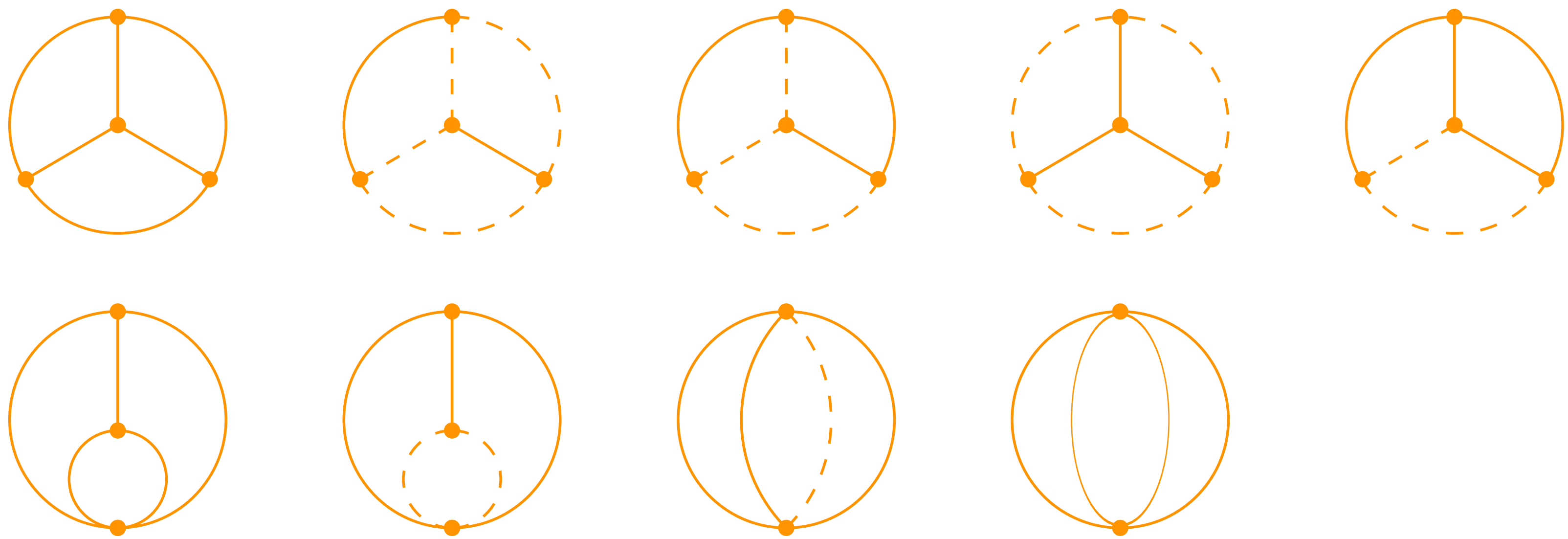
- Use Integration By Parts Identities, based on the fact that a D-dimensional integral over a total derivative is zero

$$\int d^D k \, \frac{\partial}{\partial k^\mu} f(k, ...) = 0$$

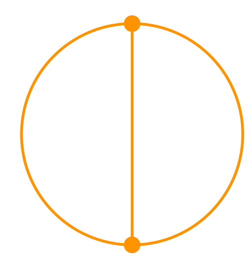
- With 3 independent momenta one can get 9 recurrence relations
- Combine them to get *reduction formulae* that reduce Feynman integrals from a given family to a set of independent *master integrals*
- Reduction of single-scale integrals crossed-checked with AIR (Automatic Integral Reduction)
  `C. Anastasiou, A. Lazopoulos, (2004)`
- Three-loop two-scale integrals reduced by AIR; they cancel in the final result so that explicit formulae for the corresponding master integrals are not needed.

Single-scale master integrals

- D.J. Broadhurst, (1999)
  J. Fleischer, M.Y. Kalmykov, (1999)



- J. van der Bij, M. Veltman, (1984)
  A.I. Davydychev, J.B.T., (1996)
  J. Fleischer, M.Y. Kalmykov, (1999)



Lagrangian

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{fermions} + \mathcal{L}_{YM} + \mathcal{L}_{scalar} + \mathcal{L}_{gf} + \mathcal{L}_{FP} \\ \mathcal{L}_{YM} &= -\frac{1}{4} W^a_{\mu\nu} W^{a,\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \\ \mathcal{L}_{scalar} &= -(D_\mu \Phi)^\dagger (D^\mu \Phi) - \frac{1}{2}\lambda \left(\Phi^\dagger \Phi\right)^2 - \mu\, \Phi^\dagger \Phi\,,\end{aligned}$$

$$\mu = -\frac{1}{2} m_H^2\,, \qquad \lambda = g^2 \frac{m_H^2}{4M_W^2}\,, \qquad v = \sqrt{2}\frac{M_W}{g}$$

