Gröbner bases as a tool to solve reduction problems for Feynman integrals: a review of recent results

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- Reduction problem for Feynman integrals
- A review of algorithmic approaches
- Gröbner bases and Buchberger algorithm
- Solving reduction problem using Gröbner bases
- Examples and results
- Perspectives

Work done in collaboration with A.V. Smirnov

[A.V. Smirnov & V.A. Smirnov, JHEP 0601 (2006) 001;

A.V. Smirnov, JHEP 0604 (2006) 026;

V.A. Smirnov, hep-ph/0601268 (a short review);

A.G. Grozin, A.V. Smirnov and V.A. Smirnov, in preparation]

see also

http://www.srcc.msu.ru/nivc/about/lab/lab4_2/index_eng.htm

Reduction problem for Feynman integrals

A given Feynman graph $\Gamma \to$ tensor reduction $\to$ various scalar Feynman integrals that have the same structure of the integrand with various distributions of powers of propagators

$$F(a_1,\ldots,a_n) \;=\; \int \cdots \int \frac{\mathrm{d}^d k_1 \ldots \mathrm{d}^d k_h}{E_1^{a_1} \ldots E_n^{a_n}}\,.$$

$d = 4 - 2\epsilon$; the denominators E_r are either quadratic or linear with respect to the loop momenta $p_i = k_i,\; i = 1,\ldots,h$ or to the independent external momenta $p_{h+1} = q_1,\ldots,p_{h+N} = q_N$ of the graph.

Methods: analytical, numerical, semianalytical . . .

An old analytical strategy:
to evaluate, by some methods, every scalar Feynman integral generated by the given graph.

A traditional strategy:

to derive, without calculation, and then apply integration by parts (IBP) relations [K.G. Chetyrkin and F.V. Tkachov'81]
between the given family of Feynman integrals as recurrence relations.

A general integral from the given family is expressed as a linear combination of some basic (master) integrals.

The whole problem of evaluation$\rightarrow$

- constructing a reduction procedure
- evaluating master integrals

No common definition of the master integrals.

After solving the reduction problem for a given family, *we know that these are master integrals because we see them*.

$F(a_1, \ldots, a_n)$ are functions of integer variables
$a_1, \ldots, a_n \in \mathbb{N}^n$
$\mathcal{F}$ is an infinitely dimensional linear space.
The simplest basis:

$$H_{a_1,\ldots,a_n}(a'_1, \ldots, a'_n) = \delta_{a_1,a'_1} \ldots \delta_{a_n,a'_n}$$

IBP:

$$\int \ldots \int \mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots \frac{\partial}{\partial k_i} \left(p_j \frac{1}{E_1^{a_1} \ldots E_n^{a_n}} \right) = 0 \,,$$

$\longrightarrow$

$$\sum \alpha_i F(a_1 + b_{i,1}, \ldots, a_n + b_{i,n}) = 0 \,,$$

Lorentz-invariance (LI) identities [T. Gehrmann and E. Remiddi'00] symmetry relations, e.g.,

$$F(a_1, \ldots, a_n) = (-1)^{d_1 a_1 + \ldots d_n a_n} F(a_{\sigma(1)}, \ldots, a_{\sigma(n)}),$$

Boundary conditions:

$$F(a_1, a_2, \ldots, a_n) = 0 \text{ when } a_{i_1} \leq 0, \ldots a_{i_k} \leq 0$$

for some subset of indices i_j;

parity conditions,...

All these relations can be described as elements of the adjoint vector space $\mathcal{F}^*$, i.e. the linear functionals on $\mathcal{F}$:

$$r \in \mathcal{F}^*,\ f \in \mathcal{F} \to \langle r, f \rangle$$

The simplest basis consists of $H^*_{a_1,\ldots,a_n}$:

$$\langle H^*_{a_1,\ldots,a_n}, f \rangle = f(a_1, \ldots, a_n) \ .$$

IBP, LI, symmetry and boundary relations,... $\to$
an infinitely dimensional vector subspace $\mathcal{R} \subset \mathcal{F}^*$.
The set of solutions of all those relations:

$$\mathcal{S} = \{ f \in \mathcal{F} : \langle r, f \rangle = 0 \ \forall\ r \in \mathcal{R} \}$$

The dimension of $\mathcal{S}$ might be infinite but, practically, it appears to be finite.

An integral $F(a_1,\ldots,a_n)$ can be expressed via some other integrals $F(a_1^1,\ldots,a_n^1)$, …, $F(a_1^k,\ldots,a_n^k)$ if there exists an element $r \in \mathcal{R}$ such that

$$\langle r, F\rangle = F(a_1,\ldots,a_n) + \sum k_{a_1',\ldots,a_n'} F(a_1',\ldots,a_n') \; .$$

The notion of the master (irreducible) integral $\to$ a priority between the points $(a_1,\ldots,a_n) \to$ *ordering*.
$(a_1',\ldots,a_n')$ is lower than the sector of $(a_1,\ldots,a_n)$
Feynman integrals are simpler, from the analytic point of view, if they have more non-positive indices.
Solving IBP relations by hand $\to$ reducing indices to zero

Sectors ('topologies'):

2^n regions labelled by subsets $\nu \subseteq \{1, \ldots, n\}$:

σ_ν = $\{(a_1, \ldots, a_n) : a_i > 0$ if $i \in \nu\,, \;\; a_i \leq 0$ if $i \notin \nu\}$

Sector $\to$ direction $\{d_1, d_2, \ldots, d_n\}$;

all d_i are 1 or -1, and $1 \leftrightarrow a_i > 0,\; -1 \leftrightarrow a_i \leq 0$

A direction $\{d_1, \ldots d_n\}$ is lower than $\{d'_1, \ldots d'_n\}$ if

$d_1 \leq d'_1, \ldots, d_n \leq d'_n$

$F(a_1, \ldots, a_n) \succ F(a'_1, \ldots, a'_n)$ if the sector of $(a'_1, \ldots, a'_n)$ is lower than the sector of $(a_1, \ldots, a_n)$.

To define an ordering completely introduce it in some way inside the sectors. At least the corner point with $a_i = 1$ and $a_i = 0$ is lower than any other point in the given sector.

$F(a_1,\ldots,a_n)$ is a master integral if there is no element $r \in \mathcal{R}$ acting on F as

$$\langle r, F\rangle = F(a_1,\ldots,a_n) + \sum k_{a'_1,\ldots,a'_n} F(a'_1,\ldots,a'_n)\;,$$

where all the points $(a'_1,\ldots,a'_n)$ are lower than $(a_1,\ldots,a_n)$.

