Unitarity Cuts

NLO six-gluon Amplitude in QCD

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in collaboration with: Ruth Britto and Bo Feng $\longrightarrow$ hep-ph/0602187 to appear in PRD

Outline

- Need for NLO
- Unitarity & Cut-Constructibility
- General Algorithm for Cuts
- NLO six-gluon amplitude in QCD

Need for NLO

- ▷ Accurate estimates for signal and background new physics processes
- ▷ Less sensitivity to unphysical input scales ($\longrightarrow$ renormalization & factorization scale)
- ▷ More realistic process modelling: initial state radiation, jet clustering, richer virtuality

- ≤ 5 legs — Bern, Campbell, Dixon, Dawson, Ellis, Glower, Kosower, Kramer, Kunszt, Nagy, Oleari, Reina, Signer, Trocsanyi, Wakeroth, Zeppefeld, ...

- $2 \longrightarrow 4$ — Denner, Dittmaier, Roth, Wieders (2005); Boudjema et al. (2005); Ellis, Giele, Znaderighi (2006)

NLO Building Blocks

- Born level n-point
- Real contribution $(n+1)$-point
- $\mathrm{Virtual}$ contribution n-point
- IR safety: $\mathrm{R} + \mathrm{V}$

)($\mathrm{Virtual}$ contribution: *Tensor Reduction* $\longrightarrow \quad I^{\mu\nu\rho\ldots} = \int d^D\ell \, \frac{\ell^\mu \ell^\nu \ell^\rho \ldots}{D_1 D_2 \ldots}$

(some) Algebraic-Semi-Numerical Approaches

- Ellis, Giele, Zanderighi $\Rightarrow$ $gg \to gggg$
- Ferroglia, Passera, Passarino, Uccirati
- Binoth, Guillet, Heinrich, Pilon, Schubert
- GRACE group
- Nagy, Soper
- Anastasiou, Daleo
- Czakon

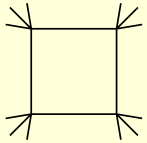
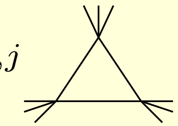
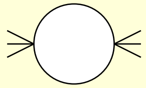
Analytic Approach

One Loop Amplitudes

Britto, Buchbinder, Cachazo & Feng; hep-ph/0503132

Britto, Feng & PM; hep-ph/0602187

Tensor Reduction

$\mathcal{A}_g = \sum_i c_{4,i}$  $+ \sum_j c_{3,j}$  $+ \sum_k c_{2,k}$  $+ \text{rational}$

Since the D-regularised scalar functions associated to **boxes** $(I_4^{(4m)}, I_4^{(3m)}, I_4^{(2m,\mathrm{e})}, I_4^{(2m,\mathrm{h})}, I_4^{(1m)})$, **triangles** $(I_3^{(3m)}, I_3^{(2m)}, I_3^{(1m)})$ and **bubbles** (I_2) are analytically known

Bern, Dixon & Kosower (1993)

- $\mathcal{A}_g$ is known, once the coefficients c_4, c_3, c_2 and the $\mathrm{rational\ term}$ are known: they all are rational functions of spinor products $\langle i\, j \rangle$, $[i\, j]$

Supersymmetric decomposition

Bern, Dixon, Dunbar & Kosower (1994)

$$\mathcal{A}_g = \underbrace{(\mathcal{A}_g + 4\mathcal{A}_f + 3\mathcal{A}_s)}_{\mathcal{N}=4} - 4\underbrace{(\mathcal{A}_f + \mathcal{A}_s)}_{\mathcal{N}=1} + \underbrace{\mathcal{A}_s}_{\mathcal{N}=0}$$

- non-zero coefficients

$\mathcal{N}$	Box	Triangle	Bubble	rational
4	$\times$			
1	$\times$	$\times$	$\times$	
0	$\times$	$\times$	$\times$	$\times$

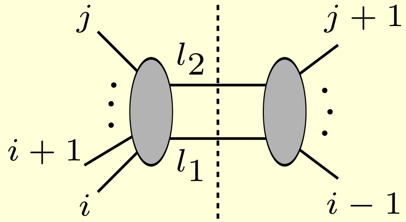
Massless SuSy 1-Loop amplitudes constructible from 4-dim tree amplitudes, *i.e.* fully determined by their discontinuities (or absorptive parts)