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} H + \sqrt{2} v + i\phi^0 \\ i\phi^1 - \phi^2 \end{pmatrix}\,, \qquad D_\mu \Phi = \left(\partial_\mu - \frac{i\,g}{2} W^a_\mu \tau^a - \frac{i\,g'}{2} B_\mu\right) \Phi$$

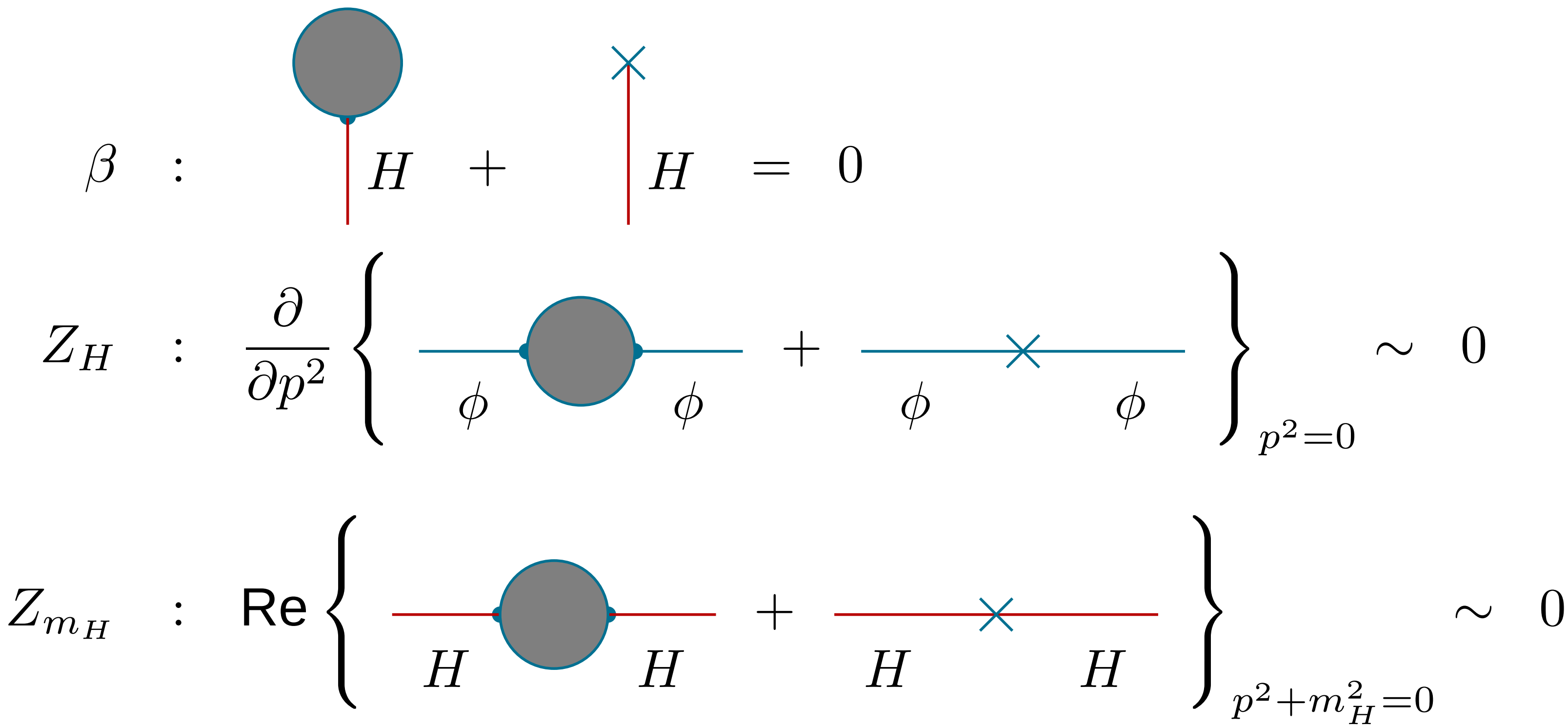
$$\begin{aligned} Z_\mu &= c_W W^3_\mu - s_W B_\mu\,, \\ A_\mu &= s_W W^3_\mu + c_W B_\mu\,, \end{aligned}$$

Renormalization

- Renormalize up to 2 loops
- Expand the renormalization factors in m_H, keeping only terms that give m_H^4 contributions to $S^{(3)}$, $T^{(3)}$, $U^{(3)}$.
- Replace in $\mathcal{L}_{inv}$:

$$
\begin{aligned}
\mu &\rightarrow \beta - \frac{1}{2} m_H^2 \\
m_H &\rightarrow Z_{m_H}\, m_H \\
\Phi &\rightarrow Z_H\, \Phi \;\Rightarrow\; \left\{ \begin{array}{ccc} M_W & \rightarrow & Z_H\, M_W \\ H & \rightarrow & Z_H\, H \\ \phi^{\pm} & \rightarrow & Z_H \phi^{\pm} \\ \phi^0 & \rightarrow & Z_H \phi^0 \end{array} \right.
\end{aligned}
$$

