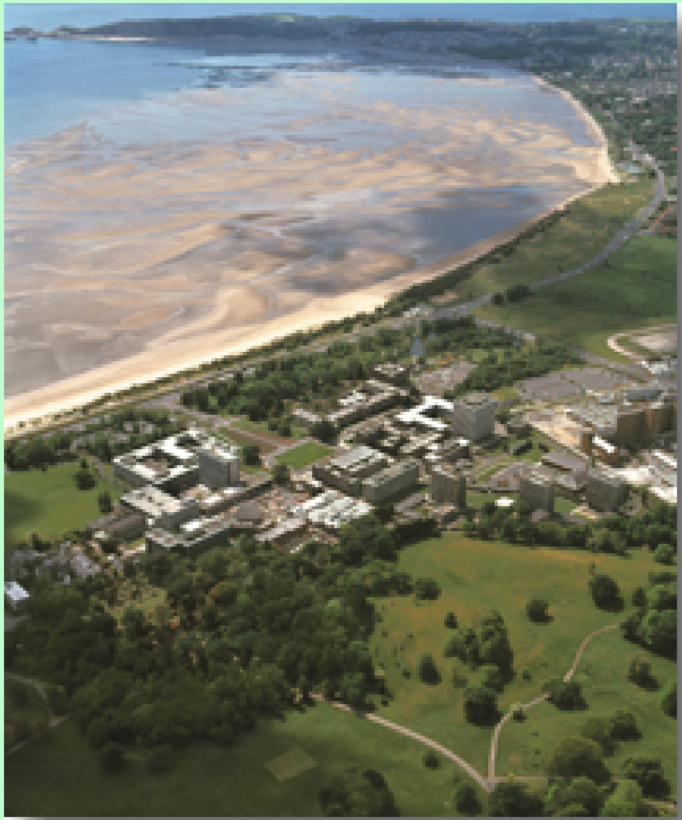
# Exploiting Twistor Techniques for QCD Calculations



David Dunbar,

Swansea University, Wales, UK

including work with

Zvi Bern, Steve Bidder, Emil Bjerrum-Bohr, Harald Ita, Warren Perkins, Kasper Risager

Loops and Legs 2006

# QCD Matrix Elements

- QCD matrix elements are an important part of calculating the QCD background for processes at LHC

- NLO calculations (at least!) are needed for precision

Talks by Binoth, Mnich

- One-Loop n-point on-shell amplitudes unknown analytically for n>5

- In this talk we discuss how recent (> Dec 03) ideas of a "weak-weak" duality might help,

  - apply to 2g → (n-2)g in massless QCD

# Duality with String Theory

In Dec 03 Witten proposed a Weak-Weak duality between

A) Yang-Mills theory ( N=4 )

B) Topological String Theory

S-matrix of two theories should be identical

- True for tree level gluon scattering

Rioban, Spradlin, Volovich

proposal constructive for N=4 but... N<4..??

# Is the duality useful?

Theory A : ⟷ Theory B:

hard, interesting ⟷ easy

Perturbative QCD, hard, interesting ⟷ Topological String Theory: harder, uninteresting

-duality may be useful indirectly

**-the existance may suggest or motivate**

# Some notation/defintions

Amplitude a function $A(\epsilon^\mu, p^\mu)$

Spinor helicity replaces polarisation by fermionic variables Xu, Zhang,Chang 87

$$\epsilon^\mu = \frac{\langle k|\gamma^\mu|p\rangle}{\langle k|p\rangle}$$

Use fermionic coordinates and replace bosonic momenta by `twistors`

$$p_{a\dot{a}} = \lambda_a \bar{\lambda}_{\dot{a}}$$

Amplitude now function entirely of fermionic variables

$$p_\mu = \sigma_\mu^{a\dot{a}} p_{a\dot{a}}$$

$$A(\lambda_a, \bar{\lambda}_{\dot{a}})$$

$$|p^+\rangle = \bar{\lambda}_{\dot{a}}$$
$$|p^-\rangle = \lambda_a$$

Fourier (Penrose transform) one of spinors,

$$\tilde{A}(\lambda_a, \mu_{\dot{a}}) = \int d\bar{\lambda} e^{\bar{\lambda}\mu} A(\lambda, \bar{\lambda})$$

Amplitude on twistor space is closest to string theory

# Inspired by duality –BCFW construction for tree amplitudes

Britto,Cachazo,Feng (and Witten)

Return of the analytic S-matrix!