Unitarity & Cut-Constructibility

- Discontinuity accross the Cut

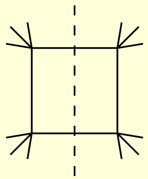
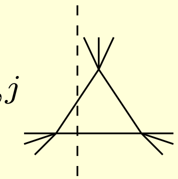
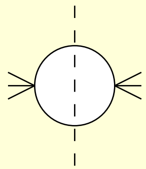
Bern, Dixon, Dunbar & Kosower (1994)

Cut Integral in the P_{ij}^2-channel



$$C_{i\ldots j} = \Delta(A_n^{\text{1-loop}}) = \int d\mu \; A^{\text{tree}}(\ell_1, i, \ldots, j, \ell_2) A^{\text{tree}}(-\ell_2, j+1, \ldots, i-1, -\ell_1)$$

with $d\mu = d^4\ell_1 \; d^4\ell_2 \; \delta^{(4)}(\ell_1 + \ell_2 - P_{ij}) \; \delta^{(+)}(\ell_1^2) \; \delta^{(+)}(\ell_2^2)$

$$C_{i\ldots j} = \Delta(A_n^{\text{1-loop}}) = \sum_i c_{4,i}$$  $$+ \sum_j c_{3,j}$$  $$+ \sum_k c_{2,k}$$ 

- The Cut carries information about the coefficients.
- In 4-dim we lose any information about the rational term

IR Divergence

In Dim-reg, $D = 4 - 2\epsilon$, the divergent behaviour of $\mathcal{A}_g$ is:

$$\mathcal{A}_g^{1-\text{loop}}|_{\text{singular}} = \frac{1}{\epsilon}\, \mathcal{A}_g^{\text{tree}}$$

- Divergence of scalar functions

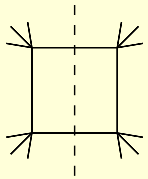
Function	$I_4^{(4m)}$	$I_4^{(3m,2m,1m)}$	$I_3^{(3m)}$	$I_3^{(2m,1m)}$	I_2
Divergence	none	$\frac{(-P^2)^{-\epsilon}}{\epsilon^2}$	none	$\frac{(-P^2)^{-\epsilon}}{\epsilon^2}$	$\frac{1}{\epsilon}$

- No need for 1m- & 2m-triangles

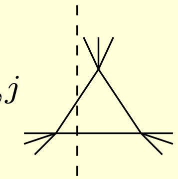
Their coefficients are related to the Box-coefficients, as required by the mutual cancelation of the corresponding divergent pieces.

Reduced Basis

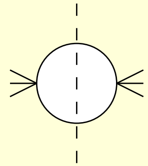
$$C_{i\ldots j} = \Delta(A_n^{\text{1-loop}}) = \sum_i c_{4,i}$$



$$+ \sum_j c_{3,j}$$



$$+ \sum_k c_{2,k}$$



with **boxes** $(I_4^{(4m)}, I_4^{(3m)}, I_4^{(2m,\mathrm{e})}, I_4^{(2m,\mathrm{h})}, I_4^{(1m)})$, **triangles** $(I_3^{(3m)})$ and **bubbles** (I_2)

The global divergence of the amplitude is carried by the **bubble** coefficients:

$$\sum_k c_{2,k} = \mathcal{A}_g^{\text{tree}}$$

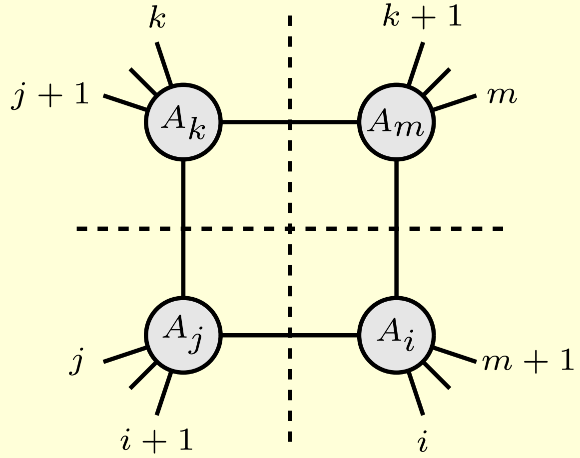
- coefficients show up entangled in a given cut: how do we disentangle them?