Renormalization conditions

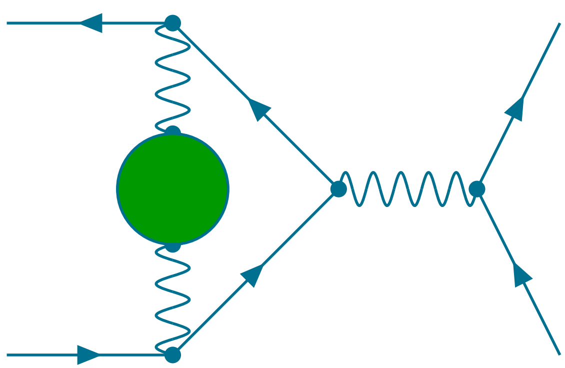


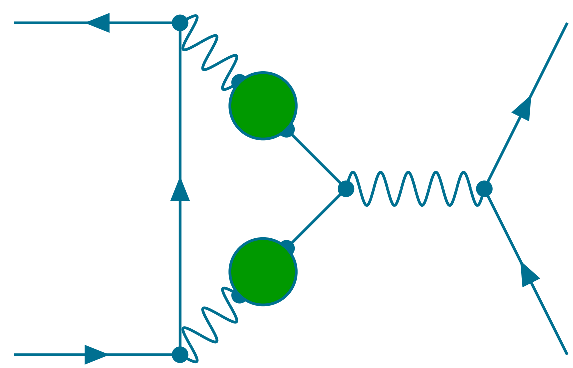
- β and Z_H are expressed in terms of vacuum integrals.
- Z_{m_H} calculated analytically by `V. Borodulin, G. Jikia, (1997)`

The counterterms remove all terms of order m_H^2 (m_H^4) in the one-loop (two-loop) self-energies

$$\Sigma^{\phi\phi}, \Sigma^{\phi^0\phi^0}, \Sigma^{\phi W}, \Sigma^{\phi^0 Z}, \Sigma^{ZZ}, \Sigma^{WW}$$

Therefore, no contributions from vertices like, e.g.





Checks

- Ward Identities checked in d dimensions, before expanding in ε
- All singular terms, of order $1/\varepsilon^j$, $j = 1, 2, 3, 4$, appearing in the ε-expansion, cancel
- Custodial symmetry implies that U should be zero at the leading order in m_H; this was verified explicitly

Results

$$
\begin{aligned}
\Delta\rho^{(1)} &= -\frac{3}{4}\,\frac{g^2}{16\pi^2}\,\frac{s_W^2}{c_W^2}\,\log\left(\frac{m_H^2}{M_W^2}\right)\,, \\
\Delta\rho^{(2)} &= \left(\frac{g^2}{16\pi^2}\right)^2 \frac{s_W^2}{c_W^2}\,\frac{m_H^2}{M_W^2}\left(-\frac{21}{64}+\frac{9}{32}\pi\sqrt{3}+\frac{3}{32}\pi^2-\frac{9}{8}C\sqrt{3}\right) \\
&= \left(\frac{g^2}{16\pi^2}\right)^2 \frac{s_W^2}{c_W^2}\,\frac{m_H^2}{M_W^2}\,(\,0.1499\,)\,, \\
\Delta\rho^{(3)} &= \left(\frac{g^2}{16\pi^2}\right)^3 \frac{s_W^2}{c_W^2}\,\frac{m_H^4}{M_W^4}\left(-\frac{21}{512}+\frac{729}{512}\,\pi\sqrt{3}-\frac{3391}{4608}\,\pi^2-\frac{9}{16}\pi\,C\right. \\
&\qquad -\frac{1577}{2304}\,\pi^3\sqrt{3}-\frac{9109}{69120}\,\pi^4+\frac{99}{16}\,\sqrt{3}\,\log 3\;C \\
&\qquad -\frac{297}{32}\sqrt{3}\,\mathsf{Ls}_3(2\pi/3)-\frac{399}{16}\sqrt{3}\,C+\frac{3043}{128}\,\zeta(3) \\
&\qquad \left.+\frac{567}{32}\,C^2+\frac{109}{8}\,U_{3,1}-36\,V_{3,1}\right) \\
&= \left(\frac{g^2}{16\pi^2}\right)^3 \frac{s_W^2}{c_W^2}\,\frac{m_H^4}{M_W^4}\,(\,-1.7282\,)\,.
\end{aligned}
$$

