Four-loop tadpoles: applications in QCD

M. Czakon

U. Würzburg

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Anomalous dimensions for b -> sγ at NNLO

$\longrightarrow$ with U. Haisch and M. Misiak

Mass corrections to $\sigma(e^+e^- \to \text{had.})$ at $O(\alpha_s^3)$

- c- and b-quark masses

$\longrightarrow$ with R. Boughezal and T. Schutzmeier

- running of the fine structure constant

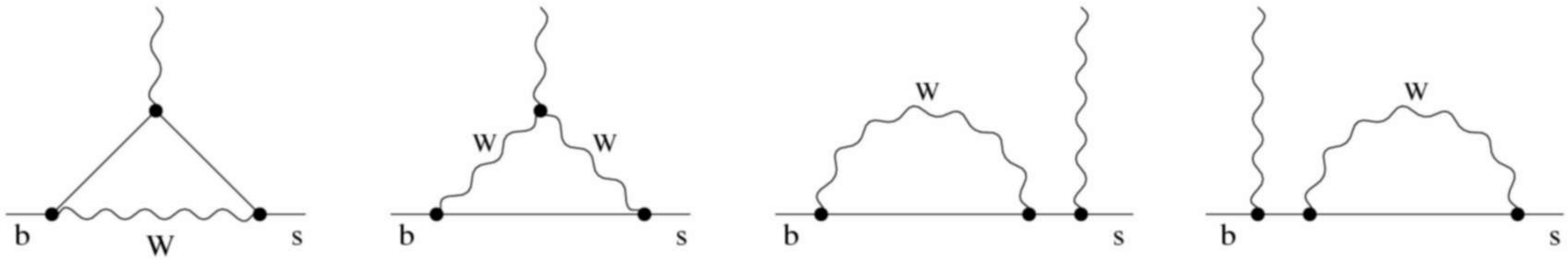
$\longrightarrow$ with T. Schutzmeier

$\Delta\rho$ at $O(\alpha\alpha_s^3)$

$\longrightarrow$ with R. Boughezal

Anomalous dimensions for b -> sγ at NNLO

Rare B decays provide strong constraints on extensions of the SM



Current exp. average: $(3.55 \pm 0.24\ ^{+0.09}_{-0.10} \pm 0.03) \times 10^{-4}$
(HFAG, hep-ph/0603003)

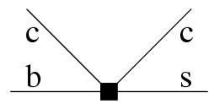
SM expectation (NLO): $(3.57 \pm 0.30) \times 10^{-4}$
(Gambino, Misiak, hep-ph/0104034, Buras et al. hep-ph/0203135)

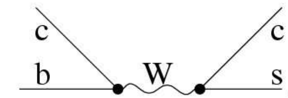
$$\mathcal{B}(\bar{B} \to X_s\gamma)_{E_\gamma > 1.6\,\mathrm{GeV}} = \mathcal{B}(\bar{B} \to X_c e\bar{\nu})_{\mathrm{exp}} \left[\frac{\Gamma(b\to s\gamma)}{\Gamma(b\to ce\bar{\nu})}\right]_{\mathrm{LO\ EW}} f\left(\frac{\alpha_s(M_W)}{\alpha_s(m_b)}\right) \times$$

$$\times \Big\{ 1 + \underbrace{\underset{\substack{\textbf{NLO}\\ \sim 25\%}}{\mathcal{O}(\alpha_s)} + \underset{\substack{\textbf{NNLO}\\ \sim 7\%}}{\mathcal{O}(\alpha_s^2)} + \underset{\sim 4\%}{\mathcal{O}(\alpha_{\mathrm{em}})}}_{\text{perturbative corrections}} + \underbrace{\underset{\sim 1\%}{\mathcal{O}\left(\frac{\Lambda^2}{m_b^2}\right)} + \underset{\sim 3\%}{\mathcal{O}\left(\frac{\Lambda^2}{m_c^2}\right)} + \underset{<\sim 5\%}{\mathcal{O}\left(\frac{\Lambda}{m_b}\alpha_s\right)}}_{\text{non-perturbative corrections}} \Big\}$$

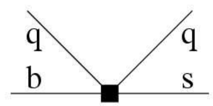
Effective Field Theory approach

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm QCD\times QED}(u,d,s,c,b) + \frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} \sum_{i=1}^{8} C_i(\mu)\, O_i + \begin{pmatrix}\text{higher-dimensional,}\\ \text{on-shell vanishing,}\\ \text{evanescent}\end{pmatrix}.$$

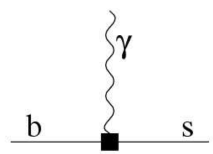
$O_{1,2} =$ 

$= (\bar{s}\Gamma_i c)(\bar{c}\Gamma'_i b),$ **from** 

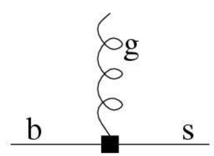
, $|C_i(m_b)| \sim 1$

$O_{3,4,5,6} =$ 

$= (\bar{s}\Gamma_i b)\Sigma_q(\bar{q}\Gamma'_i q),$ $|C_i(m_b)| < 0.07$

$O_7 =$ 

$= \frac{e m_b}{16\pi^2}\, \bar{s}_L \sigma^{\mu\nu} b_R F_{\mu\nu},$ $C_7(m_b) \simeq -0.3$