The polylogarithmic structure of boxes, 3m-triangles, and bubbles is different. Therefore their double cuts have specific signature which enable us to distinguish unequivocally among them.

Quadruple Cuts

Boxes

- Multiple Cuts   Bern, Dixon, Dunbar, Kosower (1994)



The discontinuity across the leading singularity, *via* quadruple cuts, is **unique**, and corresponds to the **coefficient** of the master **box**   Britto, Cachazo, Feng (2004)

$$c_{4,i} \quad \propto \quad A_j^{\rm tree} A_k^{\rm tree} A_m^{\rm tree} A_i^{\rm tree}$$

Double Cuts

3m-Triangles & Bubbles

$$C_{i\ldots j} = \Delta(A_n^{\text{1-loop}}) = \int d\mu \; A^{\text{tree}}(\ell_1, i, \ldots, j, \ell_2) A^{\text{tree}}(-\ell_2, j+1, \ldots, i-1, -\ell_1)$$

with $d\mu = d^4\ell_1 \; d^4\ell_2 \; \delta^{(4)}(\ell_1 + \ell_2 - P_{ij}) \; \delta^{(+)}(\ell_1^2) \; \delta^{(+)}(\ell_2^2)$

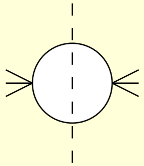
- Twistor-motivated Integration Measure — Cacahazo, Svrček & Witten (2004)

Use the $\delta^{(4)}$ integral to reduce just to a single loop momentum variable ℓ such that,

$$\ell_\mu \sigma^\mu = \begin{pmatrix} \ell_0 + \ell_3 & \ell_1 + i\ell_2 \\ \ell_1 - i\ell_2 & \ell_0 - \ell_3 \end{pmatrix} = |\ell\rangle[\ell| \equiv \begin{pmatrix} \sqrt{t}\, e^{\frac{(i\theta+\phi)}{2}} \\ \sqrt{t}\, e^{-\frac{(i\theta+\phi)}{2}} \end{pmatrix} \left(\sqrt{t}\, e^{-\frac{(i\theta-\phi)}{2}}, \sqrt{t}\, e^{\frac{(i\theta-\phi)}{2}} \right) = t\, |\lambda\rangle[\lambda| \,,$$

$$\Rightarrow \int d^4\ell \; \delta^{(+)}(\ell^2) \; f(|\ell\rangle, |\ell]) = \int_0^\infty t \; dt \; \int \langle\lambda \; d\lambda\rangle[\lambda \; d\lambda] \; t^\alpha f(|\lambda\rangle, |\lambda])$$

Cutting a Bubble

$$\Delta I_2 = $$  $$ = \int d^4\ell\, \delta^{(+)}(\ell^2)\, \delta^{(+)}((\ell - K)^2) = \int_0^\infty t\, dt \int \langle\lambda\, d\lambda\rangle [\lambda\, d\lambda]\, \delta^{(+)}((\ell - K)^2)$$

Since

$$\begin{aligned} \delta^{(+)}((\ell - K)^2) &= \delta^{(+)}(K^2 - 2\ell\cdot K) \\ &= \delta^{(+)}(K^2 - \langle\ell|K|\ell]) \quad = \frac{1}{\langle\lambda|K|\lambda]}\, \delta^{(+)}\left(t - \frac{K^2}{\langle\lambda|K|\lambda]}\right) \end{aligned}$$

- after the t-integration

$$\Delta I_2 = \int \langle\lambda\ d\lambda\rangle[\lambda\ d\lambda] \frac{K^2}{\langle\lambda|K|\lambda]^2}$$

- Derivative Form (*via* Integration-by-Parts)

$$\frac{[\lambda\ d\lambda]}{\langle\lambda|K|\lambda]^2} = [d\lambda\ \partial_{|\lambda]}]\ \frac{[\eta\ \lambda]}{\langle\lambda|K|\eta]\langle\lambda|K|\lambda]}\ , \qquad \forall \eta : \eta^2 = 0$$