Shift amplitude $A(\lambda_a, \bar\lambda_{\dot a})$ so it is a complex function of z

$$\lambda_a^1 \longrightarrow \lambda_a^1 + z\lambda_a^2$$
$$\bar\lambda_{\dot a}^2 \longrightarrow \bar\lambda_{\dot a}^2 - z\bar\lambda_{\dot a}^1$$

$$p_{a\dot a}^1 \longrightarrow p_{a\dot a}^1 - z\lambda_a^2\bar\lambda_{\dot a}^1$$

Tree amplitude becomes an analytic function of z, A(z)

-Full amplitude can be reconstructed from analytic properties

# Provided,

a) $A(z)$ analytic with simple poles at $z_i$

b) $A(z) \longrightarrow 0$ as $z \to \infty$

then

$$0 = \int_{C_\infty} dz \frac{A(z)}{z} = \sum Res \frac{A(z)}{z}$$

$$A(0) = -\sum_i \frac{Res(A(z))_i}{z_i}$$

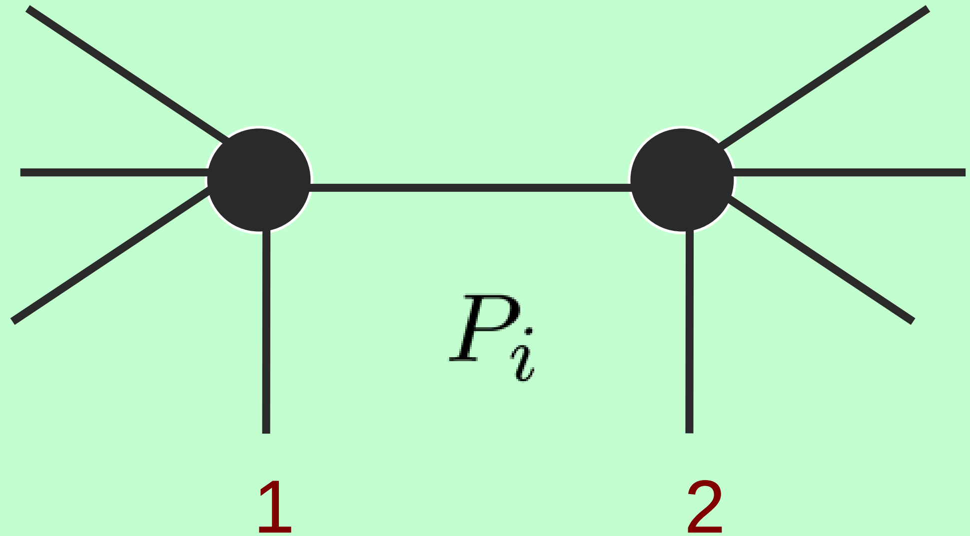
Residues occur when amplitude factorises on multiparticle pole (including two-particles)

$$P(z_i)^2 = 0$$

# -results in recursive on-shell relation

(c.f. Berends-Giele off shell recursion)

$$A = \sum$$




$$A = \sum \hat{A}(.., 1..)[z_i] \times \frac{1}{P_i^2} \times \hat{A}(.., 2, ..)[z_i]$$

$$z_i = \frac{P_i^2}{\langle 2|P|1\rangle}$$

Tree Amplitudes are on-shell but continued to complex momenta (three-point amplitudes must be included)

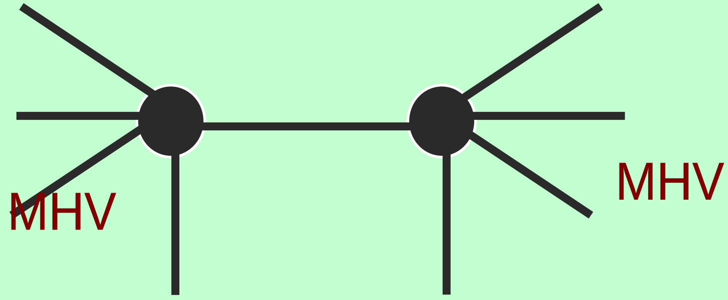
# Extended BCFW

$$A(1^-2^-3^-4^+ \ldots . n^+)$$

$$\bar\lambda^1_{\dot a} \longrightarrow \bar\lambda^1_{\dot a} + z\langle 23\rangle \bar\eta_{\dot a}$$
$$\bar\lambda^2_{\dot a} \longrightarrow \bar\lambda^2_{\dot a} + z\langle 31\rangle \bar\eta_{\dot a}$$
$$\bar\lambda^3_{\dot a} \longrightarrow \bar\lambda^3_{\dot a} + z\langle 12\rangle \bar\eta_{\dot a}$$

