New Four/Five Loop Results in QCD/QED

or

Four-loop massless propagators: applications in QCD/QED

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based on recent results obtained with: P. Baikov, J. H. Kühn

LOOPS & LEGS 2006

- **Massless propagators and physics**
- **List of theoretical and programming tools in use**
- **New Results (and Puzzles) :**

  1. **DIS: 4-loop anom. dimensions (and the matrix elements) of the non-siglet spin=twist=2 operator and the tensor current**
  2. **Higgs $\to gg$ Decay to Order α_s^5 in N^{3}LO (4 $\times$ 4 loops)**
  3. **scalar correlator in 5 loops: full result and applications**
  4. **puzzles of the quenched QED β-function at 5 loops: is it really rational? Could it contain $\zeta(3)$ or $\zeta(4)$ or $\zeta(5)$ or $\zeta(6)$ or $\zeta(7)$?**

- **Conclusion**

Massless propagators: central problem

$$\mathbf{\Pi}^{jj}(q^2 = -Q^2) = i \int \mathrm{d}x e^{iqx} \langle 0|T[\ j(x) j^\dagger(0)\ ]|0\rangle$$

related to the corresponding absorptive part $R^{jj}(s)$ through

$$R^{jj}(s) \approx \Im \Pi^{jj}(s - i\delta)$$

masslessness $\longleftrightarrow$ simplicity:

5-loop R(s) is reducible*

to 4-loop massless propagators ($\equiv$ p-integrals) $\longleftarrow$ main object to compute

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* (i) the same is true for for massive corrections like $\mathbf{m_q^2/s}$, etc.
/ J. Kühn, K.Ch (91,94)/
(ii) any 5-loop anom. dim. or β-function in any theory reducible to 4-loop p-integrals
/K. Ch., Smirnov (1984)/

Tool Box *

- **IRR / Vladimirov, (78)/ + IR R^* -operation /K. Ch., Smirnov (1984)/ + resolved combinatorics /K. Ch., (1997)/**
- **reduction to Masters: "direct and automatic" construction of CF's through $1/D$ expansion—made with BAICER—within the Baikov's representation for Feynman integrals**[1]
- **all 4-loop master p-integrals are known analytically /P. Baikov and K.Ch. (2004)/**
- **computing time and required resources: could be huge (the price for full automatization)**; to cope with it we use parallel FORM /Vermaseren, Retey, Fliegner, Tentyukov, ...(2000 – . . .)

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* NO IBP identities are use at any step!

[1]Baikov, Phys. Lett. B385 (1996) 403; B474 (2000) 385; Nucl.Phys.Proc.Suppl.116:378-381,2003

Warming up*: the current $\bar\psi \sigma_{\mu\nu} \psi$ in the 4 loops /has applications in the lattice and the QCD heavy quark effective theory/

The 4-loop piece of its $\overline{\mathsf{MS}}$ anomalous dimension reads:

$$
\begin{aligned}
\gamma_{T,4}^{\mathrm{QCD},\overline{\mathrm{MS}}} = \quad & \frac{2208517}{41472} - \frac{7733}{3888}\zeta_3 + \frac{319}{144}\zeta_4 - \frac{10465}{972}\zeta_5 \\
+ \quad & n_f \left[-\frac{1537379}{186624} - \frac{18979}{3888}\zeta_3 + \frac{437}{432}\zeta_4 + \frac{575}{216}\zeta_5 \right] \\
+ \quad & n_f^2 \left[\frac{9961}{93312} + \frac{115}{648}\zeta_3 - \frac{5}{72}\zeta_4 \right] + n_f^3 \left[\frac{7}{15552} + \frac{1}{324}\zeta_3 \right]
\end{aligned}
$$

Full agreement with previous (partial) results
by /D. Broadhurst (1999), J. Gracey (2001)/ Also we have a result for the matrix element /important for MOM-like, more lattice fiendly renormalized schemes/