$$
\begin{aligned}
S^{(1)} &= \frac{1}{12\pi} \log\left(\frac{m_H^2}{M_W^2}\right) , \\
S^{(2)} &= \frac{1}{4\pi}\left(\frac{g^2}{16\pi^2}\right)\frac{m_H^2}{M_W^2}\left(-\frac{35}{72} - \frac{1}{8}\,\pi\sqrt{3} + \frac{7}{54}\,\pi^2\right) \\
&= \frac{1}{4\pi}\left(\frac{g^2}{16\pi^2}\right)\frac{m_H^2}{M_W^2}\,(\,0.1131\,) , \\
S^{(3)} &= \frac{1}{4\pi}\left(\frac{g^2}{16\pi^2}\right)^2\frac{m_H^4}{M_W^4}\left(\frac{1153}{576} - \frac{19}{48}\,\pi\sqrt{3} + \frac{13}{16}\,\pi C + \frac{2753}{10368}\,\pi^2 - \frac{109}{432}\,\pi^3\sqrt{3}\right. \\
&\qquad -\frac{7199}{155520}\,\pi^4 + \frac{7}{4}\,\sqrt{3}\,C\log 3 - \frac{21}{8}\,\sqrt{3}\,\mathsf{Ls}_3\left(\frac{2\pi}{3}\right) \\
&\qquad \left. -\frac{105}{16}\,\sqrt{3}\,C + \frac{38525}{3456}\,\zeta(3) - \frac{25}{24}\,C^2 - \frac{17}{18}\,U_{3,1} - 2\,V_{3,1}\right) \\
&= \frac{1}{4\pi}\left(\frac{g^2}{16\pi^2}\right)^2\frac{m_H^4}{M_W^4}\,(\,1.1105\,) .
\end{aligned}
$$

$$U_{3,1} = \frac{1}{2}\zeta(4) + \frac{1}{2}\zeta(2)\log^2 2 - \frac{1}{12}\log^4 2 - \mathsf{Li}_4\left(\frac{1}{2}\right) = -0.11787599965\,,$$

$$V_{3,1} = \sum_{m>n>0} \frac{(-1)^m \cos(2\pi n/3)}{m^3 n} = -0.03901272636\,,$$

$$C = \mathsf{Cl}_2\,(\pi/3) = 1.0149416064\,,$$

$$\mathsf{Ls}_3(\frac{2\pi}{3}) = -\int_0^{2\pi/3} \mathsf{d}\phi\ \log^2\left|2\sin\frac{\phi}{2}\right| = -2.1447672126\,.$$

$\Delta\rho$

m_H/M_W	$\Delta\rho^{(1)}$	$\Delta\rho^{(2)}$	$\Delta\rho^{(3)}$
2	$-\ 0.00078$	1.14×10^{-6}	-1.33×10^{-7}
5	-0.0018	7.14×10^{-6}	-5.20×10^{-6}
6	-0.0020	0.000010	-0.000011
7	-0.0022	0.000014	-0.000020
10	-0.0026	0.000029	-0.000083
15	-0.0031	0.000064	-0.00042
20	-0.0034	0.00011	-0.0013
25	-0.0036	0.00018	-0.0032