(conserves momenta using Schouten identity)

$$A = \sum$$




Factorisation under this shift+extensions reproduces the MHV vertex construction of Cachazo, Svercek and Witten

Risager 05; Bjerrum-Bohr, Dunbar, Ita, Perkins and Risager, 05

# One-Loop Amplitudes

One Loop Gluon Scattering Amplitudes in massless QCD

-Four Point : Ellis+Sexton

-Five Point : Bern, Dixon,Kosower

-Six-Point and beyond--- present problem/meeting

talks of Binoth, Mastrolia, Zanderighi+

-more known for supersymmetric theories

n-point MHV amplitudes supersymmetric theories

six-point N=4 amplitudes

Bern, Dixon, Dunbar and Kosower 94/95

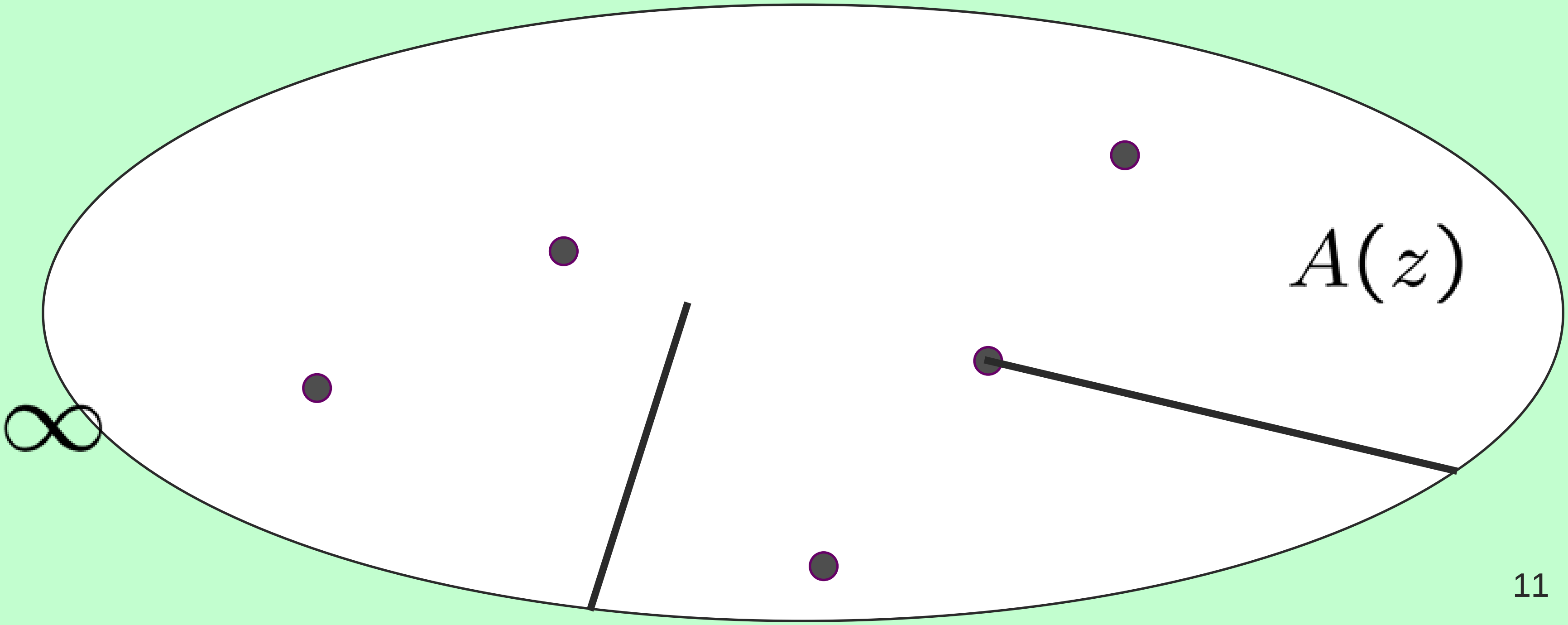
N=4, 1-loop essentially solved using twistor inspired techniques combined with unitarity method

...lots of people...