- Holomorphic Anomaly   Cachazo, Svrček, Witten (2004)

The last integration is carried out along the contour $|\lambda] = |\lambda\rangle^*$, and that give rise to a delta-function like contribution, similar to

$$\frac{\partial}{\partial \bar{z}} \frac{1}{z-a} = 2\pi \delta(z-a)$$

In our case,

$$\Delta I_2 = \int \langle \lambda \; d\lambda \rangle [d\lambda \; \partial_{|\lambda]}] \; \frac{K^2 \; [\eta \, \lambda]}{\langle \lambda | K | \eta] \langle \lambda | K | \lambda]}$$

Cauchy Residue Theorem would imply the contribution at the pole

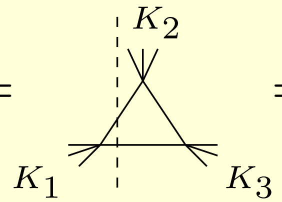
$$|\lambda\rangle = K|\eta] \quad (\Leftrightarrow |\lambda] = K|\eta\rangle)$$

With the final result

$$\Delta I_2 = \lim_{|\lambda\rangle \to K|\eta]} \langle \lambda | K | \eta] \left(\frac{K^2 [\eta \, \lambda]}{\langle \lambda | K | \eta] \langle \lambda | K | \lambda]} \right) \; = \; 1.$$

- The discontinuity of a bubble is **rational** !!!

Cutting a 3m-Triangle

$$\Delta I_3 = $$ 

$$= \int d^4\ell \; \delta^{(+)}(\ell^2) \; \frac{\delta^{(+)}((\ell - K_1)^2)}{(\ell + K_3)^2}$$

- after the t-integration

$$\Delta I_3 = \int \langle \lambda \; d\lambda \rangle [\lambda \; d\lambda] \frac{1}{\langle \lambda | K_1 | \lambda] \langle \lambda | Q | \lambda]}$$

with $\hat{Q} = (K_3^2 / K_1^2) \hat{K}_1 + \hat{K}_3$

- Feynman parameter

$$\Delta I_3 = \int_0^1 dz \; \int \langle \lambda \; d\lambda \rangle [\lambda \; d\lambda] \frac{1}{\langle \lambda | (1-z) K_1 + z Q | \lambda]^2} = \int_0^1 dz \frac{1}{[(1-z) K_1 + z Q]^2}$$

Finally

$$\Delta I_3 = \frac{1}{\sqrt{\Delta_{3m}}} \ln \left(\frac{2a + b - \sqrt{\Delta_{3m}}}{2a + b + \sqrt{\Delta_{3m}}} \right)$$

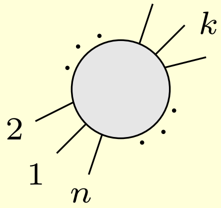
with $a = (Q - K_1)^2$, $b = 2((K_1 \cdot Q) - K_1^2)$ and

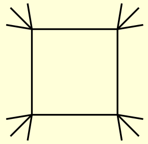
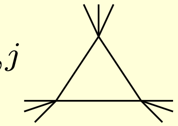
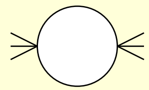
$$\Delta_{3m} = (K_1^2)^2 - 2K_1^2 K_2^2 - 2K_1^2 K_3^2 - (K_2^2 - K_3^2)$$

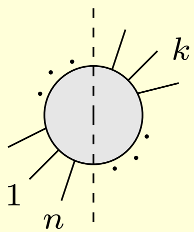
- The discontinuity a 3m-Triangle is a $\ln$**(irrational argument)** !!!

- The double cut detect box-coefficient as well. One can show that the discontinuity of a 1m-,2m-,3m-box is a $\ln$**(rational argument)**. When it appears, it is flagged and forgotten (later used as crosscheck), since we prefer to compute it from 4-ple cut.