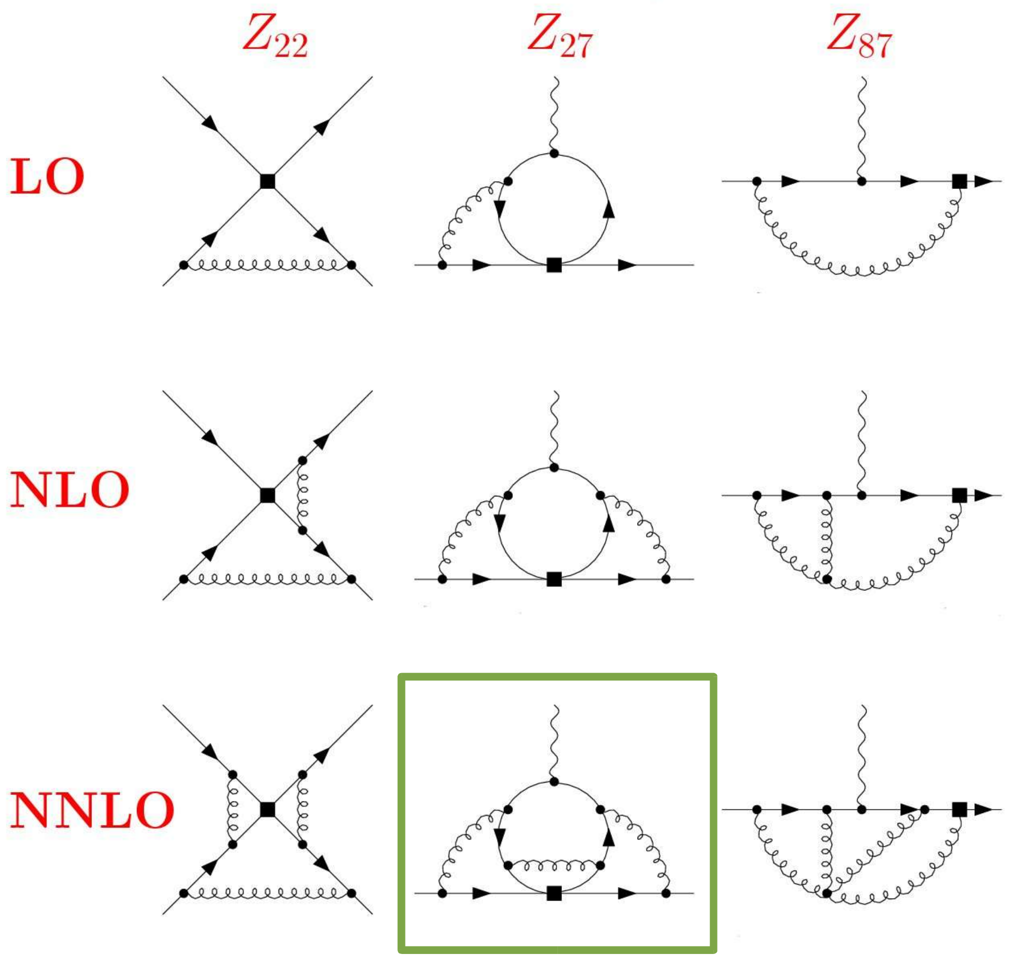
$O_8 =$ 

$= \frac{g m_b}{16\pi^2}\, \bar{s}_L \sigma^{\mu\nu} T^a b_R G^a_{\mu\nu},$ $C_8(m_b) \simeq -0.15$

Ingredients of the approach

- anomalous dimensions
- matching
- matrix elements

Resummation of large logarithms $\alpha_s \ln\left(\frac{M_W^2}{m_b^2}\right)$



$$\mu \frac{d}{d\mu} C_j(\mu) = C_i(\mu)\gamma_{ij}(\mu)$$

Obtaining divergences in the $\overline{\text{MS}}$ scheme

- infrared rearrangement by putting a single mass on all lines and setting external momenta to zero
- reduces to the evaluation of fully massive four-loop tadpoles

Original idea by

- Misiak & Münz '95 and van Ritbergen, Vermaseren, Larin '97

At four-loops used in

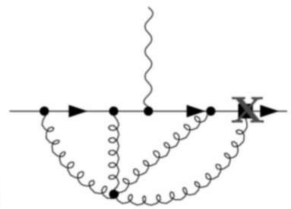
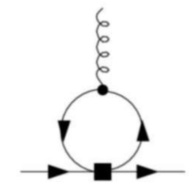
- van Ritbergen, Vermaseren, Larin '97 and M.C. '04