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* was done at 2002 with both MINCER and BAICER, to test the latter; also we have found *the Next To Renormalon contribution* at 5 loop ($1/N_F^2$ term in the terminology of J. Gracey)

anomalous dim. and the matrix element of the $O_2 = \bar{\psi}\gamma^{\{\mu}D^{\nu\}}\psi$ in 4 loops:
($\overline{\text{MS}}$ scheme, $a = \alpha_s/(4\pi)$, $n_f = 3$, Feynman gauge!)

$$\gamma_2 = \frac{32}{9}a + \frac{9440}{243}a^2 + a^3\left[\frac{3936832}{6561} - \frac{10240}{81}\zeta_3\right]$$

$$+ \quad a^4\left[\frac{1680283336}{177147} - \frac{24873952}{6561}\zeta_3 + \frac{5120}{3}\zeta_4 - \frac{56960}{243}\zeta_5\right]$$

$$< q|O_2|q >= 1 - \frac{140}{27}a + a^2\left[-\frac{113995}{1458} + \frac{280}{27}\zeta_3\right]$$

$$+ \quad a^3\left[-\frac{126410231}{52488} + \frac{1562656}{2187}\zeta_3 + \frac{4067}{81}\zeta_4 - \frac{21100}{243}\zeta_5\right]$$

$$+ \quad a^4\left[-\frac{762284085865}{7558272} + \frac{20199525605}{472392}\zeta_3 - \frac{3495127}{1458}\zeta_3^2 + \frac{30620959}{34992}\zeta_4\right.$$

$$+ \quad \left.\frac{80119519}{8748}\zeta_5 - \frac{462575}{486}\zeta_6 - \frac{105149513}{11664}\zeta_7\right]$$

Some numerics

$$\gamma_2 = \frac{8}{9}\left(a_s + 2.731 a_s^2 + 7.876 a_s^3 + 28.7067\, a_s^4 \right)$$

$$< q|O_2|q >^{\overline{\mathsf{MS}}} = 1 - 1.296\, a_s - 4.107\, a_s^2 - 24.768\, a_s^3 - {\color{red}205.205}\, a_s^4$$

$$< q|O_2|q >^{\text{G-scheme}} = 1 - 0.18512\, a_s - 0.826\, a_s^2 - 5.687 a_s^3 - {\color{green}12.495}\, a_s^4$$

Higgs Boson $\longrightarrow$ the Last Missing Piece of the SM

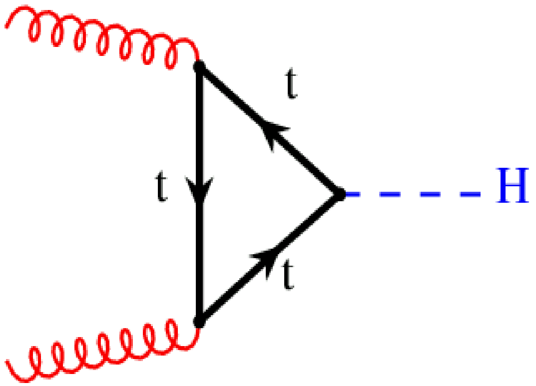
its mass M_H is constrained

by experiments + theoretical considerations

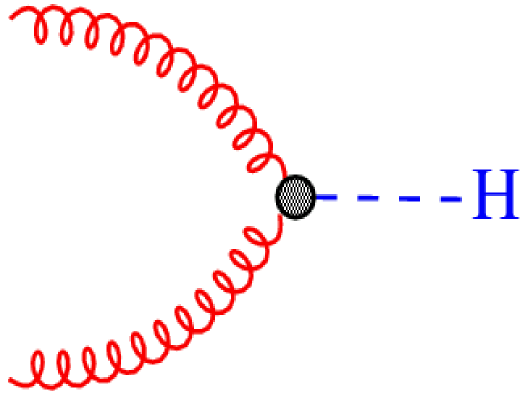
$$\mathbf{100 \quad GeV < M_H < 200 \quad GeV}$$

within this mass range the gluon fusion $\mathrm{H} \rightarrow \mathrm{gg}$ has the largest cross-section at both Tevatron and LHC