A review of algorithmic approaches

Solving reduction by hand and in an algorithmic way.

‘Laporta’s algorithm’

[S. Laporta and E. Remiddi’96; S. Laporta’00; T. Gehrmann and E. Remiddi’01]

‘When increasing the total power of the denominator and numerator, the total number of IBP equations grows faster than the number of independent Feynman integrals. Therefore this system of equations sooner or later becomes overdetermined, and one obtains the possibility to perform a reduction to master integrals’

Let $\mathcal{F}_t \subset \mathcal{F}$ be the subspace of $\mathcal{F}$ generated by $H_{a_1,\ldots,a_n}$ where $\mid a_i \mid \leq t$ and $\mathcal{R}_t$ be the intersection of $\mathcal{R}$ with the subset of $\mathcal{F}^*$ generated by $H^*_{a_1,\ldots,a_n}$ where $\mid a_i \mid \leq t$.

The limit of the difference between the dimensions of $\mathcal{F}_t$ and $\mathcal{R}_t$ equals the dimension of $\mathcal{S}$, so that there is t such that $\mathcal{R}_t$ has 'enough' relations to express any given integral $F(a_1, \ldots, a_n)$ with $\mid a_i \mid \leq t$ in terms of master integrals.

Various implementations:

- one public version `AIR` [C. Anastasiou and A. Lazopoulos'04]
- several private versions

  [T. Gehrmann and E. Remiddi, M. Czakon, P. Marquard and D. Seidel, Y. Schröder, C. Sturm, A. Onishchenko, . . .]

A lot of applications!

Baikov's method

[P.A. Baikov'96,'97; V.A. Smirnov and M. Steinhauser'03]

The basic ingredient:

$$\int \ldots \int \frac{\mathrm{d}x_1 \ldots \mathrm{d}x_n}{x_1^{a_1} \ldots x_n^{a_n}} \left[P(x_1, \ldots, x_n)\right]^{(d-h-1)/2} ,$$

where P is constructed for a given family of integrals according to some rules and h is the number of loops. This representation is used to understand which integrals are master integrals and to construct the corresponding coefficient functions $c_i(a_1, \ldots, a_n)$

$$F(a_1, \ldots, a_n) = \sum_i c_i(a_1, \ldots, a_n) I_i ,$$

$I_i = F(a_{i1}, \ldots, a_{in})$.

For a candidate for a master integral with $a_{i1}, \ldots, a_{in} = 0$ or 1, use the basic parametric representation to construct a function that satisfies the relations $\mathcal{R}$ and that vanishes if $(a_1, \ldots, a_n)$ is lower than $(a_{i1}, \ldots, a_{in})$ (in particular, if it belongs to a lower sector).

Suppose that we know that the integrals with the indices $A^1 = (a_1^1, \ldots, a_n^1)$, ..., $A^k = (a_1^k, \ldots, a_n^k)$ are master integrals and we constructed the corresponding solutions of this type.

These functions form a basis of the solution space $\mathcal{S}$:

$$F = \sum_i k_i C_i \ .$$

Substitute all A_j and solve an upper-triangle system of linear equations

$$F(A_j) = \sum_i k_i C_i(A_j) \text{ for } i \leq j \; .$$

so the coefficients k_i are expressed in terms of $F(A_i)$.
k_i and $C_i \rightarrow$
evaluation of any Feynman integral of the given class.

Applications:

[K.G. Chetrykin, talk at LL'06; V.A. Smirnov and M. Steinhauser'03; B.A. Kniehl, A. Onishchenko, J.H. Piclum and M. Steinhauser'06]

Reduction using Gröbner bases

Historically, suggested by O.V. Tarasov [O.V. Tarasov'98]

Reduce the problem to differential equations by introducing a mass for every line,

$a_i \mathbf{i}^+ \rightarrow \frac{\partial}{\partial m_i^2}$

Applications:

- Two-loop self-energy diagrams with five general masses [O.V. Tarasov'04]
  (the previous solution was reproduced [O.V. Tarasov'97])
- Massless two-loop off-shell vertex diagrams
  [F. Jegerlehner and O.V. Tarasov'05]
  (in agreement with the solution of [T.G. Birthwright, E.W.N. Glover and P. Marquard'04] by Laporta's algorithm)

Janet bases [V.P. Gerdt'04]

This algorithm works (at the moment) only for Feynman graphs with two lines.

One more approach based on Gröbner bases

[A.V. Smirnov and V.A. Smirnov'05]

Gröbner bases and Buchberger algorithm

(classical definitions)

Let $\mathcal{A} = \mathbb{C}[x_1, \ldots, x_n]$ be the ring of polynomials of n variables $x_1, \ldots, x_n$ and $\mathcal{I} \subset \mathcal{A}$ be an ideal.

A classical problem: to construct an algorithm that shows whether a given element $g \in \mathcal{A}$ is a member of $\mathcal{I}$ or not.

This problem is solved easily if we have a Gröbner basis
Ordering $\rightarrow$ reduction modulo basis
(a generalization of division of polynomial in the case of one variable)

A basis $\{f_1, f_2, \ldots, f_k\}$ is called a Gröbner basis of $\mathcal{I}$ if any $g \in \mathcal{I}$ is reduced by the reduction procedure to zero.

Gröbner basis $\rightarrow$ an algorithm to verify whether an element $g \in \mathcal{A}$ is a member of $\mathcal{I}$.

If a given basis is not a Gröbner basis, use Buchberger algorithm to construct it.
The main procedure: taking S-polynomials of pairs of elements of the given basis.
Let $\hat{f}_i = wq_i$ and $\hat{f}_j = wq_j$ where $\hat{f}_{i,j}$ are highest terms; w, q_i and q_j are monomials and w is not a constant.
Define $S(f_i, f_j) = f_i q_j - f_j q_i$.