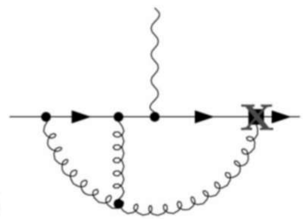
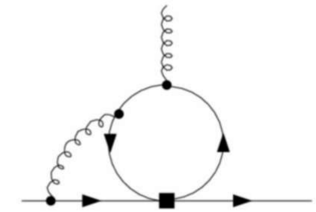
Mixing between $O_1 \ldots O_6$ and O_7, O_8 requires insertion of $\gamma^\alpha \times \gamma_\alpha$ and $\gamma^\alpha \gamma^\beta \gamma^\gamma \times \gamma_\alpha \gamma_\beta \gamma_\gamma$ and two color structures and projection on

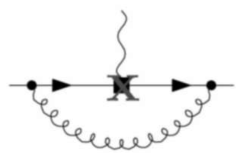
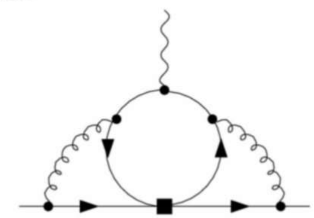
$$\boxed{\gamma_\mu \not{p} \not{k},\ \gamma_\mu \left(p \cdot k\right),\ \gamma_\mu p^2,\ \gamma_\mu k^2,\ \not{p} k_\mu,\ \not{p} p_\mu,\ \not{k} p_\mu,\ \not{k} k_\mu,\ M_b \not{k} \gamma_\mu,\ M_b \gamma_\mu \not{k},\ M_b \not{p} \gamma_\mu,\ M_b \gamma_\mu \not{p}.}$$

Large computational time -> finished on the DESY Zeuthen Grid Engine

Subtraction of subdivergences

3-loop diagrams  **with 1-loop counterterms** 

2-loop diagrams  **with 2-loop counterterms** 

1-loop diagrams  **with 3-loop counterterms** 

Types of off-shell operator counterterms (apart from the physical operators $O_1, \ldots, O_8$)

- gauge-invariant EOM-vanishing operators, e.g., $(\bar{s}_L \gamma^\mu T^a b_L) \left[D^\nu G^a_{\mu\nu} \; + \; g \sum_q (\bar{q} \gamma^\mu T^a q) \right]$,
- gauge-variant EOM-vanishing operators, e.g., $\bar{s}_L \left[-i \overleftarrow{\not{D}} \, \not{G} + \not{G}(i\not{D} - M_b) \right] b_R$
- BRS-exact operator $\delta_{\mathrm{BRS}} \left[(\bar{s}_L \gamma^\mu T^a b_L) \partial_\mu \bar{\eta}^a \right] \;=\; (\bar{s}_L \gamma^\mu T^a b_L) \left[\partial_\mu \partial^\nu G^a_\nu - g f^{abc} (\partial_\mu \bar{\eta}^b) \eta^c \right]$
- evanescent operators, e.g., $(\bar{s}_L \gamma^{[\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} \gamma^{\mu_4} \gamma^{\mu_5]} c_L)(\bar{c}_L \gamma_{[\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} \gamma_{\mu_4} \gamma_{\mu_5]} b_L)$
- so-called m^2-operators (due to propagator numerator simplification) e.g., $m^2 \bar{s}_L \not{G} b_L$

Completed result for the mixing with O_7

$$\hat{\gamma}^{(2)} = \begin{bmatrix} \frac{150994745}{1062882} + \frac{1272596}{6561}\zeta_3 \\ \frac{138336202}{177147} - \frac{2713672}{2187}\zeta_3 \\ -\frac{58397866}{177147} + \frac{3236560}{2187}\zeta_3 \\ -\frac{5108749081}{2125764} + \frac{2007886}{6561}\zeta_3 \\ \frac{5824017302}{177147} + \frac{112180720}{2187}\zeta_3 \\ \frac{3603565835}{531441} + \frac{15361912}{6561}\zeta_3 \end{bmatrix}$$