Higgs gluon fusion and its cross-process $H \longrightarrow gg$ are extremely interesting also from theoretical point of view:



$$\mathbf{m_t} \longrightarrow \infty$$



$$\mathcal{L}_{\mathbf{eff}} = -\mathbf{2}^{1/4}\ \mathbf{G_F}\ \mathbf{H}\ \mathbf{C_1}(\alpha_{\mathbf{s}}, \ln\frac{\mu^{\mathbf{2}}}{\mathbf{M_t^2}})\ \mathbf{G}^{\mathbf{a}}_{\mu\nu}\mathbf{G}^{\mathbf{a}}_{\mu\nu}$$

Hgg coupling is a device to count the number of heavy fermion generations, it is even sensitive to quark isodublets with degenerate masses

Theoretically very demanding process:

- Born approximation starts from one loop and $\sim \alpha_s^2$, e.g. :

$$\Gamma_{\mathrm{Born}}(\mathbf{H} \to \mathbf{gg}) = \frac{\mathbf{G_F M_H^3}}{\mathbf{36}\pi\sqrt{\mathbf{2}}} \left(\frac{\alpha_{\mathbf{s}}(\mu)}{\pi}\right)^{\mathbf{2}}$$

- which leads to a strong scale dependence and to an utmost importance of higher order corrections

- a lot of efforts devoted to evaluate higher order QCD effects during last 10-15 years:

  for decay:

  NLO: T. Inami, T. Kubota, and Y. Okada,Z. Phys. C **18**, 69 (1983).

  NNLO: K. Ch., B. Kniehl, M. Steinhauser, PRL, 79 (1977) 353

  for gluon fusion

  NLO: Z. Phys. C **18**, 69 (1983); TS. Dawson, Nucl. Phys. B **359**, 283 (1991); A. Djouadi, M. Spira and P. M. Zerwas, Phys. Lett. B **264**, 440 (1991).

  NNLO: R. V. Harlander and W. B. Kilgore, Phys. Rev. Lett. **88**, 201801 (2002); C. Anastasiou and K. Melnikov, Nucl. Phys. B **646**, 220 (2002) V. Ravindran, J. Smith and W. L. van Neerven, Nucl. Phys. B **665**, 325 (2003);

Net Effects of Higher Orders

- indeed NLO results to a 60-70% increase of the both $\sigma(pp \rightarrow H + X)$ and $\Gamma_{\mathrm{Born}}(H \rightarrow gg)$ while NNLO adds approximately about 20% more (for both, the production and decay rates!)

- even at NNLO the residual scale dependence amounts to about 15-20% (and again for both processes)

- note the striking similarity of QCD radiative corrections to the Higgs gluon fusion and Higgs gluon decay!

Very recently as a spin-off of the heroic calculation of the 3-loop splitting function* S. Moch and A. Vogt have succeeded even in finding "leading" set of NNNLO corrections to $\sigma(gg \to H)$. This leads to a significant stabilization of the scale dependence: with $\mu = M_H/2 — 2M_H$ the relative change of the production cross-section is now about 4% only !

*** / S. Moch, J.A.M. Vermaseren and A. Vogt (2004-2005)/**

Another important observation was recently made by Anastasiou, Melnikov and Petriello (hep-ph/0509014):

"We point out that the appropriate uncertainty in the $gg \to H$ channel which enters the analysis of Higgs couplings should instead be $\pm 5\%$, which is smaller by a factor of four.

This reduction relies upon the observation that the theoretical input for the Higgs coupling determination is the ratio

$$\sigma_{gg}^{SM} / \Gamma_{gg}^{SM}$$

The QCD corrections to σ_{gg}^{SM} and Γ_{gg}^{SM} track each other, and a large portion of the uncertainty cancels when the ratio is taken"

Even working only in NNLO they conclude: **"Consequently, the ratio of these two quantities has a theoretical uncertainty smaller than the uncertainty in the cross section alone by a factor of two."**

Hgg coupling in the heavy top-quark limit is described by an extra term in the $n_f = 5$ effective topless Lagrangian ($\alpha_s' \equiv \alpha_s^{(5)}$, $\alpha_s \equiv \alpha_s^{(6)}$)