Corrections to ρ as a function of m_H/M_W

$\Delta \sin^2 \theta_{\mathrm{eff}}^{\mathrm{lept}}$

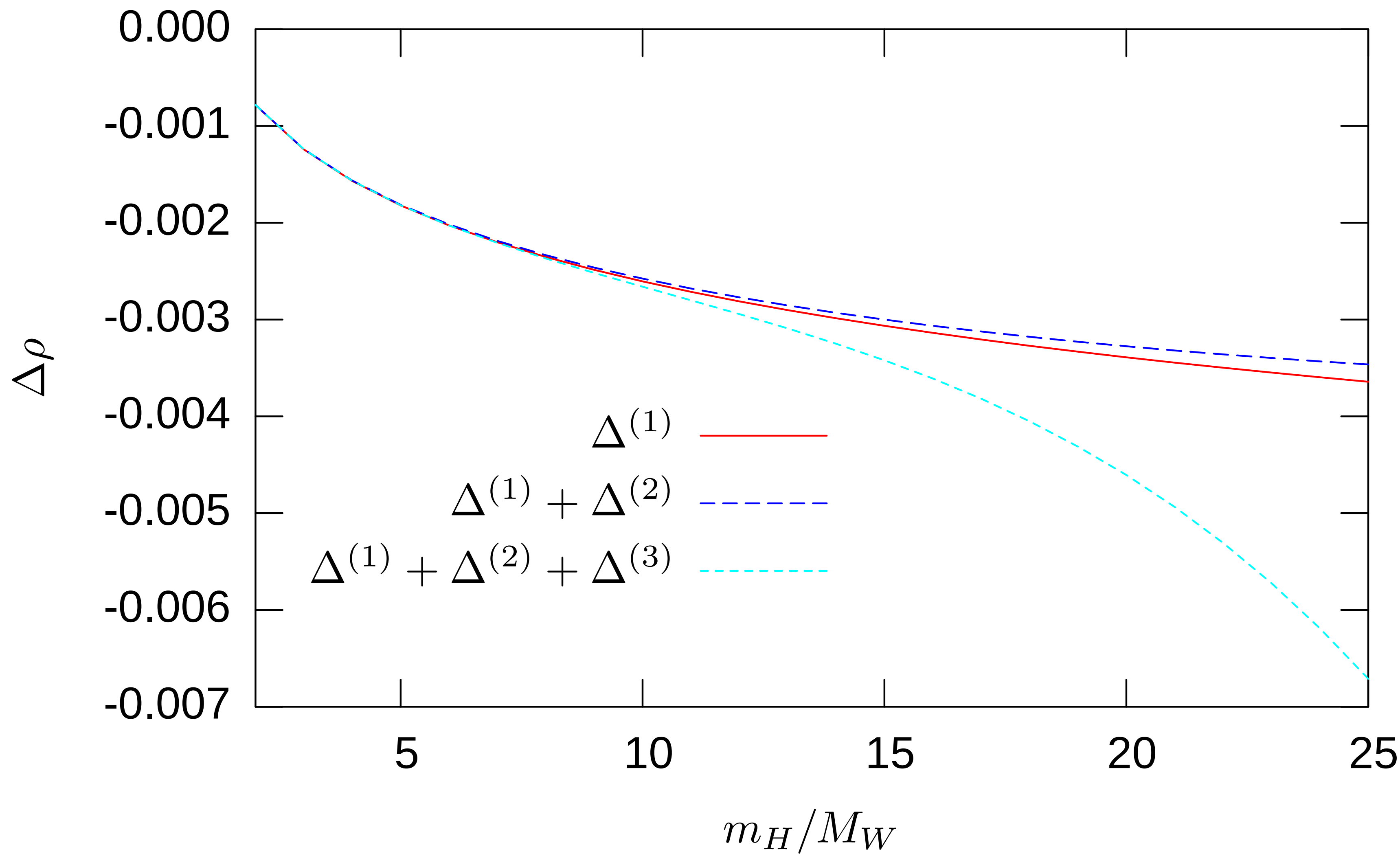
m_H/M_W	$\Delta^{(1)} \sin^2 \theta_{\mathrm{eff}}^{\mathrm{lept}}$	$\Delta^{(2)} \sin^2 \theta_{\mathrm{eff}}^{\mathrm{lept}}$	$\Delta^{(3)} \sin^2 \theta_{\mathrm{eff}}^{\mathrm{lept}}$
2	3.8×10^{-4}	-6.7×10^{-8}	7.4×10^{-8}
5	8.9×10^{-4}	-4.2×10^{-7}	2.9×10^{-6}
10	1.3×10^{-3}	-1.7×10^{-6}	4.6×10^{-5}