Solving reduction problem using Gröbner bases

Our algorithm: ordering on master integrals $\leftrightarrow$ ordering on the algebra of shift operators appearing in IBP relations

$$\int \ldots \int \mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots \frac{\partial}{\partial k_i} \left(p_j \frac{1}{E_1^{a_1} \ldots E_N^{a_N}} \right) = 0$$

$\longrightarrow$

$$\sum c_i F(a_1 + b_{i,1}, \ldots, a_n + b_{i,n}) = 0$$

The left-hand sides of IBP relations $\quad f_i \cdot F = 0$
are determined by some operators f_i expressed in terms of the operators of multiplication by the indices a_i and operators $Y_i = \mathbf{i}^+, Y_i^- = \mathbf{i}^-$:

$$(Y_i^\pm \cdot F)(a_1, \ldots, a_n) = F(a_1, \ldots, a_i \pm 1, \ldots, a_n)$$

Ordering with two additional conditions:
i) for any $a \in \mathbb{N}^n$ not equal to $(0, \ldots 0)$ one has $a \prec (0, \ldots 0)$
ii) for any $a, b, c \in \mathbb{N}^n$ one has $a \prec b$ if and only if $a + c \prec b + c$.

E.g., lexicographical ordering:
A set $(i_1, \ldots, i_n)$ is higher than a set $(j_1, \ldots, j_n)$,
$(i_1, \ldots, i_n) \succ (j_1, \ldots, j_n)$
if there is $l \leq n$ such that $i_1 = j_1$, $i_2 = j_2$, ..., $i_{l-1} = j_{l-1}$ and $i_l > j_l$.

Degree-lexicographical ordering: $(i_1, \ldots, i_n) \succ (j_1, \ldots, j_n)$ if $\sum i_k > \sum j_k$, or $\sum i_k = \sum j_k$ and $(i_1, \ldots, i_n) \succ (j_1, \ldots, j_n)$ in the sense of the lexicographical ordering.

An ordering can be defined by a matrix.

Lexicographical, degree-lexicographical and reverse degree-lexicographical ordering for $n = 5$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The ideal $\mathcal{I}$ of IBP relations generated by the elements f_i.

Let us think of $a_i > 0$,

$$F(a_1, a_2, \ldots, a_n) = Y_1^{a_1-1} \ldots Y_n^{a_n-1} F(1, 1, \ldots, 1)$$

In this case it is reasonable to get rid of the operators Y_i- by multiplying (from the left) the operators f_i by sufficiently large powers of the operators Y_i.

variables $x_i \to$ shift operators Y_i

Ordering of points $(a_1,\ldots,a_n)$ in $\mathbb{N}^n \rightarrow$
ordering of monomials of operators Y_i

For two monomials $M_1 = Y_1^{i_1-1}\ldots Y_n^{i_n-1}$ and
$M_2 = Y_1^{j_1-1}\ldots Y_n^{j_n-1}$
$(M_1\cdot F)(1,\ldots,1) \succ (M_2\cdot F)(1,\ldots,1)$ if and only if $M_1 \succ M_2$

The reduction problem $\rightarrow$
reduce the monomial $Y_1^{a_1-1}\ldots Y_n^{a_n-1}$ modulo the ideal of the IBP relations

$$Y_1^{a_1-1}\ldots Y_n^{a_n-1} = \sum r_i f_i + \sum c_{i_1,\ldots,i_n} Y_1^{i_1-1}\ldots Y_n^{i_n-1}$$

Apply to F at $a_1 = 1, \ldots, a_n = 1$ to obtain

$$F(a_1, a_2, \ldots, a_n) = \sum c_{i_1,\ldots,i_n} F(i_1, i_2, \ldots, i_n)\,,$$

where integrals on the right-hand side are master integrals.

Our algorithm: to build a set of special bases of the ideal of IBP relations (Gröbner-type bases)

Building elements with lowest possible degrees
$\leftrightarrow$
master integrals with minimal possible degrees.

Our algorithm

[A.S.& V.S'05]

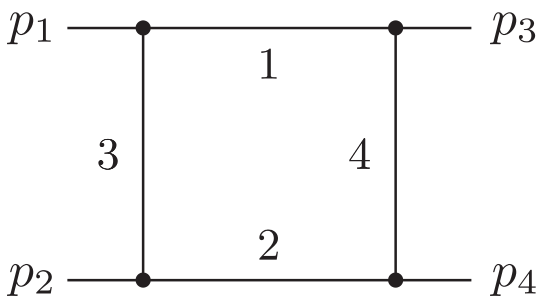
- sectors
  $\sigma_\nu = \{(a_1, \ldots, a_n) : a_i > 0 \text{ if } i \in \nu\,, \;\; a_i \leq 0 \text{ if } i \notin \nu\}$
- In the sector $\sigma_{\{1,\ldots,n\}}$, consider Y_i as basic operators.
  In the sector σ_ν, consider Y_i for $i \in \nu$ and Y_i^- for other i as basic operators.
- Construct sector bases (s-bases), rather than Gröbner bases for all the sectors.
  An s-basis for a sector σ_ν is a set of elements of a basis which provides the possibility of a reduction to master integrals *and* integrals whose indices lie in *lower* sectors, i.e. $\sigma_{\nu'}$ for $\nu' \subset \nu$. (It is most complicated to construct s-bases for minimal sectors.)

- The construction can be terminated when the Gröbner basis is not yet constructed but the 'current' basis already provides us the needed reduction.
- The basic operations are the same, i.e. calculating S-polynomials and reducing them modulo current basis, with a chosen ordering.

After constructing s-bases for all non-trivial sectors one obtains a recursive (with respect to the sectors) procedure to evaluate $F(a_1, \ldots, a_n)$ at any point and thereby reduce a given integral to master integrals.

Description of the algorithm (implemented in `Mathematica`): [A.V. Smirnov, JHEP 0604 (2006) 026]

Examples and results



$$
\begin{aligned}
sa_1Y_1 &= a_1 + 2a_2 + a_3 + a_4 - d + (a_1Y_1 + a_3Y_3 + a_4Y_4)Y_2^- = 0 \, , \\
sa_2Y_2 &= 2a_1 + a_2 + a_3 + a_4 - d + (a_2Y_2 + a_3Y_3 + a_4Y_4)Y_1^- = 0 \, , \\
ta_3Y_3 &= a_1 + a_2 + a_3 + 2a_4 - d + (a_1Y_1 + a_2Y_2 + a_3Y_3)Y_4^- = 0 \, , \\
ta_4Y_4 &= a_1 + a_2 + 2a_3 + a_4 - d + (a_1Y_1 + a_2Y_2 + a_4Y_4)Y_3^- = 0 \, ,
\end{aligned}
$$