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\mathbf{QCD}}^{\mathbf{n_f=5}} \quad - \quad \mathbf{2^{1/4} G_F^{1/2} H\, C_1} \; [\mathbf{O}_1'] \,, \qquad \mathbf{O}_1' = \mathbf{G}^{\mathbf{0}\prime}_{\mathbf{a}\mu\nu} \mathbf{G}^{\mathbf{0}\prime\mu\nu}_{\mathbf{a}}$$

Due to the optical theorem:

$$\mathbf{\Gamma(H \to gg) = 2 G_F\, C_1^2\, M_H^3\; R^{GG}(M_H^2)}, \qquad \mathbf{R^{GG}(q^2)} \equiv \frac{\pi}{\mathbf{2\, q^4}} \mathrm{Im} \mathbf{\Pi^{GG}(q^2)}$$

where

$$\mathbf{\Pi^{GG}(q^2)} = \int \mathbf{e^{iqx}} \langle \mathbf{0} | \mathbf{T} \left([\mathbf{O}_1'] \, (\mathbf{x}) \, [\mathbf{O}_1'(\mathbf{0})] \right) | \mathbf{0} \rangle \, \mathrm{dx}$$

Important: the coef. function C_1 comes from massive tadpoles and known since long /K.Ch, B. Kniehl, M.Steinhauser (1997)/

Π^{GG} obviously from massless propagator; N^{3}LO means 4 loops for both

Π^{GG} **is contributed by 10240 4-loop diagrams:**

- **all diagrams are "native", no need for IR transformation and no squared propagator $\longleftarrow$ simple task for BAICER** *
- **done for about only 4 weeks of work of SGI cluster** †
- **in numerical form** R^{GG} **reads** $(\mu^2 = q^2,\ a_s = \alpha_s/\pi)$

$$R^G = 1 + 12.4167\, a_s + 68.6482\, a_s^2 - 212.447\, a_s^3$$

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* set of FORM3 programs implementing $1/D$ expansion to express any 4-loop massless propagator in terms of masters

† cmp. to 15(!) months taken by five loop SS-correlator (due to resulting after IR transformations non-native 4-loop diagrams)

RESULTS for K FACTOR

$$\Gamma(\mathbf{H} \to \mathbf{gg}) \;=\; \Gamma_{\rm Born}(\mathbf{H} \to \mathbf{gg})\,\mathbf{K}, \quad \mu = \mathbf{M_H}, \quad \mathbf{a_s'} \equiv \alpha_s^{(5)}/\pi$$