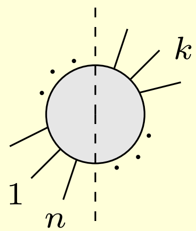
Cut-Constructible $\mathcal{A}_g$

$\mathcal{A}_g =$ 

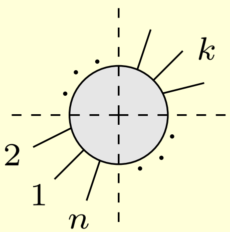
$= \sum_i c_{4,i}$  $+ \sum_j c_{3,j}$  $+ \sum_k c_{2,k}$ 

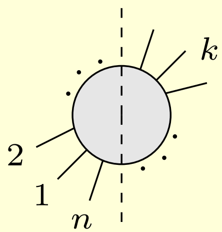
$c_{2,i} = \Bigg[$ 

$\Bigg]_{\propto \text{ rational}}$

$c_{3,i} = \Bigg[$ 

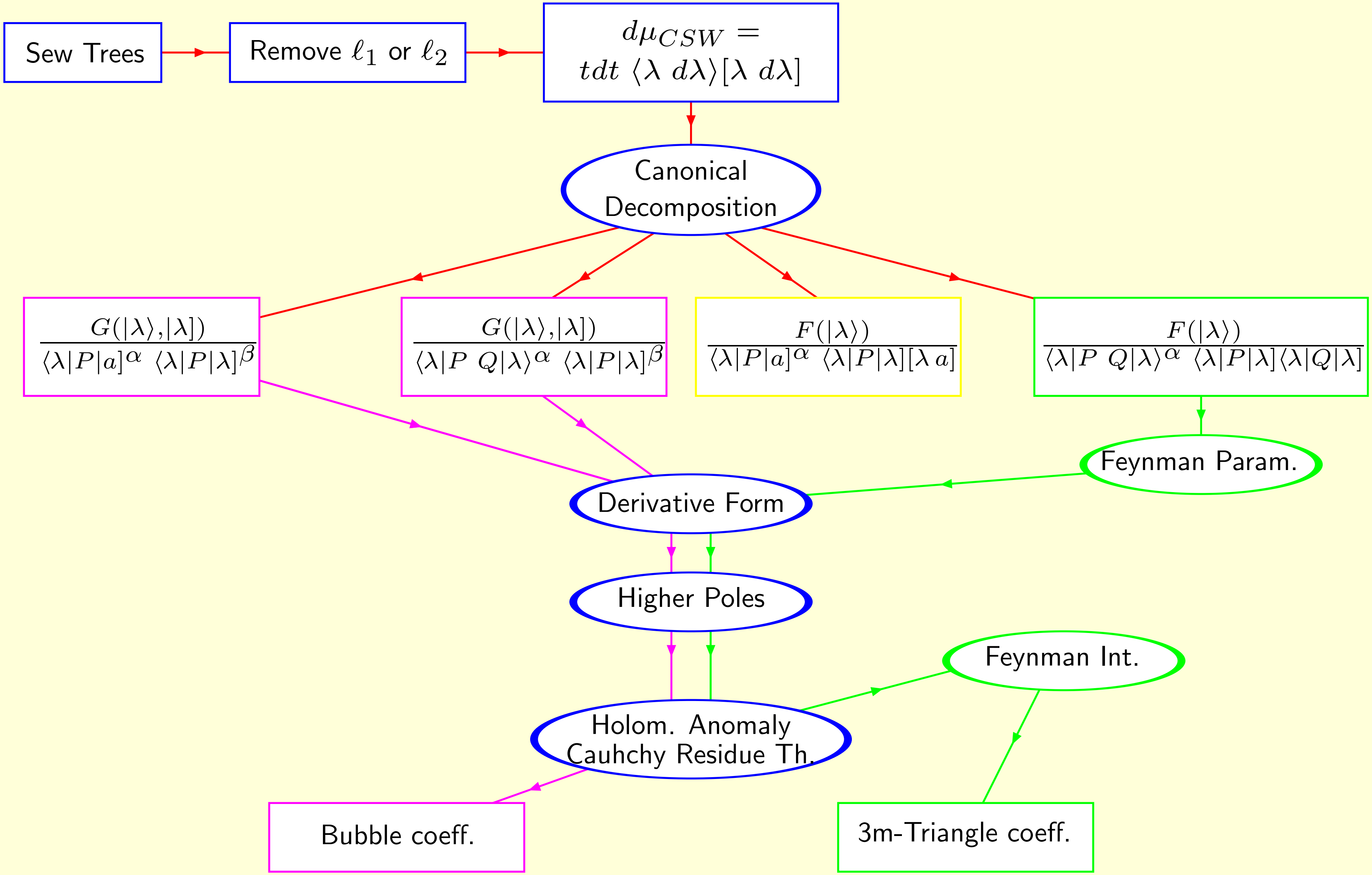
$\Bigg]_{\propto \ln(\text{irr.})}$

$c_{4,i} =$ 

$= \Bigg[$ 

$\Bigg]_{\propto \ln(\text{rat.})}$

General Algorithm



Master Formula

From the anticommutation of the γ-matrices one has (Schouten identity):