Numerical effect on the branching ratio

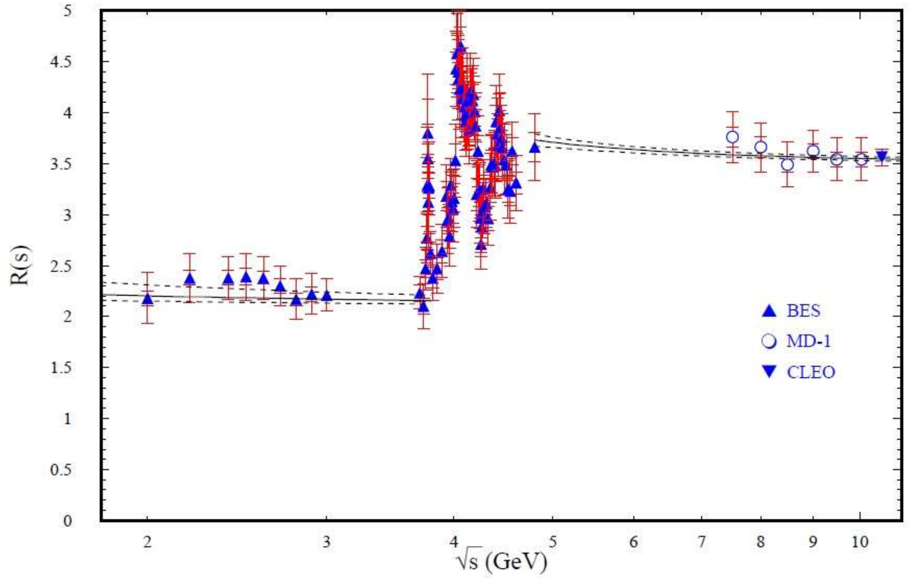
$\boxed{-2.4\ \%}$

Impact of the remaining mixing with O_8 expected to be 10 times smaller

Charm and bottom quark masses

Quark masses may be derived from the measurement of the production cross section around threshold

Example for charm quark threshold region



Idea: Subtract the contribution of the massless quarks evaluated in pQCD and compare the experimental predicition with theory

Starting from the definitions of the R-ratio and the current corr.

$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma_{\text{pt}}}, \quad \left(-q^2 g_{\mu\nu} + q_\mu q_\nu\right) \Pi(q^2) = i \int \mathrm{d}x\, e^{iqx} \langle 0|T j_\mu(x) j_\nu^\dagger(0)|0\rangle$$

One obtains the classic dispersion relation

$$\Pi_c(q^2) = \frac{q^2}{12\pi^2} \int \mathrm{d}s \frac{R_c(s)}{s(s-q^2)} + Q_c^2 \frac{3}{16\pi^2} \bar{C}_0$$

The moment can then be calculated or taken from experiment

$$\mathcal{M}_n \equiv \frac{12\pi^2}{n!} \left(\frac{\mathrm{d}}{\mathrm{d}q^2}\right)^n \Pi_c(q^2)\Big|_{q^2=0} \qquad \mathcal{M}_n^{\text{exp}} = \int \frac{\mathrm{d}s}{s^{n+1}} R_c(s)$$

The three-loop analysis done by Kühn & Steinhauser '01

The smallest error comes from the first moment

The calculation requires an expansion of propagators with respect to external momenta -> tadpoles

- 1st physical moment is equivalent to the second term in the Taylor expansion
- up to eight dots and six power of numerators
- very expensive computationally, but only 2 weeks real time
- parts of the calculation done at the DESY-Zeuthen Grid Engine
- next moment seems hardly reachable by the method because of resources usage

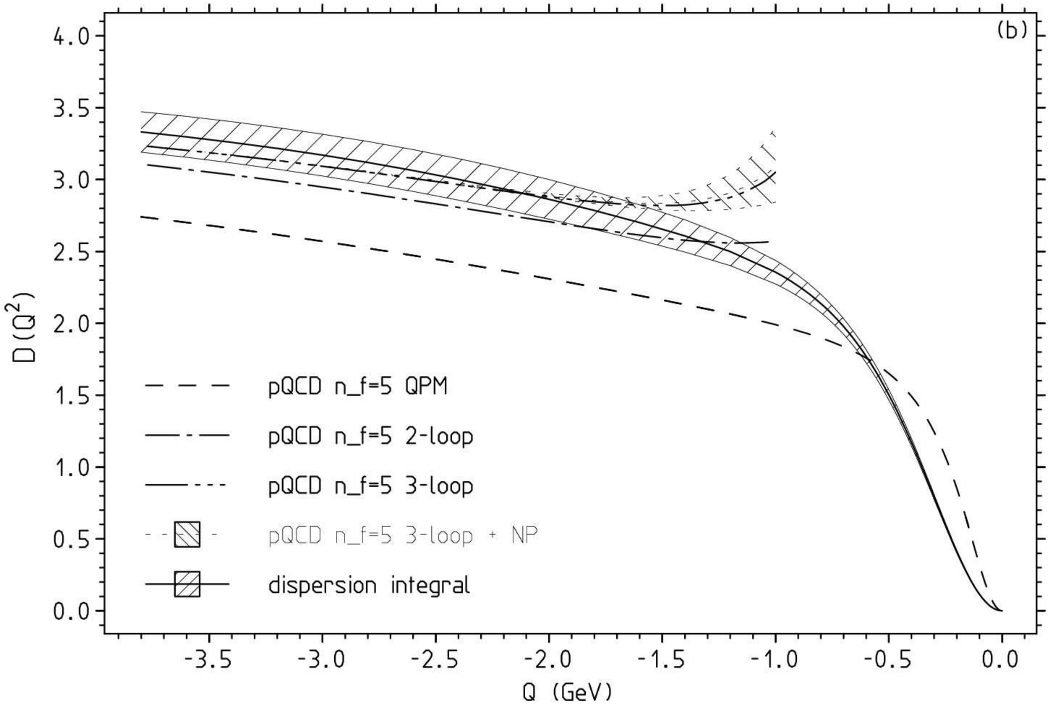
The result will be published soon after the Conference