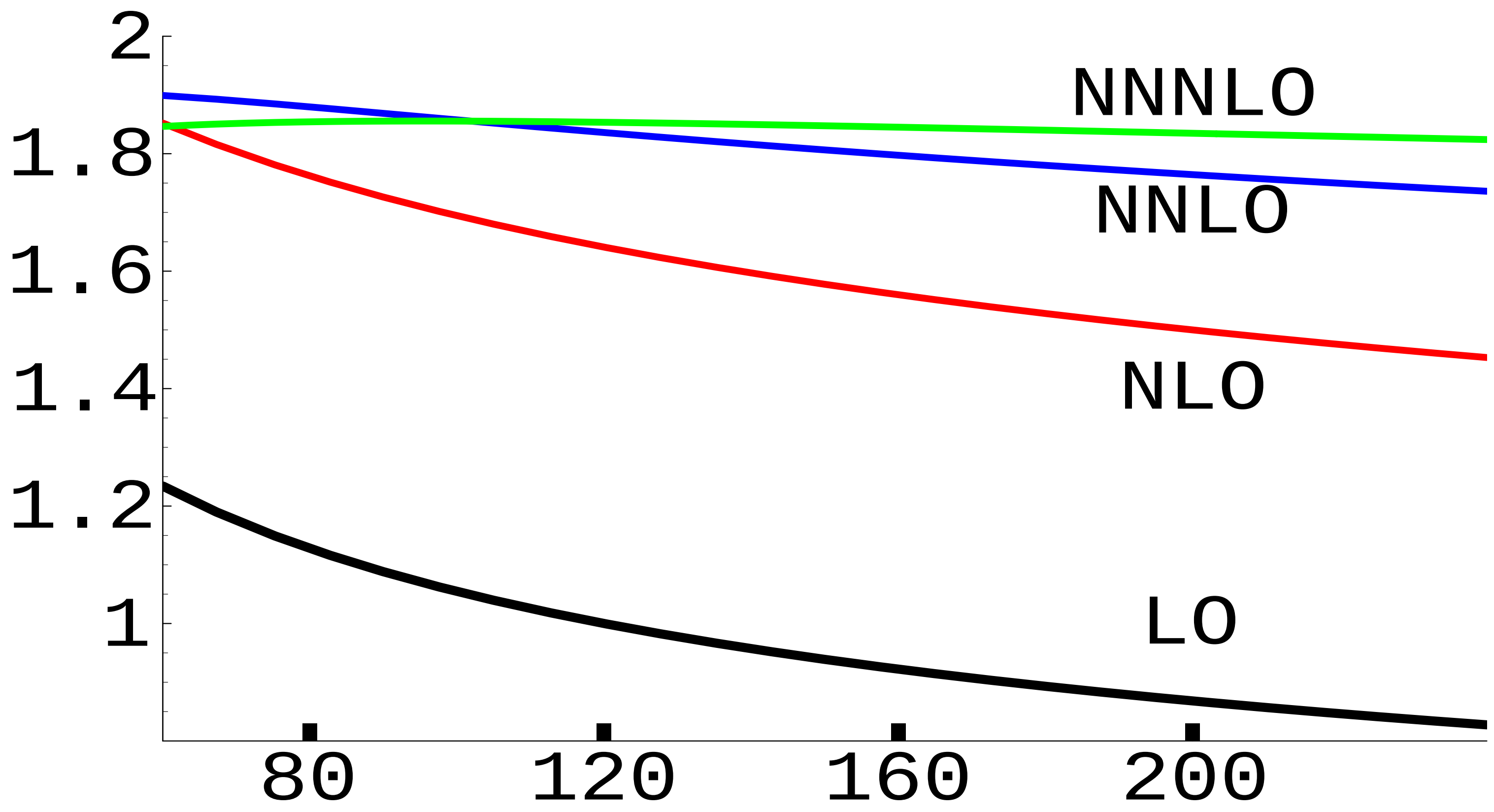
$$\mathbf{K} = \mathbf{1} + \mathbf{17.91}\,\mathbf{a_s'} + (\mathbf{156.8} - \mathbf{5.71}\ln\frac{\mathbf{M_t^2}}{\mathbf{M_H^2}})(\mathbf{a_s'})^2$$

$$+(\mathbf{467.9} - \mathbf{122.4}\ln\frac{\mathbf{M_t^2}}{\mathbf{M_H^2}} + \mathbf{10.9}\ln^2\frac{\mathbf{M_t^2}}{\mathbf{M_H^2}})\;(\mathbf{a_s'})^3$$

for $M_t = 175$ GeV, $M_H = 120$ GeV and $\alpha_s/\pi = .036$

$$\mathbf{K} = \mathbf{1} + \mathbf{17.9167}\;\mathbf{a_s'} + \mathbf{152.5}\;(\mathbf{a_s'})^2 + \mathbf{381.5}\;(\mathbf{a_s'})^3$$

$$= \;\mathbf{1} + \mathbf{0.65575} + \mathbf{0.2043} + \mathbf{0.0187}$$



Depedence of $\Gamma(H \to gg)(\mu)/\Gamma(H \to gg)(M_H)$ on the renormalization scale
$\mu = M_H/2 — 2M_H$

for $M_H = 120$ GeV and $M_t = 175$ GeV

$\delta\Gamma/\Gamma =$ **(residual scale dependence)**

LO: $\pm$ 24% **NLO: ± 22%** **NNLO: ± 10%** **NNNLO: $\pm$ 3% !**

Scalar Correlator in 5 loops and Higgs Decay into b-quarks

Higgs boson decays into quark–antiquark pair $(\bar{f}f)$ via its coupling to the corresponding quark scalar current:

$$\Gamma(H \to \bar{f}f) = \frac{G_F\, M_H}{4\sqrt{2}\pi} m_f^2 \widetilde{R}(s = M_H^2), \tag{1}$$

where $\widetilde{R}(s) = \mathrm{Im}\,\widetilde{\Pi}(-s - i\epsilon)/(2\pi\, s)$ is the absorptive part of the scalar two-point correlator:

$$\widetilde{\Pi}(Q^2) = (4\pi)^2 i \int dx e^{iqx} \langle 0|\, T[\, J_f^{\mathrm{S}}(x) J_f^{\mathrm{S}}(0)\,]\, |0\rangle$$

Consider Adler function:

$$\widetilde{D}(Q^2) = \frac{Q^2}{6} \frac{\mathrm{d}}{\mathrm{d}Q^2} \frac{\widetilde{\Pi}(Q^2)}{Q^2} = \int_0^\infty \frac{Q^2\ \widetilde{R}(s) ds}{(s + Q^2)^2}\ ,$$

$$\widetilde{D}(Q^2) = 1 + \sum_{i=1}^{\infty}\ \tilde{d}_i a_s^i(Q^2), \quad \tilde{R}(s) = 1 + \sum_{i=1}^{\infty}\ \tilde{r}_i a_s^i(s)\ ,$$

$$
\begin{aligned}
d_4 = \; & n_f^3 \left[-\frac{520771}{559872} + \frac{65}{432}\zeta_3 + \frac{1}{144}\zeta_4 + \frac{5}{18}\zeta_5 \right] \\
+ \; & n_f^2 \left[\frac{220313525}{2239488} - \frac{11875}{432}\zeta_3 + \frac{5}{6}\zeta_3^2 + \frac{25}{96}\zeta_4 - \frac{5015}{432}\zeta_5 \right] \\
+ \; & n_f \left[-\frac{1045811915}{373248} + \frac{5747185}{5184}\zeta_3 - \frac{955}{16}\zeta_3^2 - \frac{9131}{576}\zeta_4 \right. \\
+ \; & \left. \frac{41215}{432}\zeta_5 + \frac{2875}{288}\zeta_6 + \frac{665}{72}\zeta_7 \right] \\
+ \; & \left[\frac{10811054729}{497664} - \frac{3887351}{324}\zeta_3 + \frac{458425}{432}\zeta_3^2 \right. \\
+ \; & \left. + \frac{265}{18}\zeta_4 + \frac{373975}{432}\zeta_5 - \frac{1375}{32}\zeta_6 - \frac{178045}{768}\zeta_7 \right]
\end{aligned}
$$

the resulting $\widetilde{R}$ reads

$$\begin{aligned} \widetilde{R} &= 1 + 5.6667 a_s + [35.94 - 1.359\, n_f]\, a_s^2 \\ &+ a_s^3 \left[164.14 - 25.77\, n_f + 0.259\, n_f^2\right] \\ &+ a_s^4 \left[39.34 - 220.9\, n_f + 9.685\, n_f^2 - 0.0205\, n_f^3\right]. \end{aligned} \quad (2)$$

and with "kinematical" π^2 terms explicilty separated and underlined:

$$\begin{aligned} \widetilde{R} = 1{+}5.667 a_s{+}a_s^2 &\left[51.57 - \underline{15.63} - n_f(1.907 - \underline{0.548})\right] \\ + \; a_s^3 &\left[648.7 - \underline{484.6} - n_f(63.74 - \underline{37.97}) + n_f^2(0.929 - \underline{0.67})\right] \\ +a_s^4 &\left[9471. - \underline{9431.} - n_f(1454.3 - \underline{1233.4}) + n_f^2(54.78 - \underline{45.10}) \right. \\ - \; &\left. n_f^3(0.454 - \underline{0.433})\right] \end{aligned}$$

remarkable mutual cancellations in all n_f powers!!!
for $n_f = 3 \longrightarrow a_s^4(5589 - 6126) = -536.8$