# Continuation of One-Loop amplitudes Have Cuts

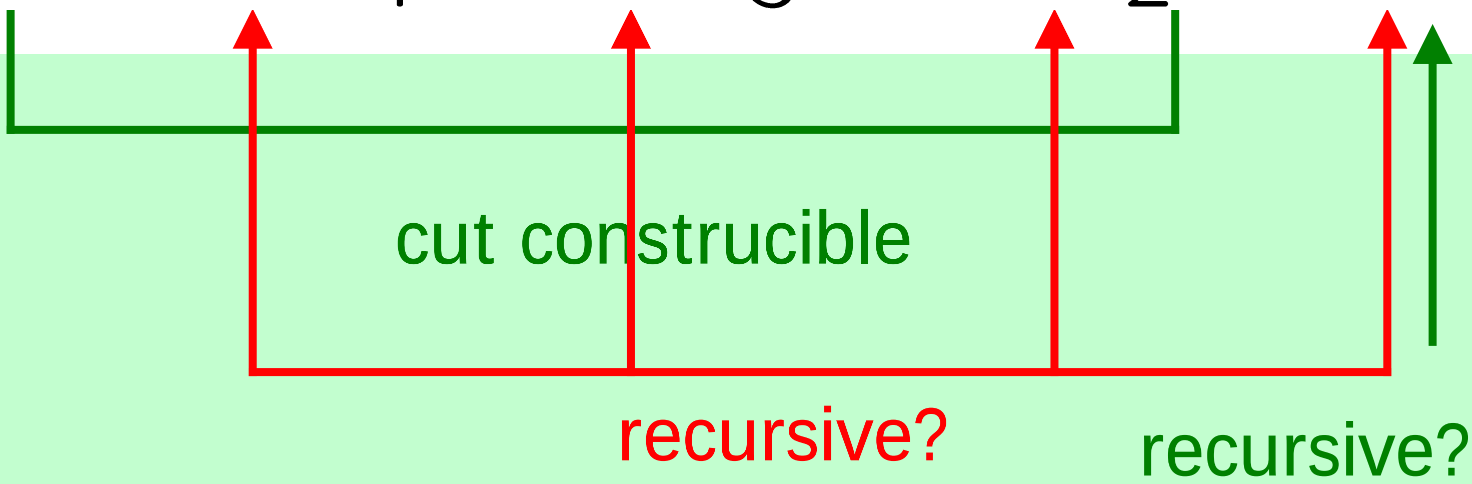
Analytically continuing the 1-loop amplitude in momenta leads to a function with both poles and cuts in z

$$\ln(s) \longrightarrow \ln(s(z))$$



# Expansion in terms of Integral Functions

$$A^{1-loop} = \sum c_i I_4^{(i)} + d_i I_3^{(i)} + e_i I_2^{(i)} + R$$




- R is rational and not cut constructible (to O(ħ))

-amplitude is a mix of cut constructible pieces and rational

# Recursion for Rational terms

$$A - \sum_i c_i I^{(i)} = R$$

-can we shift R and obtain it from its factorisation?

-YES see Berger, Bern, Dixon, Forde and Kosower

1) Function must be rational
2) Function must have simple poles
3) We must understand these poles