Running of the fine structure constant

$\Delta\alpha(M_Z)$ contributes to any predicition in the electroweak sector

The low energy tail is evaluated with dispersion relations from data

Possible improvement of the prediction by moving to the Euclidean



Ideally the error would be smaller by a half

Eidelman, Jegerlehner, Kataev, Veretin '98

It is necessary to have the full mass dependence of the correlator

- At three loop done with Pade approximants and other tricks
- It looks hopless in view of the experience with direct expansion of tadpoles
- Solution: use differential equations on propagators
  inspired by Czyz, Caffo, Laporta, Remiddi '98 – '02
- Requires reduction of four-loop propagators with massive lines
- As a first step we implemented automatic expansion

It takes about $\frac{1}{2}$ hour to obtain 30 terms of the expansion of the three-loop result

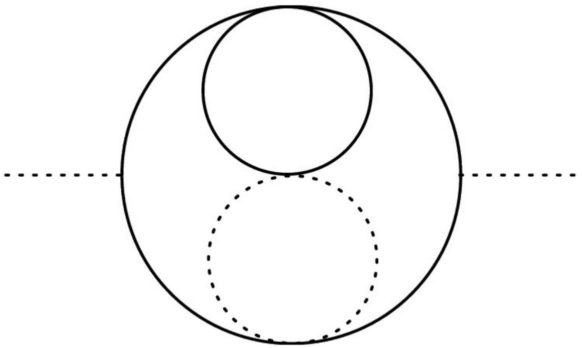
$$
\begin{aligned}
+a^2 \ &((2.03953 + 2.64911\, nl)\ x + (1.41227 + 0.454915\, nl)\ x^2 + (0.088057 + 0.107216\, nl)\ x^3 \\
&+ (-0.0299336 + 0.0265519\, nl)\ x^4 + (-0.0155361 + 0.00662735\, nl)\ x^5 \\
&+ (-0.00493409 + 0.00165382\, nl)\ x^6\ (-0.00132002 + 0.000412023\, nl)\ x^7 \\
&+ (-0.000319571 + 0.000102483\, nl)\ x^8 + (-0.0000713887 + 0.0000254583\, nl)\ x^9 \\
&+ \left(-0.0000146235 + 6.31819\, 10^{-6}\, nl\right)\ x^{10} + \left(-2.64326\, 10^{-6} + 1.56698\, 10^{-6}\, nl\right)\ x^{11} \\
&+ \left(-3.68366\, 10^{-7} + 3.88445\, 10^{-7}\, nl\right)\ x^{12} + \left(-1.1456\, 10^{-8} + 9.62635\, 10^{-8}\, nl\right)\ x^{13} \\
&+ \left(1.87424\, 10^{-8} + 2.38512\, 10^{-8}\, nl\right)\ x^{14} + \left(1.0366\, 10^{-8} + 5.909\, 10^{-9}\, nl\right)\ x^{15} \\
&+ \left(4.06362\, 10^{-9} + 1.46386\, 10^{-9}\, nl\right)\ x^{16} + \left(1.39426\, 10^{-9} + 3.62652\, 10^{-10}\, nl\right)\ x^{17} \\
&+ \left(4.43833\, 10^{-10} + 8.98461\, 10^{-11}\, nl\right)\ x^{18} + \left(1.35515\, 10^{-10} + 2.22607\, 10^{-11}\, nl\right)\ x^{19} \\
&+ \left(3.91083\, 10^{-11} + 5.51592\, 10^{-12}\, nl\right)\ x^{20} + \left(1.09139\, 10^{-11} + 1.36691\, 10^{-12}\, nl\right)\ x^{21} \\
&+ \left(1.81899\, 10^{-12} + 3.38778\, 10^{-13}\, nl\right)\ x^{22} + \left(8.39732\, 10^{-14}\, nl\right)\ x^{23} \\
&+ \left(-3.63798\, 10^{-12} + 2.08171\, 10^{-14}\, nl\right)\ x^{24}\ \left(-3.63798\, 10^{-12} + 5.16127\, 10^{-15}\, nl\right)\ x^{25} \\
&+ \left(-1.81899\, 10^{-12} + 1.27982\, 10^{-15}\, nl\right)\ x^{26} + \left(-3.63798\, 10^{-12} + 3.17392\, 10^{-16}\, nl\right)\ x^{27} \\
&+ \left(-1.81899\, 10^{-12} + 7.87226\, 10^{-17}\, nl\right)\ x^{28} + \left(1.9528\, 10^{-17}\, nl\right)\ x^{29} \\
&+ \left(-1.81899\, 10^{-12} + 4.84476\, 10^{-18}\, nl\right)\ x^{30})
\end{aligned}
$$