Application: Higgs Decay into b quarks

$$\Gamma(H \to \bar{f} f) = \frac{G_F\, M_H}{4\sqrt{2}\pi} m_f^2 R^S(s = M_H^2)$$

$$\begin{aligned} R^S &= 1 + 5.66677\, a_s + 29.147\, a_s^2 + 41.758\, a_s^3 - 825.7\, a_s^4 \\ &= 1 + 0.2075 + 0.0391 + 0.0020 - 0.00148 \end{aligned}$$

where we set $a_s = \alpha_s/\pi = 0.0366$ **(for the Higgs mass value** $M_H = 120$ **GeV)**

Application: m_s from QCD sum rules for the PP-correlator*

$$\mathbf{m}_s(\mathbf{2\ GeV}) = \mathbf{105} \pm \mathbf{5}\Big|_{\mathbf{param}} \pm \mathbf{6}\Big|_{\mathbf{nonp}} \pm \mathbf{7}\Big|_{hadr}$$

$param$: Λ_{QCD}, scale, condensates

$nonpt$: instanton correction

$hadr$: hadronic input

important: new $\mathcal{O}({\alpha_s}^4)$ term amounts to a small

-2 MeV shift of m_s $\Longrightarrow$ good PT stability!

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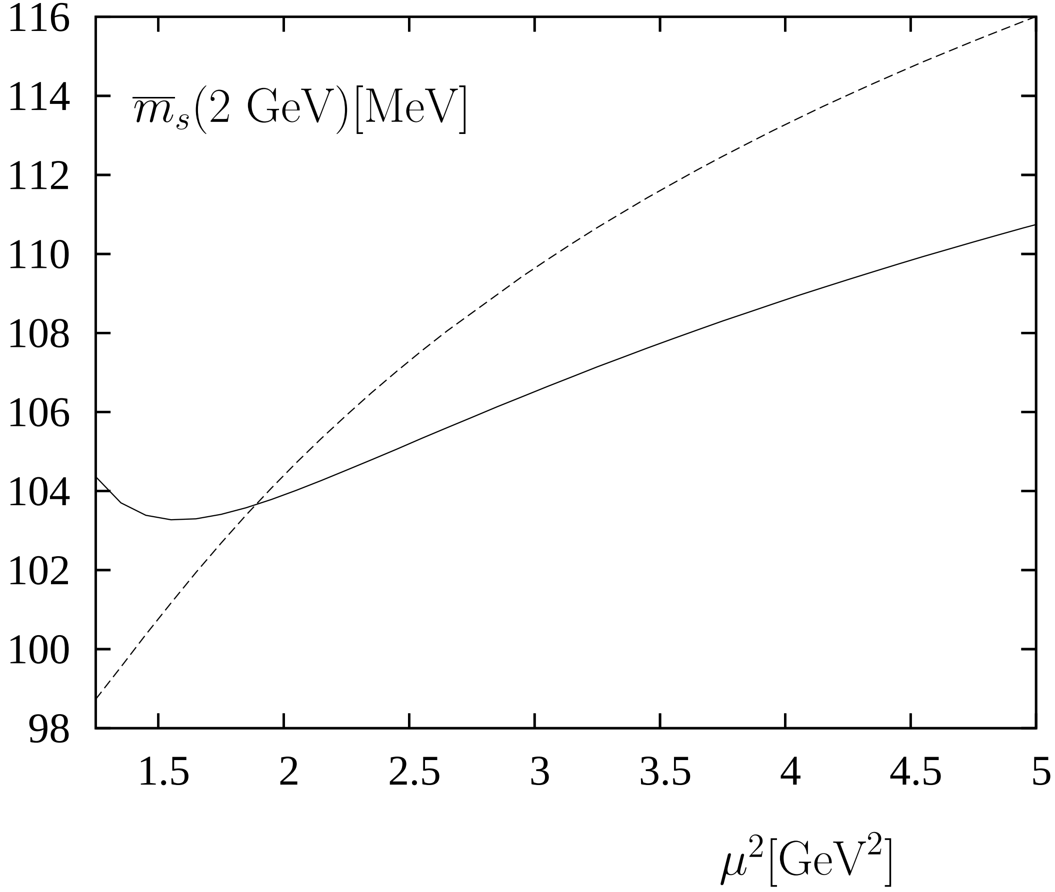
*K.Ch., A. Khodjamirian, hep=ph/0512295