15	1.5×10^{-3}	-3.8×10^{-6}	2.3×10^{-4}
20	1.6×10^{-3}	-6.7×10^{-6}	7.4×10^{-4}
25	1.8×10^{-3}	-1.1×10^{-5}	1.8×10^{-3}

ΔM_W

m_H/M_W	$\Delta^{(1)}M_W$	$\Delta^{(2)}M_W$	$\Delta^{(3)}M_W$
2	-0.055	0.000041	-0.000010
5	-0.13	0.00025	-0.00039
10	-0.18	0.0010	-0.0063
15	-0.21	0.0023	-0.032
20	-0.24	0.0041	-0.10
25	-0.26	0.0064	-0.25

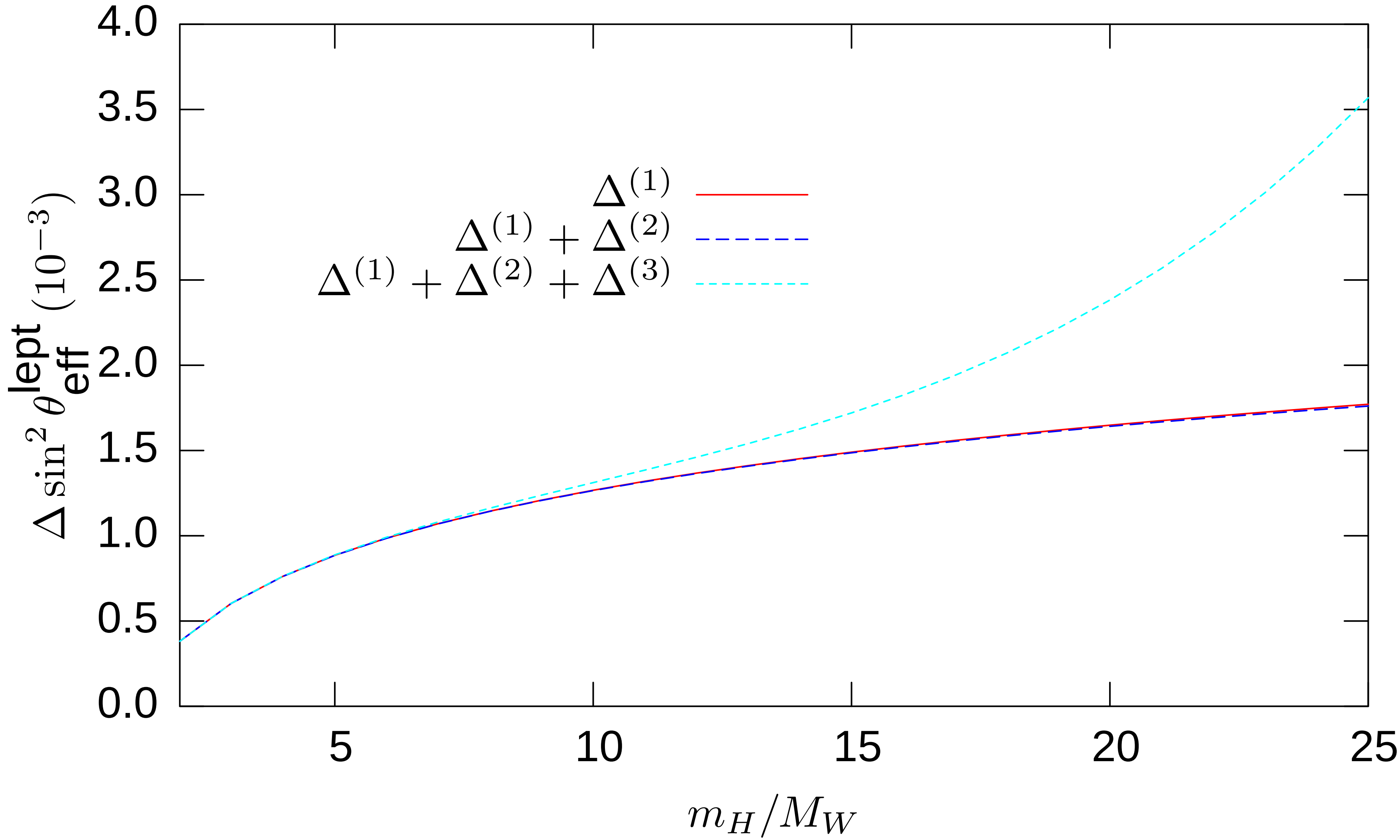
ΔM_W in GeV as a function of m_H/M_W.