where $s = (p_1 + p_2)^2$ and $t = (p_1 + p_3)^2$

Other examples can be found in [A.V. Smirnov & V.A. Smirnov'06; http://www.srcc.msu.ru/nivc/about/lab/lab4_2/index_eng.htm]

```
In[7]:= f1 = (a[1] + 2 ** a[2] + a[3] + a[4] - d) +
            (a[1] ** Y[1] + a[3] ** Y[3] + a[4] ** Y[4]) ** Ym[2] - s ** a[1] ** Y[1];
        f2 = (2 ** a[1] + a[2] + a[3] + a[4] - d) +
            (a[2] ** Y[2] + a[3] ** Y[3] + a[4] ** Y[4]) ** Ym[1] - s ** a[2] ** Y[2];
        f3 = (a[1] + a[2] + a[3] + 2 ** a[4] - d) +
            (a[1] ** Y[1] + a[2] ** Y[2] + a[3] ** Y[3]) ** Ym[4] - t ** a[3] ** Y[3];
        f4 = (a[1] + a[2] + 2 ** a[3] + a[4] - d) +
            (a[1] ** Y[1] + a[2] ** Y[2] + a[4] ** Y[4]) ** Ym[3] - t ** a[4] ** Y[4];

In[11]:= RESTRICTIONS = {{-1, 0, -1, 0 }, {-1, 0, 0, -1 }, {0, -1, -1, 0 }, {0, -1, 0, -1 }}

Out[11]= {{-1, 0, -1, 0}, {-1, 0, 0, -1}, {0, -1, -1, 0}, {0, -1, 0, -1}}

In[12]:= startinglist = {f1, f2, f3, f4}

Out[12]= {-d + a[1] + 2 a[2] + a[3] + a[4] - s a[1] Y[1] + a[1] Y[1] Ym[2] +
           a[3] Y[3] Ym[2] + a[4] Y[4] Ym[2], -d + 2 a[1] + a[2] + a[3] + a[4] -
           s a[2] Y[2] + a[2] Y[2] Ym[1] + a[3] Y[3] Ym[1] + a[4] Y[4] Ym[1],
          -d + a[1] + a[2] + a[3] + 2 a[4] - t a[3] Y[3] + a[1] Y[1] Ym[4] +
           a[2] Y[2] Ym[4] + a[3] Y[3] Ym[4], -d + a[1] + a[2] + 2 a[3] + a[4] -
           t a[4] Y[4] + a[1] Y[1] Ym[3] + a[2] Y[2] Ym[3] + a[4] Y[4] Ym[3]}

In[13]:= SYMMETRIES = {{2, 1, 3, 4}, {1, 2, 4, 3}, {2, 1, 4, 3}}

Out[13]= {{2, 1, 3, 4}, {1, 2, 4, 3}, {2, 1, 4, 3}}
```

In[14]:= **BuildBasis**$\left[\{1, 1, 1, 1\}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\right]$

```
No even restrictions set

No regularized lines

Initial data protected. Use ClearBasis to clear it and the basis from memory.

Dimension = 4

Using the code 76 search style (MinimizingLengthWhenSearchingQ = -1)

Evaluation limit is 200000

New element of length 9

Degree is {1, 0, 0, 0}

New element of length 9

Degree is {0, 1, 0, 0}

New element of length 9

Degree is {0, 0, 1, 0}

New element of length 9

Degree is {0, 0, 0, 1}

Saved element 1 of length 9

Saved element 2 of length 9

Saved element 3 of length 9

Saved element 4 of length 9

All tests done

Sorting

Permutation = {4, 3, 2, 1}

Sorting over

Evaluation time: 0.048991
```

In[15]:= **BuildBasis**$\left[\{-1, 1, 1, 1\}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\right]$

```
No even restrictions set
No regularized lines
Dimension = 4
Using the code 76 search style (MinimizingLengthWhenSearchingQ = -1)
Evaluation limit is 200000
New element of length 9
Degree is {0, 0, 0, 0}
New element of length 9
Degree is {1, 1, 0, 0}
New element of length 9
Degree is {0, 0, 1, 0}
New element of length 9
Degree is {0, 0, 0, 1}
Saved element 1 of length 9
Saved element 2 of length 9
Saved element 3 of length 9
Saved element 4 of length 9
All tests done
Sorting
Permutation = {1, 4, 3, 2}
Sorting over
Evaluation time: 0.051992
```

In[16]:= **BuildBasis**$\left[\{1, 1, -1, 1\}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\right]$

```
No even restrictions set
No regularized lines
Dimension = 4
Using the code 76 search style (MinimizingLengthWhenSearchingQ = -1)
Evaluation limit is 200000
```

```
New element of length 9

Degree is {1, 0, 0, 0}

New element of length 9

Degree is {0, 1, 0, 0}

New element of length 10

Degree is {1, 0, 0, 0}

New element of length 9

Degree is {1, 0, 1, 0}

Saved element 1 of length 9

Saved element 2 of length 9

Test results: {True, True, False, False}

Sorting

Permutation = {2, 1, 3, 4}

Sorting over

Trying to reduce elements

Reducing basis element 4 of length 9

{1, 0, 1, 0}

{d, 1, 10, 3, 10}

{0, 1, 1, 0}

{d, 1, 19, 1, 10}

{0, 0, 1, 1}

Reduction over

Degree is {0, 0, 1, 1}

Element reduced. New length: 21

Test results: {True, True, False, False}

Sorting

Permutation = {1, 2, 3, 4}

Sorting over

Reducing basis element 2 of length 9

{1, 0, 0, 0}

{d, 3, 10, 1, 11}

{0, 1, 0, 0}
```

```
same degree

{d, 1, 18, 1, 10}

{0, 0, 0, 0}

Reduction over

Degree is {0, 0, 0, 0}

Element reduced. New length: 21

Saved element 3 of length 21

Saved element 4 of length 21

All tests done

New element of length 9

Degree is {1, 0, 0, 0}

Sorting

Permutation = {2, 1, 3, 5, 4}

Sorting over

Evaluation time: 0.162975
```