$$\frac{\langle \lambda\, a\rangle}{\langle \lambda\, b\rangle\ \langle \lambda\, c\rangle} = \frac{\langle b\, a\rangle}{\langle b\, c\rangle}\frac{1}{\langle \lambda\, b\rangle} + \frac{\langle c\, b\rangle}{\langle c\, b\rangle}\frac{1}{\langle \lambda\, c\rangle}$$

which holds in the more general case of $|i\rangle = P|j],\ (\forall P : P^2 \neq 0)$

Trifold Usefulness

- NO Tensor Reduction: the power of the loop (spinor) variable decreases.
- Canonical Decomposition: when $|b\rangle = P_{\rm cut}|\lambda]$,

$$\frac{\langle \lambda\, a\rangle}{\langle \lambda|P|\lambda]\ \langle \lambda\, c\rangle} = \frac{[\lambda|P|a\rangle}{[\lambda|P|c\rangle}\frac{1}{\langle \lambda|P|\lambda]} + \frac{\langle c|P|\lambda]}{\langle c|P|\lambda]}\frac{1}{\langle \lambda\, c\rangle}$$

it allows $|\lambda\rangle$ vs $|\lambda]$, needed to recast the integrand in a Derivative Form wrt $|\lambda\rangle$.

- Higher Poles lifting: Algebraic extraction of the single poles hidden beneath the higher ones.

$$\mathcal{I}_{2-\rm pole} = \frac{1}{\langle \lambda\, \eta\rangle^2}\frac{\prod_{j=1}^{N}\langle \lambda\, a_j\rangle}{\prod_{k=1}^{N}\langle \lambda\, b_k\rangle} = (\text{after } N \text{ iteration}) =$$

$$= \frac{1}{\langle \lambda\, \eta\rangle}\sum_{m=0}^{N-1}\frac{\prod_i^m\langle \eta\, a_i\rangle}{\prod_i^m\langle \eta\, b_i\rangle}\frac{\langle a_{m+1}\, b_{m+1}\rangle}{\langle \eta\, b_{m+1}\rangle\langle \lambda\, a_{m+1}\rangle}\frac{\prod_{j=m+1}^{N}\langle \lambda\, a_j\rangle}{\prod_{k=m+1}^{N}\langle \lambda\, b_k\rangle} + \frac{1}{\langle \lambda\, \eta\rangle^2}\frac{\prod_i^N\langle \eta\, a_i\rangle}{\prod_i^N\langle \eta\, b_i\rangle}$$

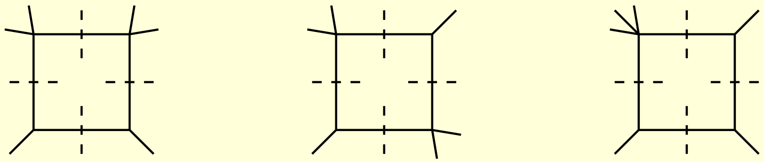
where the last term, being a pure double pole, has zero residue.

6-gluon Amplitude in QCD

- Numerical Status: known! Ellis, Giele, Zanderighi (2006)
- Analytical Status:

Amplitude	$\mathcal{N}=4$	$\mathcal{N}=1$	$\mathcal{N}=0\vert_{\rm CC}$	$\mathcal{N}=0\vert_{\rm rat}$
$(--++++)$	BDDK'94	BDDK'94	BDDK'94	BDK'05, KF'05
$(-+-+++)$	BDDK'94	BDDK'94	BBST'04	
$(-++-++)$	BDDK'94	BDDK'94	BBST'04	
$(---+++)$	BDDK'94	BBDD'04	BBDI'05, BFM'06	BBDFK$^{\rm two\ days\ ago}$
$(--+-++)$	BDDK'94	BBCF'05, BBDP'05	BFM'06	
$(-+-+-+)$	BDDK'94	BBCF'05, BBDP'05	BFM'06	