# Recursion on Integral Coefficients

Integral coefficients are rational functions

There are many choices of basis

The coefficients can encode stange effect of the Duality

For example, an rather unexpected results is that NMHV

tree amplitudes are coplanar in twistor space,

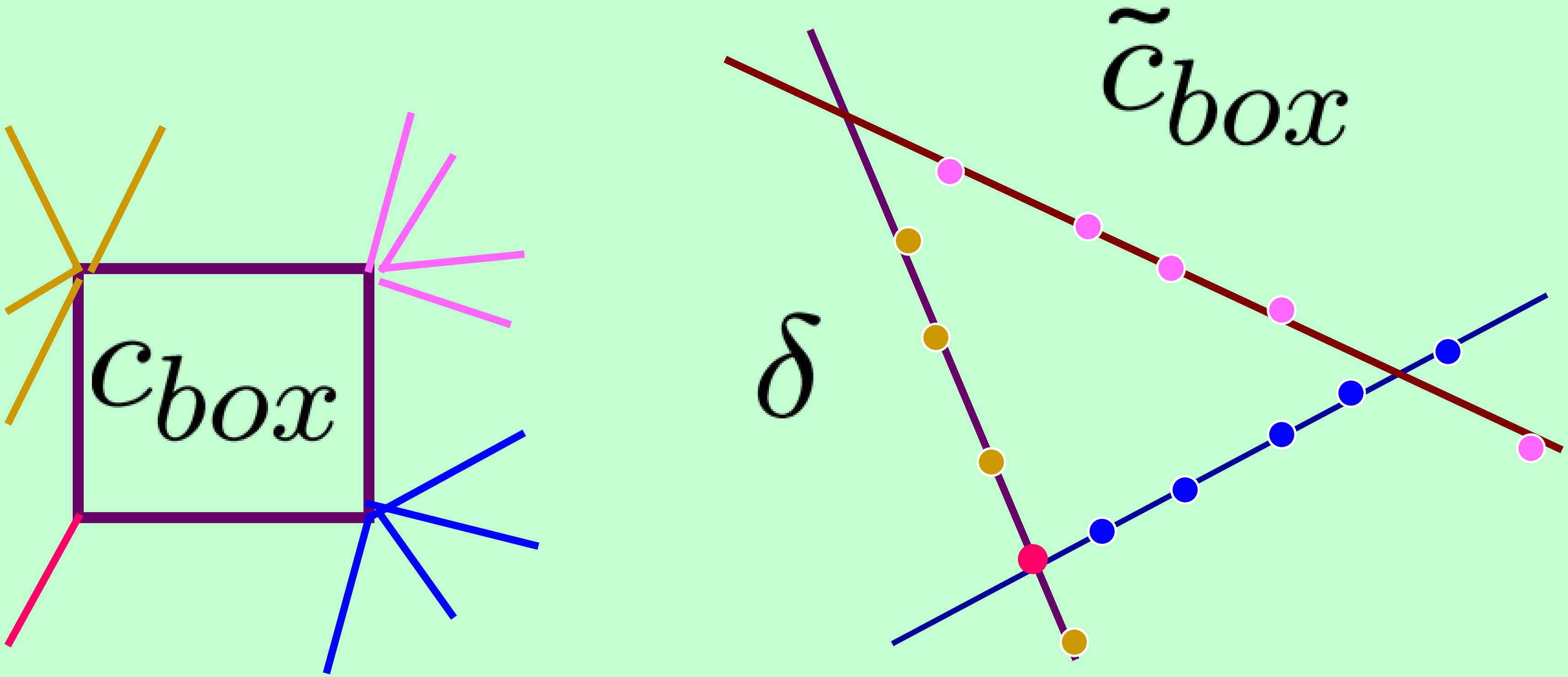
$$\tilde{A}^{NMHV,tree}(\lambda_a^i, \mu_{\dot{a}}^i) = 0$$

unless

$$Z_i \equiv (\lambda_{\dot{a}}^i, \mu_{\dot{a}}^i)$$ lie on a plane

# Box Coefficients-Twistor Structure

Similarly Box coefficients have coplanar support for NMHV 1-loop amplitudes




-true for both N=4 and QCD!!!

# -to carry out recursion we must understand poles of coefficients

-multiparticle factorisation theorems

Bern,Chalmers

$$A_n^{\text{one-loop}} \xrightarrow{K^2 \rightarrow 0}$$

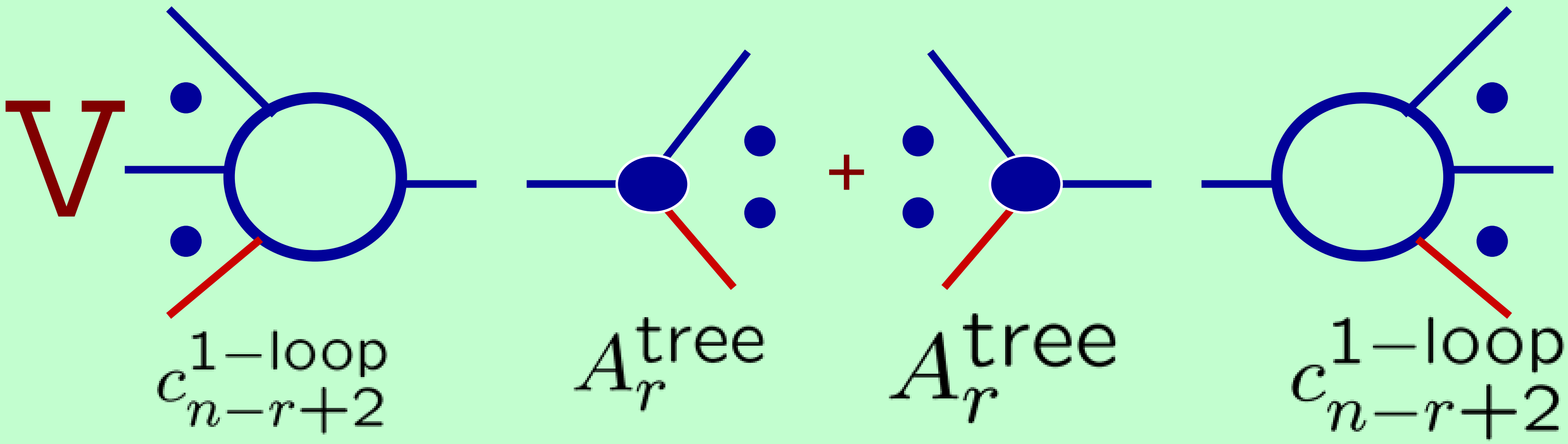
$$A^{\text{one-loop}} \times \frac{i}{K^2} \times A^{\text{tree}}$$