In[17]:= **BuildBasis**$\left[\{-1, -1, 1, 1\}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\right]$

```
No even restrictions set

No regularized lines

Dimension = 4

Using the code 76 search style (MinimizingLengthWhenSearchingQ = -1)

Evaluation limit is 200000

New element of length 9

Degree is {0, 1, 1, 0}

New element of length 9

Degree is {1, 0, 1, 0}

New element of length 9

Degree is {0, 0, 1, 0}

New element of length 9

Degree is {0, 0, 0, 1}

Saved element 3 of length 9
```

```
Saved element 4 of length 9

Test results: {False, False, True, True}

Sorting

Permutation = {4, 3, 1, 2}

Sorting over

Trying to reduce elements

Reducing basis element 4 of length 9

{1, 0, 1, 0}

{d, 1, 10, 3, 10}

{1, 0, 0, 1}

Reduction over

Degree is {1, 0, 0, 1}

Element reduced. New length: 18

Test results: {False, False, True, True}

Sorting

Permutation = {1, 2, 4, 3}

Sorting over

Reducing basis element 4 of length 9

{0, 1, 1, 0}

{d, 1, 10, 3, 10}

{0, 1, 0, 1}

Reduction over

Degree is {0, 1, 0, 1}

Element reduced. New length: 18

Test results: {False, False, True, True}

Sorting

Permutation = {1, 2, 4, 3}

Sorting over
```

$$\begin{pmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 2 & 2 \\ 1 & 2 & 0 & 2 \\ 1 & 2 & 2 & 0 \end{pmatrix}$$

```
{1, 3}
```

```
{0, 1, 1, 0}

{d, 1, 24, 3, 10}

{0, 1, 0, 0}

New element of length 30

Degree is {0, 1, 0, 0}

Saved element 2 of length 30

Testing element 1

Testing element 2

Testing element 3

Testing element 4

Testing element 5

Test results: {False, True, True, True}

Sorting

Permutation = {1, 2, 5, 3, 4}

Sorting over

Trying to reduce elements

No elements reduced
```

$$\begin{pmatrix} 0 & 2 & 2 & 0 & 1 \\ 2 & 0 & 2 & 2 & 2 \\ 2 & 2 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 & 2 \\ 1 & 2 & 2 & 2 & 0 \end{pmatrix}$$

```
{3, 4}

{0, 1, 0, 2}

New element of length 94

Degree is {0, 1, 0, 2}

Testing element 6

Test results: {False, True, True, True}

Sorting

Permutation = {1, 2, 3, 4, 5, 6}

Sorting over

Trying to reduce elements

No elements reduced
```

$$\begin{pmatrix} 0 & 2 & 0 & 0 & 1 & 1 \\ 2 & 0 & 2 & 2 & 2 & 2 \\ 0 & 2 & 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 & 2 & 1 \\ 1 & 2 & 2 & 2 & 0 & 2 \\ 1 & 2 & 1 & 1 & 2 & 0 \end{pmatrix}$$

```
{1, 5}

{1, 0, 1, 0}

{d, 1, 24, 3, 10}

{1, 0, 0, 0}

New element of length 30

Degree is {1, 0, 0, 0}

Testing element 7

Saved element 1 of length 30

All tests done

Sorting

Permutation = {1, 2, 3, 7, 4, 5, 6}

Sorting over

Evaluation time: 0.6679
```

In[18]:= **BuildBasis**$\left[\{1, 1, -1, -1\}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]$

```
No even restrictions set

No regularized lines

Dimension = 4

Using the code 76 search style (MinimizingLengthWhenSearchingQ = -1)

Evaluation limit is 200000

New element of length 9

Degree is {1, 0, 0, 0}

New element of length 9

Degree is {0, 1, 0, 0}

New element of length 9

Degree is {0, 1, 0, 1}

New element of length 9

Degree is {0, 1, 1, 0}
```

```
Saved element 1 of length 9

Saved element 2 of length 9

Test results: {True, True, False, False}

Sorting

Permutation = {1, 2, 4, 3}

Sorting over

Trying to reduce elements

Reducing basis element 4 of length 9

{0, 1, 0, 1}

{d, 1, 10, 3, 10}

{1, 0, 0, 1}

Reduction over

Degree is {1, 0, 0, 1}

Element reduced. New length: 18

Test results: {True, True, False, False}

Sorting

Permutation = {1, 2, 4, 3}

Sorting over

Reducing basis element 4 of length 9

{0, 1, 1, 0}

{d, 1, 10, 3, 10}

{1, 0, 1, 0}

Reduction over

Degree is {1, 0, 1, 0}

Element reduced. New length: 18

Test results: {True, True, False, False}

Sorting

Permutation = {1, 2, 4, 3}

Sorting over
```

$$\begin{pmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 2 & 2 \\ 1 & 2 & 0 & 2 \\ 1 & 2 & 2 & 0 \end{pmatrix}$$

```
{1, 3}

{0, 1, 1, 0}

{d, 1, 24, 3, 10}

{0, 0, 1, 0}

New element of length 30

Degree is {0, 0, 1, 0}

Saved element 3 of length 30

Testing element 1

Testing element 2

Testing element 3

Testing element 4

Testing element 5

Test results: {True, True, True, False}

Sorting

Permutation = {1, 2, 5, 3, 4}

Sorting over

Trying to reduce elements

No elements reduced
```

$$\begin{pmatrix} 0 & 2 & 2 & 0 & 1 \\ 2 & 0 & 2 & 2 & 2 \\ 2 & 2 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 & 2 \\ 1 & 2 & 2 & 2 & 0 \end{pmatrix}$$

```
{3, 4}

{2, 0, 1, 0}

New element of length 94

Degree is {2, 0, 1, 0}

Testing element 6

Test results: {True, True, True, False}

Sorting

Permutation = {1, 2, 3, 4, 5, 6}

Sorting over

Trying to reduce elements

No elements reduced
```

$$\begin{pmatrix} 0 & 2 & 0 & 0 & 1 & 1 \\ 2 & 0 & 2 & 2 & 2 & 2 \\ 0 & 2 & 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 & 2 & 1 \\ 1 & 2 & 2 & 2 & 0 & 2 \\ 1 & 2 & 1 & 1 & 2 & 0 \end{pmatrix}$$

```
{1, 5}

{0, 1, 0, 1}

{d, 1, 24, 3, 10}

{0, 0, 0, 1}

New element of length 30

Degree is {0, 0, 0, 1}

Testing element 7

Saved element 4 of length 30

All tests done

Sorting

Permutation = {1, 2, 3, 7, 4, 5, 6}

Sorting over

Evaluation time: 0.644902
```

```
In[19]:= F[{1, 1, 1, 1}]
```

```
Direction set to {1, 1, 1, 1}

{0, 0, 0, 0}

Trying to reduce with lower members

G[{1, 1, 1, 1}] is a master integral

Coefficient: 1
```

```
Out[19]= G[{1, 1, 1, 1}]
```

```
In[21]:= F[{1, 1, 1, 2}] /. {G[{1, 1, 1, 1}] → I1, G[{1, 1, 0, 0}] → I2, G[{0, 0, 1, 1}] → I3}
```