$\Delta\rho$

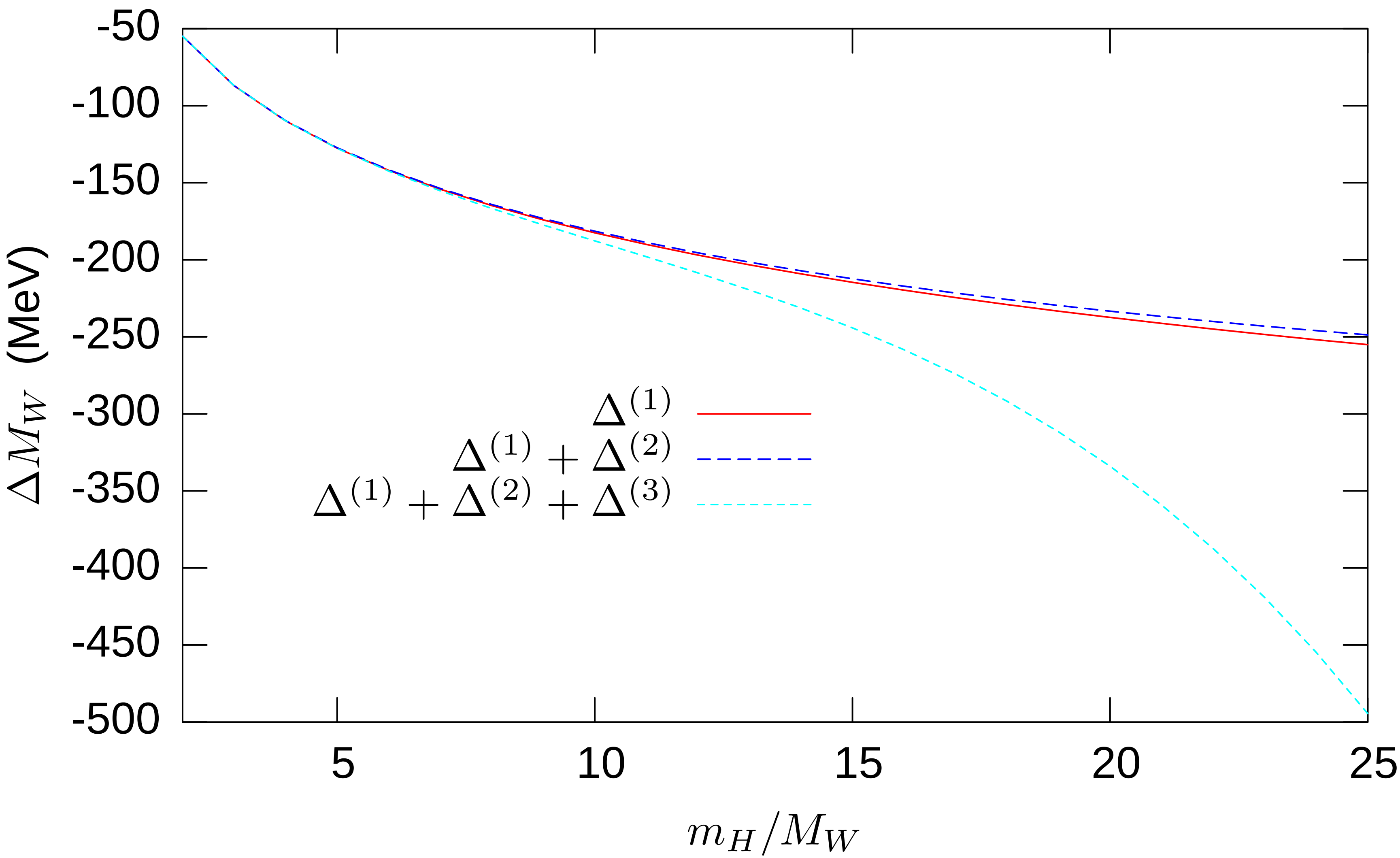


0.000
-0.001
-0.002
-0.003
-0.004
-0.005
-0.006
-0.007
$\Delta\rho$
5
10
15
20
25
m_H/M_W
$\Delta^{(1)}$
$\Delta^{(1)} + \Delta^{(2)}$
$\Delta^{(1)} + \Delta^{(2)} + \Delta^{(3)}$

The shift in $\sin^2\theta_{eff}$



The shift in M_W



Conclusion

- We have obtained analytical results for the leading three-loop contributions to the S and T parameters in the large Higgs mass limit.
- For the effective weak mixing angle, we find a positive shift:

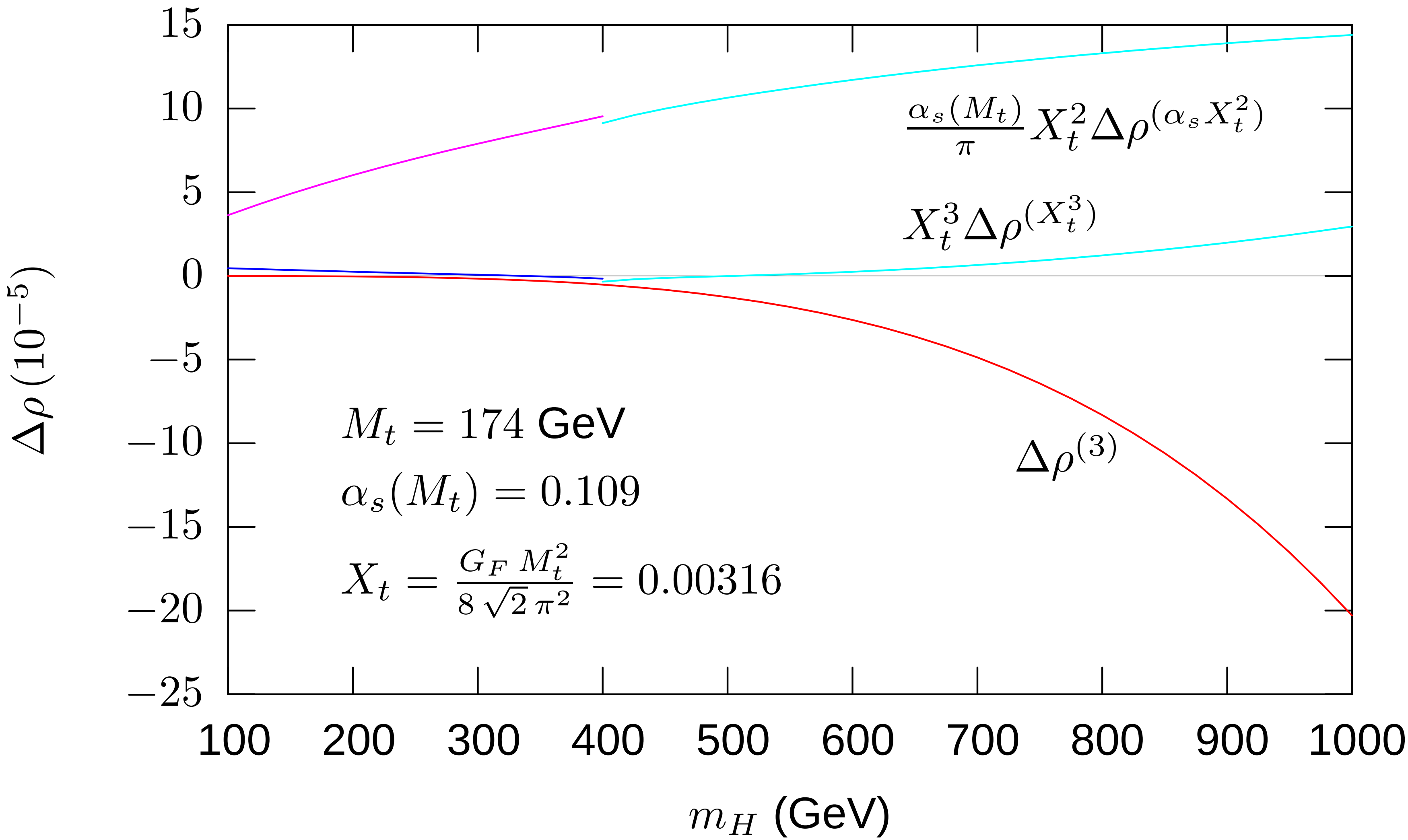
$$\Delta^{(3)} \sin^2 \theta^{\text{lept}}_{\text{eff}} = 4.6 \times 10^{-9} \times m_H^4 / M_W^4$$