The 30th term of the expansion

$$
\begin{aligned}
&C_AC_F\Bigg(\frac{180859378850195691751750576523624402795753654564433230787270805191831949224136249217543143953184253712635714861447218126928556683117922510935160119}{1367932042459691620307436436594741725673364967600876741484160608505412131882028680361086966965582961708319465850518991602064328908760678400000000} \\
&\quad+\frac{1293415826360748028230444795561130382649926766110075535974885354331365173096444 37\,\zeta_3}{11759441208929267239889435877553653487049746953702726731961447704128946543001600}+\frac{806199376405209741519082847471962 8061\,\log(m^2)}{612652347429339234584008743993125710947607099363200000}-\frac{\log(m^2)^2}{3332043892622 89188}\Bigg) \\
&+C_F^2\Bigg(\frac{5452442682630450296642824032103939456655504765440407784934641427818025912549344757664277666932329289657359507785778454301397660514082981623265892971}{78487904075556076575016844722649115407488153878738829429419051307687581337493448873177121055402301081624887384865843780446313953781350400000000000} \\
&\quad+\frac{391017464232100196470520308890453148772440390084547069175166690802047037716992589 2631\,\zeta_3}{676601264152582593587495981972021328090878253779689442864145458504765615898624000}-\frac{27697863738158790844313383640803 08061\,\log(m^2)}{124799552254124658896742521924525607785623668388800 00}+\frac{15\log(m^2)^2}{101812452274588363}\Bigg) \\
&-C_FTn_l\Bigg(\frac{361881205526166411299344838699390555223129809673524504696 1313919}{497969025010575669793979151955857508569923872510111280688822999477335891520000000}-\frac{572187047516013914294764281428602 8061\,\log(m^2)}{16847939554306828951060240459810957051059195232488 00000}+\frac{\log(m^2)^2}{916312070471295267}\Bigg) \\
&+C_FT\Bigg(\frac{632311719038467538274507225719631996054924112237114247584089620611774352771374642 30373}{679749971204285601184425700590889619059575904898137817715408420215930495406987400921151208161280000000}+\frac{789459381769239507159701086942265363888 9\,\zeta_3}{103443449008447497939188424332645869894264890747393146880} \\
&\quad-\frac{572187047516013914294764281428602 8061\,\log(m^2)}{16847939554306828951060240459810957051059195232488 00000}+\frac{\log(m^2)^2}{916312070471295267}\Bigg)
\end{aligned}
$$

46 masters with up to two irreducible numerators needed for the four-loop n_f^2 contribution. All expanded to 30 terms within $\frac{1}{2}$ hour.

Example 8-liner



Finite part

$$
\begin{aligned}
&+2078.45 + 1542.65\,x - 36.0108\,x^2 - 25.175\,x^3 - 27.7474\,x^4 - 13.2539\,x^5 - 4.80407\,x^6 \\
&-1.53205\,x^7 - 0.455313\,x^8 - 0.129593\,x^9 - 0.0358494\,x^{10} - 0.00972235\,x^{11} - 0.00259905\,x^{12} \\
&-0.000687345\,x^{13} - 0.000180272\,x^{14} - 0.0000469721\,x^{15} - 0.0000121751\,x^{16} - 3.14227\,10^{-6}\,x^{17} \\
&-8.08124\,10^{-7}\,x^{18} - 2.07217\,10^{-7}\,x^{19} - 5.30008\,10^{-8}\,x^{20} - 1.35273\,10^{-8}\,x^{21} - 3.44616\,10^{-9}\,x^{22} \\
&-8.76528\,10^{-10}\,x^{23} - 2.22632\,10^{-10}\,x^{24} - 5.64775\,10^{-11}\,x^{25} - 1.43117\,10^{-11}\,x^{26} - 3.62314\,10^{-12}\,x^{27} \\
&-9.16439\,10^{-13}\,x^{28} - 2.31625\,10^{-13}\,x^{29} - 5.85011\,10^{-14}\,x^{30}
\end{aligned}
$$