Direction set to {1, 1, 1, 1}

{0, 0, 1, 0}

{0, 0, 0, 0}

G[{1, 1, 1, 1}] is a master integral

Coefficient: $-\frac{-5+d}{t}$

Direction set to {1, 1, -1, 1}

{1, 0, 0, 0}

{0, 0, 0, 1}

{0, 0, 0, 0}

Direction set to {1, 1, -1, -1}

{1, 0, 0, 0}

{0, 0, 0, 0}

Trying to reduce with lower members

G[{1, 1, 0, 0}] is a master integral

Coefficient: $\frac{4\,(-5+d)\,(-3+d)}{(-6+d)\,s^2\,t}$

Out[21]= $-\frac{(-5+d)\,I1}{t}+\frac{4\,(-5+d)\,(-3+d)\,I2}{(-6+d)\,s^2\,t}$

```
In[22]:= F[{1, 0, 1, 2}] /. {G[{1, 1, 1, 1}] → I1, G[{1, 1, 0, 0}] → I2, G[{0, 0, 1, 1}] → I3}
```

Direction set to {-1, 1, 1, 1}

{0, 0, 1, 0}

Direction set to {-1, -1, 1, 1}

{0, 0, 2, 0}

{0, 0, 1, 1}

{0, 0, 1, 0}

{0, 0, 0, 1}

{0, 0, 0, 0}

Trying to reduce with lower members

G[{0, 0, 1, 1}] is a master integral

Coefficient: $\frac{2\,(-3+d)}{t^2}$

Out[22]= $\frac{2\,(-3+d)\,I3}{t^2}$

```
In[23]:= F[{1, 1, 0, 2}] /. {G[{1, 1, 1, 1}] → I1, G[{1, 1, 0, 0}] → I2, G[{0, 0, 1, 1}] → I3}

         Direction set to {1, 1, -1, 1}

         {0, 0, 0, 1}

         {0, 0, 0, 0}

         Direction set to {1, 1, -1, -1}

         {0, 0, 0, 0}

         G[{1, 1, 0, 0}] is a master integral
```

Coefficient: $\frac{4\,(-3+d)}{(-6+d)\,s^2}$

Out[23]= $\frac{4\,(-3+d)\,I2}{(-6+d)\,s^2}$

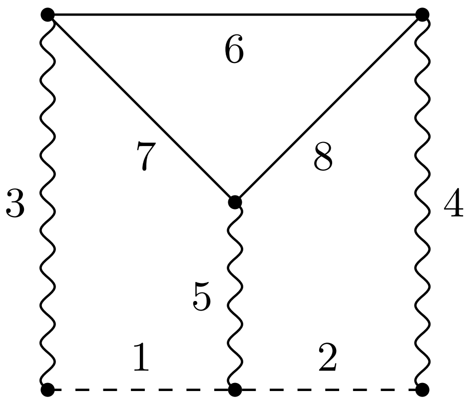
```
In[24]:= F[{2, 3, 4, 2}]

In[25]:= %
```

Out[25]= $-\big((-11+d)\,(-9+d)\,(-7+d)\,(-5+d)\,(-3+d)\,(-1536\,s^2+544\,d\,s^2-60\,d^2\,s^2+2\,d^3\,s^2-12640\,s\,t+4824\,d\,s\,t-676\,d^2\,s\,t+42\,d^3\,s\,t-d^4\,s\,t-5760\,t^2+1776\,d\,t^2-180\,d^2\,t^2+6\,d^3\,t^2)\,G[\{0,0,1,1\}]\big)/(3\,(-8+d)\,(-6+d)\,s^4\,t^7)-\big((-11+d)\,(-9+d)\,(-7+d)\,(-5+d)\,(-3+d)\,(53760\,s^3-20416\,d\,s^3+2864\,d^2\,s^3-176\,d^3\,s^3+4\,d^4\,s^3+155520\,s^2\,t-71712\,d\,s^2\,t+13056\,d^2\,s^2\,t-1184\,d^3\,s^2\,t+54\,d^4\,s^2\,t-d^5\,s^2\,t+32256\,s\,t^2-14400\,d\,s\,t^2+2320\,d^2\,s\,t^2-160\,d^3\,s\,t^2+4\,d^4\,s\,t^2-3072\,t^3+1088\,d\,t^3-120\,d^2\,t^3+4\,d^3\,t^3)\,G[\{1,1,0,0\}]\big)/(3\,(-12+d)\,(-8+d)\,(-6+d)\,s^7\,t^5)+\frac{1}{12\,s^4\,t^5}\big((-11+d)\,(-10+d)\,(-9+d)\,(-7+d)\,(-5+d)\,(-56\,s^2+4\,d\,s^2-176\,s\,t+24\,d\,s\,t-d^2\,s\,t-72\,t^2+6\,d\,t^2)\,G[\{1,1,1,1\}]\big)$

Reduction of a family of Feynman integrals relevant to the three-loop static quark potential [A.V. Smirnov and V.A. Smirnov'05]

A family of Feynman integrals with 9 indices

[A.V. Smirnov & A.G. Grozin, A.V. Smirnov and V.A. Smirnov, in preparation]



$$F(a_1,\ldots,a_9) = \int\int \frac{\mathrm{d}^d k\,\mathrm{d}^d l}{(-2v\cdot k)^{a_1}(-2v\cdot l)^{a_2}(-k^2)^{a_3}(-l^2)^{a_4}[-(k-l)^2]^{a_5}}$$

$$\times \int \frac{(2v\cdot r)^{-a_9}\mathrm{d}^d r}{(-r^2+m^2)^{a_6}[-(k+r)^2+m^2]^{a_7}[-(l+r)^2+m^2]^{a_8}}\,.$$

Symmetry:
$(1 \leftrightarrow 2, 3 \leftrightarrow 4, 7 \leftrightarrow 8)$
Boundary conditions: $F(a_1, \ldots, a_9) = 0$ if one of the following sets of lines has non-positive indices: $\{5,7\}$, $\{5,8\}$, $\{6,7\}$, $\{6,8\}$, $\{7,8\}$, $\{3,4,6\}$.
Master integrals:

$$
\begin{aligned}
I_1 &= F(1,1,0,1,1,1,1,0,0), \quad I_2 = F(1,1,1,1,0,0,1,1,0), \\
I_3 &= F(1,1,0,0,0,1,1,1,0), \\
I_4 &= F(0,1,1,0,1,1,0,1,0), \quad \bar{I}_4 = F(-1,1,1,0,1,1,0,1,0), \\
I_5 &= F(0,0,0,1,1,1,1,0,0), \quad I_6 = F(0,1,0,0,0,1,1,1,0), \\
I_7 &= F(0,1,0,0,1,1,1,0,0), \quad \bar{I}_7 = F(0,2,0,0,1,1,1,0,0), \\
I_8 &= F(0,0,0,0,0,1,1,1,0).
\end{aligned}
$$

```
In[13]:= BuildBasis[{-1, -1, -1, -1, -1, 1, 1, 1, -1},
```