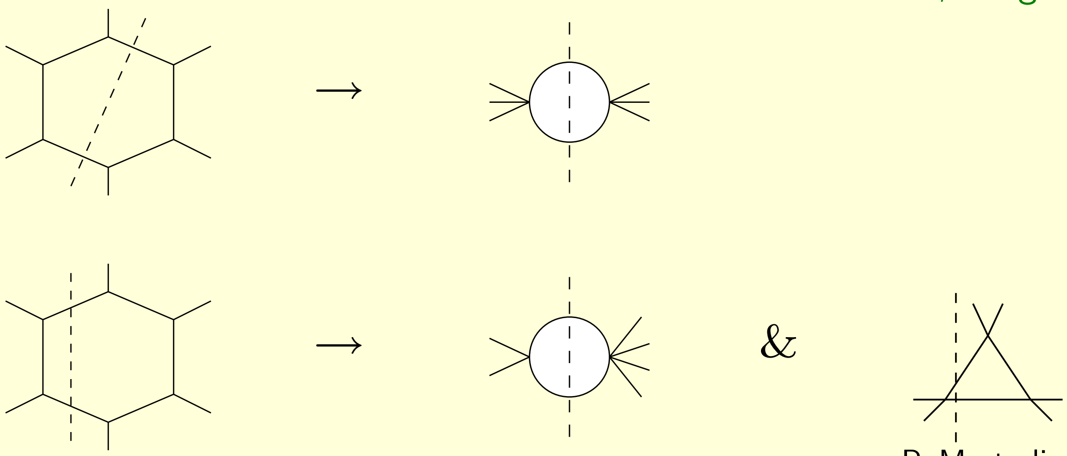
Quadruple Cuts



Bidder, Bjerrum-Bohr, Dunbar & Perkins (2005)

Double Cuts

Britto, Feng & PM [hep-ph/0602187]



Tandem Tools for One-Loop Amplitudes

Unitarity-based methods

Bern, Dixon, Dunbar & Kosower (Jurassical)

Brandhuber, Mc Namara, Spence, & Travaglini (2004,2005)

Quigley & Rozali (2004)

Britto, Buchbinder, Cachazo, Svrček & Witten (2004, 2005)

Britto, Feng & PM (2006)

▷ terms with discontinuities $\Leftarrow \text{input} : 4\text{D} - \text{Cuts}$

on-shell Recurrence Relations

Britto, Cachazo, Feng & Witten (2004)

Bern, Dixon & Kosower (2005)

Bern, Bjerrum-Bohr, Dunbar & Ita (2005)

Forde & Kosower (2005)

Berger, Bern, Dixon, Forde & Kosower (2006)

▷ rational terms $\Leftarrow \begin{cases} \text{input1} : \text{rational term @ less number of legs} \\ \text{input2} : \text{cut term @ same number of legs} \end{cases}$

▷ terms with discontinuities $\Leftarrow \text{input} : \text{cut term @ less number of legs w/in the same class of polylog}$

Summary & Outlook

- We developed a systematic technique for finite cuts applicable to generic helicity configuration and external states in non-supersymmetric gauge theories
  - CSW integration measure
  - Canonical Decomposition: algebraic spinor reduction to Box, Bubble and 3m-Triangle functions
  - One-Feynman Parameter for the residual integration
  - (purely) Algebraic treatment of higher poles (characteristic of non-SYM)

- Advantages:
  - You do not encounter the main difficulties coming form the *tensor reduction*
  - The computational problem is reduced by trivial spinor algebra to the the extraction of residues

- Application: $gg \longrightarrow gggg$
  - The *cut-terms* of the amplitude are an important ingredient for the reconstruction of the *rational term*
  - The 6-point amplitude contains the complete polylog structure of the all-n amplitude ($I_4^{4m} \longrightarrow via$ 4ple-cut): is a bootstrapp point to tackle more legs

multileg NLO QCD: squeezing out of the bottleneck!