$$+A^{\text{tree}} \times \frac{i}{K^2} \times A^{\text{one-loop}}$$

$$+A^{\text{tree}} \times \frac{i}{K^2} \times A^{\text{tree}} \times \mathcal{F}_n$$

# Potential Recursion Relation

$$c_n^{1-\text{loop}} =$$



# Complication,

The coefficients of integral functions may contain additional Spurious Singularities which are not present in the full amplitude

It is important and non-trivial to find shift(s) which avoid these spurious singularities whilst still affecting the full R/coefficient

# Example of Spurious singularities

$$A_6^{N=1\ \text{chiral}}(1^-,2^-,3^-,4^+,5^+,6^+) = a_1 K_0[s_{61}] + a_2 K_0[s_{34}]$$

$$-\frac{i}{2}\left[b_1 \frac{L_0[s_{345}/s_{61}]}{s_{61}} + b_2 \frac{L_0[s_{234}/s_{34}]}{s_{34}} + b_3 \frac{L_0[s_{234}/s_{61}]}{s_{61}} + b_4 \frac{L_0[s_{345}/s_{34}]}{s_{34}}\right]$$

**Bidder, Bjerrum-Bohr, Dixon and Dunbar**

$$L_0(r) = \frac{\ln(r)}{1-r},\quad K_0(r) = \frac{1}{\epsilon} + \ln(r)$$

$$b_1 = \frac{\langle 6|\not{P}|3\rangle^2 \langle 6^+|(\not{2}\not{P}\not{P} - \not{P}\not{2}\not{P})|3^+\rangle}{\langle 2|\not{P}|5\rangle\,[6\,1]\,[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,P^2},\quad P = P_{345}$$

$$b_4 = \frac{\langle 6|\not{P}|3\rangle^2 \langle 6^+|(\not{5}\not{P}\not{P} - \not{P}\not{5}\not{P})|3^+\rangle}{\langle 2|\not{P}|5\rangle\,[6\,1]\,[1\,2]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,P^2},\quad P = P_{345}$$

Co-planar singularity

Collinear Singularity

Multi-particle pole

$$P_{34} = \alpha k_2 + \beta k_5$$

# -coplanar singularity is spurious

$$\frac{b_1'}{s_{61}} \longrightarrow \frac{b_4'}{s_{34}} \qquad b_i' = \langle 2|P|5\rangle b_i$$

$$t_{345}/s_{61} \longrightarrow s_{34}/t_{234}$$

$$\frac{b_1'}{\langle 2|P5\rangle}\frac{L_0[s_{345}/s_{61}]}{s_{61}} + \frac{b_4'}{\langle 2|P|5\rangle}\frac{L_0[s_{234}/s_{34}]}{s_{34}} \longrightarrow \frac{1}{\langle 2|P|5\rangle} \times 0$$

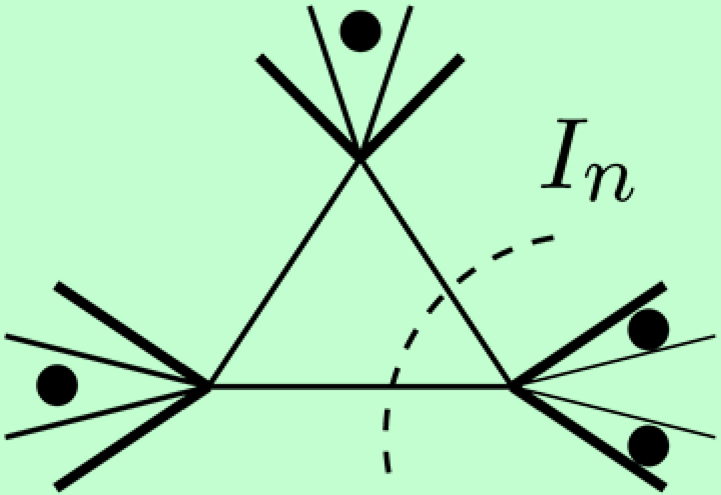
# Spurious Singularities...