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

```
, SearchStyle → 0, UsingSymmetries → 1]
```

```
Even restrictions set

No regularized lines

Initial data protected. Use ClearBasis to clear it and the basis from memory.

Dimension = 9

Local symmetries: {{{2, 1, 4, 3, 5, 6, 8, 7, 9}, {1, 1, 1, 1, 1, 1, 1, 1, 1}}}

Using the code 100 search style (MinimizingLengthWhenSearchingQ = 0)

Evaluation limit is 200000

New element of length 6

Degree is {1, 0, 0, 0, 0, 0, 1, 0, 0}

New element of length 6

Degree is {0, 1, 0, 0, 0, 0, 0, 1, 0}

New element of length 6

Degree is {1, 0, 0, 0, 0, 0, 1, 0, 0}

New element of length 9

Degree is {0, 0, 1, 0, 0, 0, 1, 0, 0}

New element of length 9

Degree is {0, 0, 0, 1, 0, 0, 0, 1, 0}

New element of length 13

Degree is {0, 0, 0, 0, 0, 1, 0, 0, 0}

New element of length 10

Degree is {0, 0, 0, 0, 1, 0, 1, 0, 0}

New element of length 10

Degree is {0, 0, 0, 0, 1, 0, 0, 1, 0}

New element of length 11

Degree is {0, 0, 0, 0, 0, 0, 1, 0, 0}
```

```
New element of length 11

Degree is {0, 0, 0, 0, 0, 0, 0, 1, 0}

New element of length 11

Degree is {0, 0, 1, 0, 0, 1, 0, 0, 0}

New element of length 11

Degree is {0, 0, 0, 1, 0, 1, 0, 0, 0}

Saved element 6 of length 13

Saved element 7 of length 11

Saved element 8 of length 11

Test results: {False, False, False, False, False, True, True, True, False}

Sorting

Permutation = {10, 9, 6, 8, 7, 5, 12, 4, 11, 2, 1, 3}

Sorting over

Trying to reduce elements

Reducing basis element 11 of length 6

{1, 0, 0, 0, 0, 0, 1, 0, 0}

{d, 1, 7, 1, 12}

{1, 0, 0, 0, 0, 0, 0, 0, 0}

Reduction over

Degree is {1, 0, 0, 0, 0, 0, 0, 0, 0}

Element reduced. New length: 17

Saved element 1 of length 17

Test results: {True, False, False, False, False, True, True, True, False}

Sorting

Permutation = {1, 2, 3, 11, 4, 5, 6, 7, 8, 9, 10, 12}

Sorting over

Reducing basis element 12 of length 6

{1, 0, 0, 0, 0, 0, 1, 0, 0}

{d, 1, 7, 1, 12}

{0, 1, 0, 0, 0, 0, 0, 1, 0}

{d, 1, 18, 1, 12}

{1, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
same degree

{d, 1, 28, 1, 18}

{0, 1, 0, 0, 0, 0, 0, 0, 0}

Reduction over

Degree is {0, 1, 0, 0, 0, 0, 0, 0, 0}

Element reduced. New length: 19

Saved element 2 of length 19

Test results: {True, True, False, False, False, True, True, True, False}

Sorting

Permutation = {1, 2, 3, 12, 4, 5, 6, 7, 8, 9, 10, 11}

Sorting over

Reducing basis element 12 of length 6

{0, 1, 0, 0, 0, 0, 0, 1, 0}

{d, 1, 7, 1, 12}

{0, 1, 0, 0, 0, 0, 0, 0, 0}

same degree

Basis element 4 replaced

Degree is {0, 1, 0, 0, 0, 0, 0, 0, 0}

{d, 1, 18, 1, 20}

{0, 0, 0, 0, 0, 1, 0, 0, 1}

{d, 1, 7, 1, 14}

{0, 0, 0, 0, 0, 0, 1, 0, 1}

{d, 1, 19, 1, 12}

{0, 0, 0, 0, 0, 0, 0, 1, 1}

Reduction over

Degree is {0, 0, 0, 0, 0, 0, 0, 1, 1}

Element reduced. New length: 20

Saved element 2 replaced, new length: 17

Test results: {True, True, False, False, False, True, True, True, False}

Sorting

Permutation = {1, 12, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}

Sorting over
```

```
Reducing basis element 12 of length 11

{0, 0, 1, 0, 0, 1, 0, 0, 0}

{d, 1, 12, 1, 14}

{0, 0, 1, 0, 0, 0, 1, 0, 0}

{d, 1, 24, 1, 12}

{0, 0, 1, 0, 0, 0, 0, 1, 0}

{1, 0, 0, 0, 0, 0, 0, 0, 0}

same degree

Basis element 6 replaced

Degree is {1, 0, 0, 0, 0, 0, 0, 0, 0}

{d, 3, 26, 7, 18}

{0, 0, 1, 0, 0, 0, 0, 0, 0}

Reduction over

Degree is {0, 0, 1, 0, 0, 0, 0, 0, 0}

Element reduced. New length: 141

Saved element 3 of length 141

Test results: {True, True, True, False, False, True, True, True, False}

Sorting

Permutation = {1, 2, 3, 4, 12, 5, 6, 7, 8, 9, 10, 11}

Sorting over

Reducing basis element 12 of length 9

{0, 0, 1, 0, 0, 0, 1, 0, 0}

{d, 1, 10, 1, 12}

{0, 0, 1, 0, 0, 0, 0, 0, 0}

same degree

Basis element 5 replaced

Degree is {0, 0, 1, 0, 0, 0, 0, 0, 0}

{0, 0, 0, 0, 0, 0, 1, 0, 1}

Reduction over

Degree is {0, 0, 0, 0, 0, 0, 1, 0, 1}

Element reduced. New length: 159

Saved element 3 replaced, new length: 21
```

```
Test results: {True, True, True, False, False, True, True, True, False}

Sorting

Permutation = {1, 2, 3, 12, 4, 5, 6, 7, 8, 9, 10, 11}

Sorting over

Reducing basis element 12 of length 11

{0, 0, 0, 1, 0, 1, 0, 0, 0}

{d, 1, 12, 1, 14}

{0, 0, 0, 1, 0, 0, 1, 0, 0}

{d, 1, 24, 1, 12}

{0, 0, 0, 1, 0, 0, 0, 1, 0}

{0, 1, 0, 0, 0, 0, 0, 0, 0}

same degree

Basis element 7 replaced

Degree is {0, 1, 0, 0, 0, 0, 0, 0, 0}

{d, 3, 26, 7, 18}

{0, 0, 0, 1, 0, 0, 0, 0, 0}

Reduction over

Degree is {0, 0, 0, 1, 0, 0, 0, 0, 0}

Element reduced. New length: 141

Saved element 4 of length 141

Test results: {True, True, True, True, False, True, True, True, False}

Sorting

Permutation = {1, 2, 3, 4, 5, 12, 6, 7, 8, 9, 10, 11}

Sorting over

Reducing basis element 12 of length 9

{0, 0, 0, 1, 0, 0, 0, 1, 0}

{d, 1, 10, 1, 12}

{0, 0, 0, 1, 0, 0, 0, 0, 0}

same degree

Basis element 6 replaced

Degree is {0, 0, 0, 1, 0, 0, 0, 0, 0}

{0, 0, 0, 0, 0, 0, 0, 1, 1}
```

```
Reduction over

Degree is {0, 0, 0, 0, 0, 0, 0, 1, 1}

Element reduced. New length: 159

Saved element 4 replaced, new length: 21

Test results: {True, True, True, True, False, True, True, True, False}

Sorting

Permutation = {1, 2, 12, 3, 4, 5, 6, 7, 8, 9, 10, 11}

Sorting over

Reducing basis element 12 of length 10

{0, 0, 0, 0, 1, 0, 1, 0, 0}

{d, 1, 11, 1, 12}

{0, 0, 0, 0, 1, 0, 0, 0, 0}

Reduction over

Degree is {0, 0, 0, 0, 1, 0, 0, 0, 0}

Element reduced. New length: 22

Saved element 5 of length 22

Testing element 1

Testing element 2

Testing element 3

Testing element 4

Testing element 5

Testing element 6

Testing element 7

Testing element 8

Testing element 9

Testing element 10

Testing element 11

Testing element 12

Test results: {True, True, True, True, True, True, True, True, False}

Sorting

Permutation = {1, 2, 3, 4, 5, 6, 12, 7, 8, 9, 10, 11}

Sorting over
```

```
Reducing basis element 12 of length 10

{0, 0, 0, 0, 1, 0, 0, 1, 0}

{d, 1, 11, 1, 12}

{0, 0, 0, 0, 1, 0, 0, 0, 0}

Reduction over

Degree is {0, 0, 0, 0, 1, 0, 0, 0, 0}

Element reduced. New length: 22

Testing element 12

Test results: {True, True, True, True, True, True, True, True, False}

Sorting

Permutation = {1, 2, 3, 4, 5, 6, 7, 12, 8, 9, 10, 11}

Sorting over

Symmetry number 1 of element 1 (length 11)

Equal to basis element 4

Symmetry number 1 of element 2 (length 20)

{0, 0, 0, 0, 0, 0, 1, 0, 1}

New element of length 20

Degree is {0, 0, 0, 0, 0, 0, 1, 0, 1}

Testing element 13

Test results: {True, True, True, True, True, True, True, True, False}

Sorting

Permutation = {1, 2, 3, 4, 5, 13, 6, 7, 8, 9, 10, 11, 12}

Sorting over

Trying to reduce elements

No elements reduced

Symmetry number 1 of element 3 (length 159)

Equal to basis element 5
```