The renormalization scale dependence of m_s



solid $O(\alpha_s^4)$, dashed $O(\alpha_s^3)$

$\simeq 2$ MeV increase of the central value, if the $O(\alpha_s^4)$ terms are removed

BEAUTY of β^{qQED}

- it is scheme independent in all orders
- the coefficients are simple rational numbers at 1,2,3 and four loops: (4/3, 4,-2,-46)
- if

$$\beta^{\text{qQED}}(\alpha_0) \equiv 0$$

  tnen $\alpha = \alpha_0$ leads to self-consistent **finite** solution of (massless) QED /K.Johnson and M. Baker, (1973)/
- its a piece ($\sim C_F^4 a_s^4$) of the full $R(s)$, but its calculation is significantly(?) simpler

some people undestand the observed rationality of β^{qQED} at 1,2,3 and 4 loops and hope that it is not a pure coincidence.

For instance: David Broadhurst: (in hep-th/9909185)

"Noting the profound work of Alain Connes and Dirk Kreimer [1],

one arrives at the nub of the rationality of quenched QED:

dimensional regularization of the *derivative* of

the scheme-independent single-fermion-loop Gell-Mann-Low function,

via Fock-Feynman-Schwinger formalism"

Clealy, for *practioners of QFT* (like me and my collaborators) there is a less profound way to contribute to the discussion: just to attack the next order!

$\beta^{\mathbf{qQED}}$ at 5 loops is done

Some Details about Complexity of Calculations

Scalar $R(s)$ versus. β^{qQED} /both at 5 loops/

- CPU-time consumption: 5 vs. 25 years on 3GH PC

- Machines: [SGI cluster (of 32 parallel SMP CPU of 1.5 GH frequency each) + individual PC's]
  $\approx$ (very roughly) 25 vs. 50 effective 3GH PC as maximum

- Calendar time: 15 vs. 7 months

- Basically computers were shared with other users, but in qQED case were used almost exclusively (summer time). Also for qQED algorithmical improvements implemented (factor 2 (or even more) in total).

Our result (still very :) preliminary) reads:

$$\frac{4}{3}, \quad 4, \quad -2, \quad -46, \quad \frac{\Box\Box\Box\Box}{\Box} + \Box\Box\Box \cdot \zeta(3)$$

where $\Box$ stands for a digit

Comments :

the rationality spell seems to be broken at five loops

more testing is needed before we could make public the content of green boxes

any clarification of the sutiation with the rationality from the experts would be of

HUGE importance and help for us

CONCLUSION

PRESENT:

- systematic evaluation of 4-loop p-integrals $\longleftrightarrow$ 5-loop UV counterterms $+$
- 4-loop anom.dim. and the $<q|$matrix elements$|q>$ of tensor and twist=spin=2 current $+$
- ImΠ^{SS} in $\mathcal{O}(\alpha_s^4)$ (H$\longrightarrow b\bar{b}$)$+$
- $\Gamma(H \rightarrow 2$ gluons) in $\mathcal{O}\left(\alpha_s^5\right)$ $+$

FUTURE:

- Im$\Pi^{VV} = R^V$ in $\mathcal{O}(\alpha_s^4)$ within reach
- calculations for higher spin (twist=2) operators /up to n=8/ should be possible
- 5-loop QCD β-function and anomalous dimensions are *in principle* analytically computable

CONCLUSION
/cont-ed/

BUT:

- **Huge demands on computer power and storage!**

- **the calculation of the $\beta^{\mathbf{qQED}}$ ($\equiv C_F^4$ part of $R(s)$) is finished BUT the testing is NOT!**

  The obtained result is puzzling and any help/hints from "theoreticial theory" is badly needed!