$\Delta\rho$ at $O(\alpha\alpha_s^3)$

Measures the relative strength of the charged and neutral currents

$$\boxed{\Delta\rho = \frac{\Pi_{ZZ}(0)}{M_Z^2} - \frac{\Pi_{WW}(0)}{M_W^2}}$$

Small contributions imply shifts in the electroweak observables by

$$\boxed{\Delta M_W = \frac{M_W}{2}\frac{c_W^2}{c_W^2 - s_W^2}\Delta\rho}$$

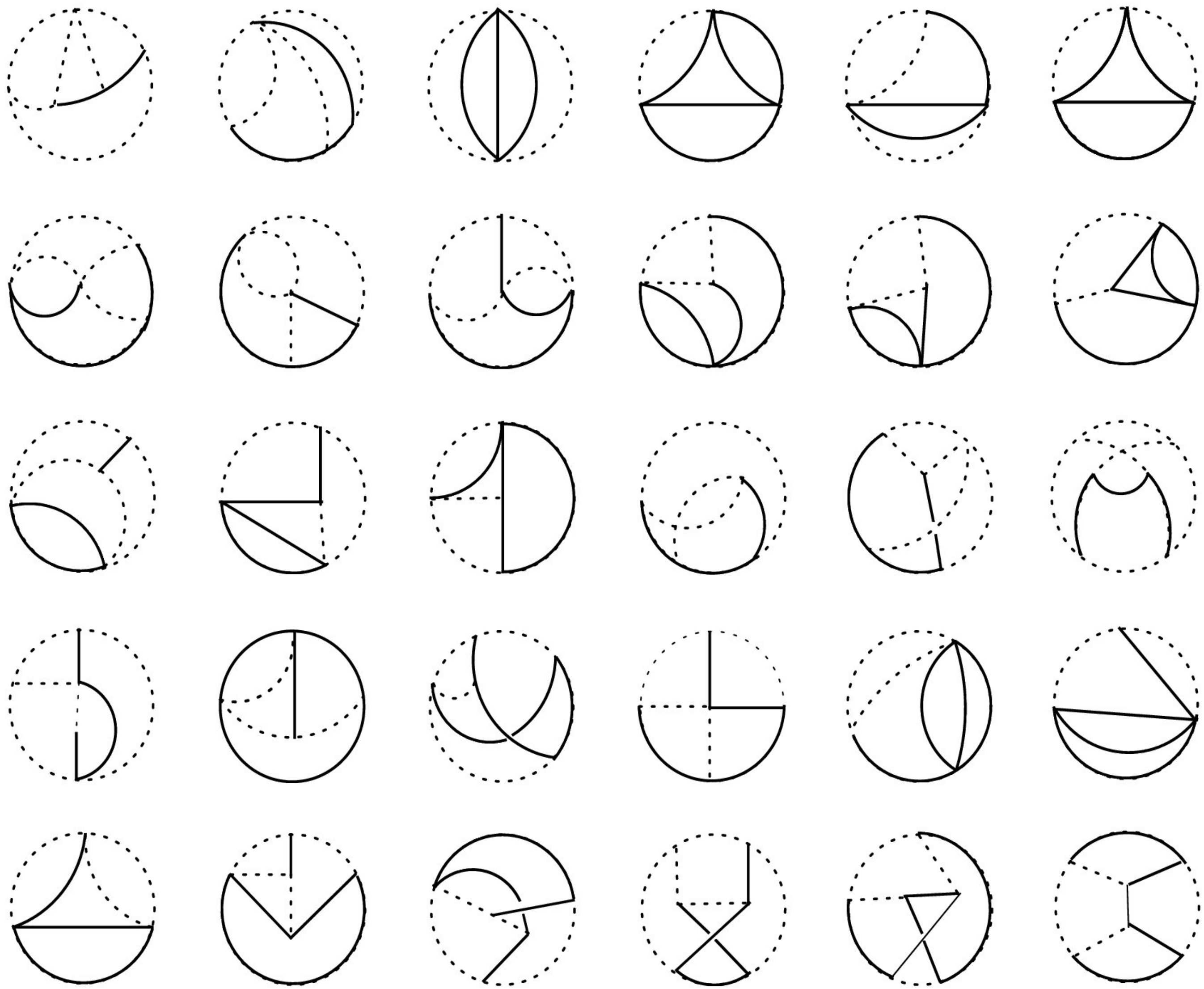
and

$$\boxed{\sin^2\theta_{\mathrm{eff}}^{\mathrm{lept}} = -\frac{c_W^2 s_W^2}{c_W^2 - s_W^2}\Delta\rho}$$

Recently the singlet term at $O(\alpha\alpha_s^3)$ has become available from Schröder & Steinhauser '05

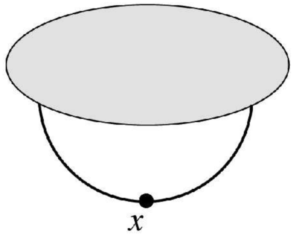
Surprisingly small value has been found. Need for complete result.