$$\begin{pmatrix} 0 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 0 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 1 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 1 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 1 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 0 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 \end{pmatrix}$$

```
{1, 2}

{1, 0, 0, 0, 0, 0, 0, 0, 0}

New element of length 20

Degree is {1, 0, 0, 0, 0, 0, 0, 0, 0}

Testing element 14

Saved element 9 of length 20

All tests done

New element of length 17

Degree is {1, 0, 0, 0, 0, 0, 0, 0, 0}

New element of length 17

Degree is {0, 1, 0, 0, 0, 0, 0, 0, 0}

Sorting

Permutation = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 16, 13, 14, 15}

Sorting over

Evaluation time: 6.27005
```

In[14]:= **F[{-2, -2, -1, 0, -1, 2, 3, 4, -2}]**

In[15]:= **%**

Out[15]=
$$\frac{2\,(-2+d)\,(576+1864\,d+1010\,d^2+463\,d^3-319\,d^4-5\,d^5+11\,d^6)\,G[\{0,0,0,0,0,1,1,1,0\}]}{3\,d^2\,(2+d)\,(4+d)}$$

Examples of reduction:

$$F(1,\ldots,1,0) = -\frac{3(d-4)(3d-10)}{8(d-5)(2d-9)} I_1 - \frac{3(d-4)(3d-10)}{16(d-5)(2d-9)} I_2$$
$$-\frac{(d-3)(3d-10)(3d-8)}{8(d-5)(3d-13)(3d-11)} I_3 - \frac{3(d-2)(3d-11)(3d-10)(3d-8)}{64(d-5)(2d-9)(2d-7)(3d-13)} \bar{I}_4$$
$$+\frac{9(d-4)(d-2)(3d-10)(3d-8)}{64(d-5)(2d-9)(2d-7)(3d-13)} I_5 - \frac{3(3d-10)(3d-8)}{32(d-5)(2d-9)(2d-7)} \bar{I}_7\ ,$$

$$F(1,\ldots,1,-1) = \frac{3(d-3)(3d-11)}{16(d-5)(d-4)(2d-9)} I_4$$
$$-\frac{(d-2)(2d-7)(2d-5)}{8(d-3)(2d-9)(3d-13)} I_6 - \frac{3(2d-7)^2(2d-5)(3d-11)(3d-7)}{256(d-4)^2(d-3)(2d-9)} I_7\ .$$

Perspectives

- The algorithm can works successfully at the level of modern calculations, e.g., in problems with 12 indices.
- There are various interesting practical and mathematical problems. Which orderings are optimal for a given sector? What is the order of CPU time needed for the construction of the corresponding s-basis? Will the algorithm work for a given problem?
- Further improvements are necessary for more sophisticated calculations.
- Combining our algorithm with other ideas. Janet basis?