- For the W boson mass, the shift is negative:

$$\Delta^{(3)} M_W = -6.3 \times 10^{-4} \text{MeV} \times m_H^4 / M_W^4$$

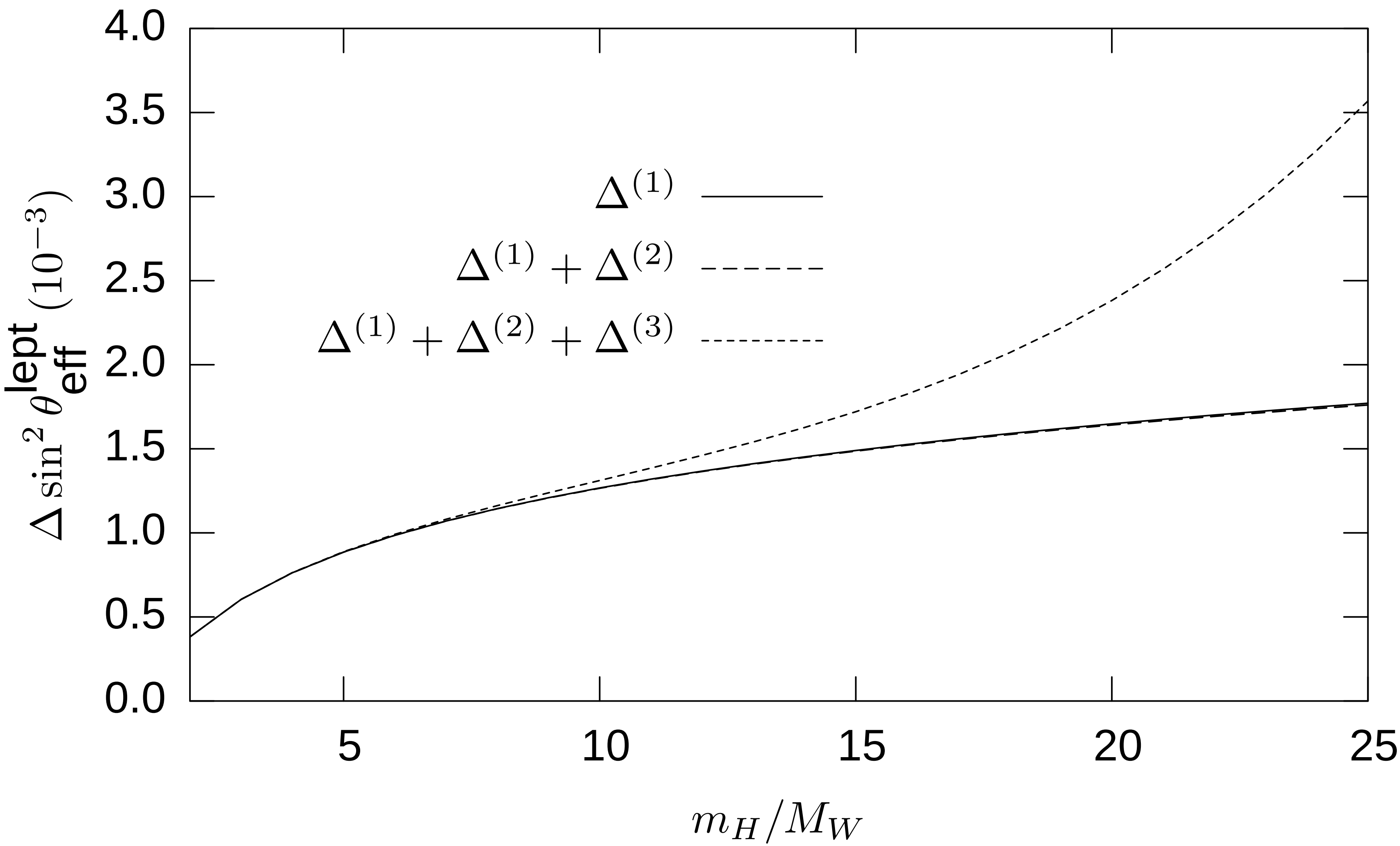
- In both cases, the two-loop contributions are very small, and the sign of the three-loop contributions is the same as the sign of the one-loop contributions.
- Radiative corrections due to a heavy Higgs boson do not mimic the effects of a light Higgs boson.

Large M_t vs Large m_H



Large M_t corrections : M. Faisst et al, Nucl.Phys. B665 (2003) 649

The shift in $\sin^2\theta_{eff}$



The shift in M_W