- Cancellations occur  between integral functions so that integral coefficients may contain these term
- For tree amplitudes these singularities are not a problem
- Must try to avoid shifts which for which, e.g

$$\langle 2|P|5\rangle[z] = 0$$

- For some z

# Criteria to avoid Spurious Singularities




Consider an integral coefficient and isolate a coefficient and consider the cut. Consider shifts in the cluster.

- Shift must send tree to zero as z -> 4
- Shift must not affect cut legs

-such shifts will generate a recursion formulae

# Example: Split Helicity Amplitudes

Consider the colour-ordered n-gluon amplitude

$$A(-,-,-,\cdots -,-,+,+,\cdots,+)$$

-two minuses gives MHV

-BCFW can be used for tree amplitudes

Britto, Rioban Spradlin and Volovich

-for N=4 SYM essentially solved

-use as a example of generating coefficients recursively

# Supersymmetric Decomposition

Supersymmetric gluon scattering amplitudes are the linear combination of QCD amplitudes

$$A_n^{\mathcal{N}=4} \equiv A_n^{[1]} + 4A_n^{[1/2]} + 3A_n^{[0]}$$

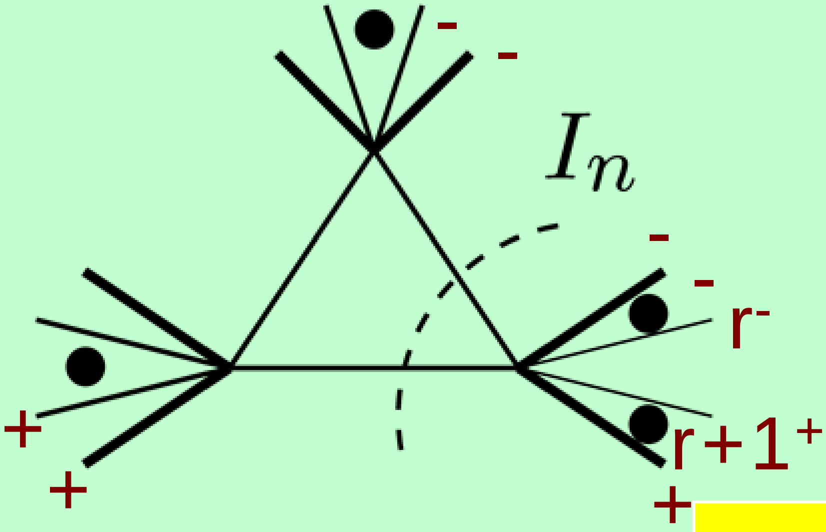
$$A_n^{\mathcal{N}=1\ \text{vector}} \equiv A_n^{[1]} + A_n^{[1/2]}$$

$$A_n^{\mathcal{N}=1\ \text{chiral}} \equiv A_n^{[1/2]} + A_n^{[0]}$$

-this can be inverted

$$A_n^{[1]} = A_n^{\mathcal{N}=4} - 4A_n^{\mathcal{N}=1\ \text{chiral}} + A_n^{[0]}$$

$$A_n^{[1/2]} = A_n^{\mathcal{N}=1\ \text{chiral}} - A_n^{[0]}$$




-look at cluster on corner with “split”

-shift the adjacent – and + helicity legs

-criteria satisfied for a recursion

$$\lambda^r_a \longrightarrow \lambda^r_a + z\lambda^{r+1}_a$$
$$\bar\lambda^{r+1}_{\dot a} \longrightarrow \bar\lambda^{r+1}_{\dot a} - z\bar\lambda^r_{\dot a}$$

-we obtain formulae for integral coefficients for both the N=1 and scalar cases which together with N=4 cut give the QCD case (with, for n>6 rational pieces outstanding)