30 nontrivial new masters



Masters computed with Laporta's difference equation method

- chose a massive propagator and assign a power to it

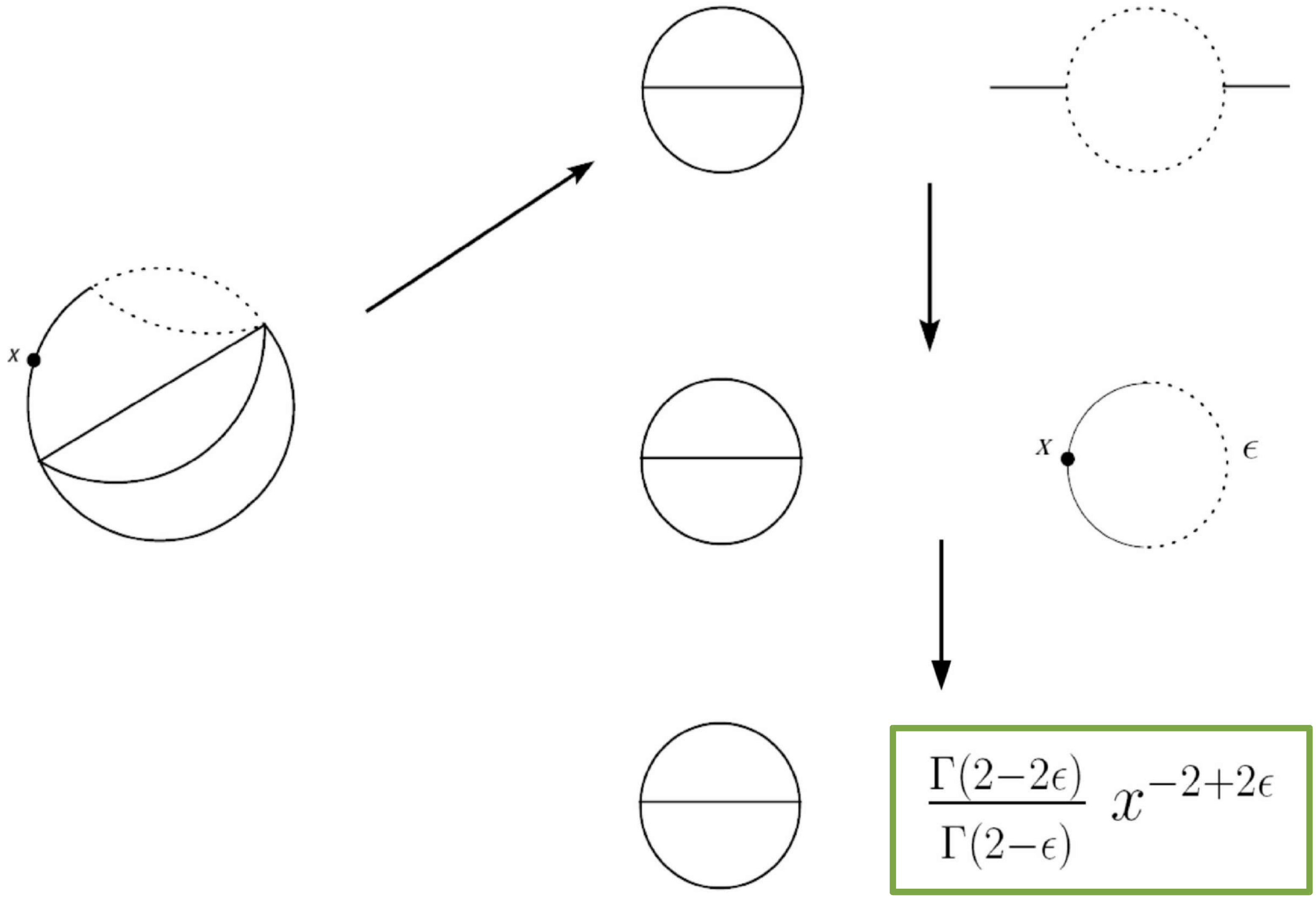


- write down a recurrence relation in x, by using IBP relations (highest order encountered was 4)

- find the behaviour of the integral for large x from the expansion of the propagator subloop

- solve the recurrence with the boundary condition at large x, using an ansatz for the solution as factorial series $\boxed{\sum_{s=0}^{\infty} \frac{a_s \Gamma(x+1)}{\Gamma(x-K+s+1)}}$

Example of a propagator subgraph with a threshold at 0



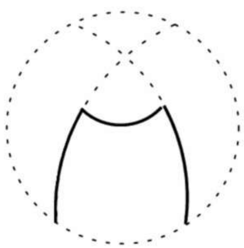
$$\boxed{\frac{\Gamma(2-2\epsilon)}{\Gamma(2-\epsilon)}\; x^{-2+2\epsilon}}$$

Very efficient computational method

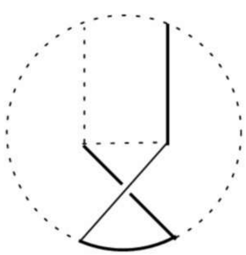
- fast derivation of recurrence relations
  (no longer than 12 hours on 1 CPU)
- fast numerical evaluation
  (a few hours for 40 digits)

28 masters computed, 2 9-liners remain to be done

Example results for two of the 9-liners



$$= \quad 6.72847056008568105554 + 26.0876465999666155538\,\epsilon$$



$$= \quad 3.71140264536682392668 + 2.11520599545877264 75\,\epsilon$$

Conclusions

- Completed ADM mixing of four-quark operators with the magnetic penguin operator in $b \to s\gamma$
- Completed calculation of the 1st physical moment of the heavy quark current correlator
- Developed automatized method to calculate the complete mass dependence of the quark current correlator
- Almost completed calculation of the masters needed for four-loop QCD corrections to $\Delta\rho$