## -for , special case of 3 minuses,

$$A_n^{\mathcal{N}=1\ \text{chiral}}(1^-,2^-,3^-,4^+,5^+,\cdots,n^+) =$$
$$\frac{A^{\text{tree}}}{2}\left(K_0(s_{n1})+K_0(s_{34})\right) - \frac{i}{2}\sum_{r=4}^{n-1}\hat{d}_{n,r}\,\frac{L_0[t_{3,r}/t_{2,r}]}{t_{2,r}}$$
$$-\frac{i}{2}\sum_{r=4}^{n-2}\widehat{g}_{n,r}\,\frac{L_0[t_{2,r}/t_{2,r+1}]}{t_{2,r+1}} - \frac{i}{2}\sum_{r=4}^{n-2}\hat{h}_{n,r}\,\frac{L_0[t_{3,r}/t_{3,r+1}]}{t_{3,r+1}},$$

$$\hat{d}_{n,r} = \frac{\langle 3|K_{3,r}K_{2,r}|1\rangle^2\,\langle 3|K_{3,r}\big[k_2,K_{2,r}\big]K_{2,r}|1\rangle}{[2|K_{2,r}|r\rangle[2|K_{2,r}|r{+}1\rangle\,\langle 3\,4\rangle\ldots\langle r{-}1\,r\rangle\langle r{+}1\,r{+}2\rangle\ldots\langle n\,1\rangle\,t_{2,r}\,t_{3,r}},$$
$$\widehat{g}_{n,r} = \sum_{j=1}^{r-3}\frac{\langle 3|K_{3,j+3}K_{2,j+3}|1\rangle^2\,\langle 3|K_{3,j+3}K_{2,j+3}\big[k_{r+1},K_{2,r}\big]|1\rangle\langle j{+}3\,j{+}4\rangle}{[2|K_{2,j+3}|j{+}3\rangle[2|K_{2,j+3}|j{+}4\rangle\,\langle 3\,4\rangle\langle 4\,5\rangle\ldots\langle n\,1\rangle\,t_{3,j+3}\,t_{2,j+3}},$$
$$\hat{h}_{n,r} = (-1)^n\,\widehat{g}_{n,n-r+2}\big|_{(123..n)\to(321n..4)}\,. \qquad (1)$$

$$A_n^{[0]}(1^-,2^-,3^-,4^+,5^+,\cdots,n^+) =$$
$$\frac{1}{3}A_n^{\mathcal{N}=1\ \text{chiral}}(1^-,2^-,3^-,4^+,5^+,\cdots,n^+)$$
$$-\frac{i}{3}\sum_{r=4}^{n-1}\ \hat{d}_{n,r}\,\frac{L_2[t_{3,r}/t_{2,r}]}{t_{2,r}^3} - \frac{i}{3}\sum_{r=4}^{n-2}\ \hat{g}_{n,r}\,\frac{L_2[t_{2,r}/t_{2,r+1}]}{t_{2,r+1}^3}$$
$$-\frac{i}{3}\sum_{r=4}^{n-2}\ \hat{h}_{n,r}\frac{L_2[t_{3,r}/t_{3,r+1}]}{t_{3,r+1}^3} + \mathrm{R}\,,$$

$$\hat{d}_{n,r} = \frac{\langle 3|K_{3,r}k_2|1\rangle\,\langle 3|k_2K_{2,r}|1\rangle\,\langle 3|K_{3,r}[k_2,K_{2,r}]K_{2,r}|1\rangle}{[2|K_{2,r}|r\rangle[2|K_{2,r}|r+1\rangle\,\langle 3\,4\rangle\ldots\langle r-1\,r\rangle\,\langle r+1\,r+2\rangle\ldots\langle n\,1\rangle},$$

$$L_2(r) = \frac{ln(r)-(r-1/r)}{(1-r)^3}$$

For R see Berger, Bern, Dixon, Forde and Kosower

# Conclusions

-Optimism in computing one-loop QCD matrix elements

-Recent progress uses UNITARITY and FACTORISATION as key features of on-shell amplitudes

-Inspired by Weak-Weak duality but not dependant upon it

-after much progress in highly super-symmetric theories the (harder) problem of QCD beginning to yield results

-many approaches complementing each other

-cut constructibility: BDDK; Britto, Buchbinder, Cachazo Feng Mastrolia

-on-shell recursion: Berger,Bern,Dixon,Forde Kosower;

Bidder, Bjerrum-Bohr, Dunbar, Ita,

-numerical: Binoth et al; Ellis, Giele, Zanderighi

-also fermions, masses, multiloop……..???…….