Three-loop planar integrals for four-point one-mass processes

Nikolaos Syrrakos

based on 2112.14275(JHEP), in collaboration with Dhimiter Canko.

Technical University Munich

Loops and Legs in Quantum Field Theory, 26.03.2022

Table of Contents

Table of Contents

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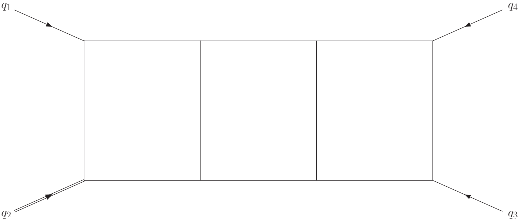
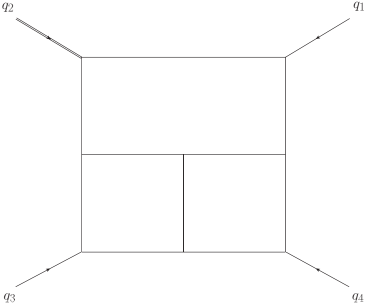
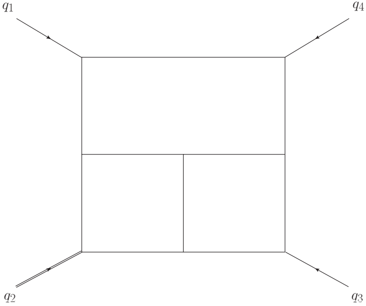
- This talk: **Analytic results for all three-loop planar master integrals (ladder & tennis-court topologies) with one massive leg for scattering kinematics.**

Table of Contents

Integral families (F1, F2, F3)







Master integrals and kinematics

- 117(F2) and 166(F3) master integrals (`KIRA2`, `FIRE6`).
- $\sum_{i=1}^{4} q_i = 0,\ q_2^2 = m^2,\ q_i^2 = 0$ for $i = 1, 3, 4$.
- $S_{12} = (q_1 + q_2)^2,\ S_{23} = (q_2 + q_3)^2,\ S_{13} = m^2 - S_{12} - S_{23}$.
- Euclidean region: $S_{12} < 0, \quad S_{23} < 0, \quad m^2 < 0$.
- Scattering kinematics

$$\text{s-channel}: m^2 > 0, \quad S_{12} \geq m^2, \quad S_{23} \leq 0, \quad S_{13} \leq 0 \tag{1}$$

$$\text{t-channel}: m^2 > 0, \quad S_{12} \leq 0, \quad S_{23} \geq m^2, \quad S_{13} \leq 0 \tag{2}$$

$$\text{u-channel}: m^2 > 0, \quad S_{12} \leq 0, \quad S_{23} \leq 0, \quad S_{13} \geq m^2. \tag{3}$$

Table of Contents

Main steps

- Construct a canonical basis[1] $\mathbf{g}$.
- Apply the Simplified Differential Equations approach[2].
- Compute necessary boundary terms.
- Results in terms of GPLs (MPLs) and analytic continuation.

$$\mathcal{G}(a_1, a_2, \ldots, a_n; x) = \int_0^x \frac{\mathrm{dt}}{t - a_1} \mathcal{G}(a_2, \ldots, a_n; t) \tag{4}$$

$$\mathcal{G}(0, \ldots, 0; x) = \frac{1}{n!} \log^n(x) \tag{5}$$

[1] Henn, Phys. Rev. Lett. **110** (2013), 251601.

[2] Papadopoulos, JHEP **07** (2014), 088

Canonical basis

- Up to seven propagators: Magnus series expansions[3] (Federico Gasparotto, Luca Mattiazzi).
- Up to nine propagators: `Mathematica` package `DlogBasis`[4].
- Top sector: Analyse leading singularities in 4D loop-by-loop and use known 1-,2-loop results as building blocks[5].

[3] Argeri et al., JHEP **03** (2014), 082.
[4] Henn, Mistlberger, Smirnov, Wasser, JHEP04(2020)167.
[5] P. Wasser, PhD thesis.

Simplified Differential Equations (SDE)

- Parametrise the external momenta by introducing a dimensionless parameter x in the following manner

$$q_1 = xp_1, \quad q_2 = p_1 + p_2 - xp_1, \quad q_3 = p_3, \quad q_4 = p_4 \tag{6}$$

with $\sum_{i=1}^{4} p_i = 0,\ p_i^2 = 0$.

- Mapping for the kinematic invariants between the two momentum configurations

$$S_{12} = s_{12}, \quad S_{23} = s_{23}x, \quad m^2 = s_{12}(1-x) \tag{7}$$

with $s_{12} = (p_1 + p_2)^2,\ s_{23} = (p_2 + p_3)^2$.

- Note: $x \to 1$ yields massless momentum configuration.

SDE kinematics

- Introduce $y = s_{23}/s_{12}$.
- Euclidean region: $0 < x < 1,\ s_{12} < 0,\ 0 < y < 1$.
- Physical regions:

$$\text{s-channel}: 0 < x < 1,\ s_{12} > 0,\ -1 \leq y \leq 0 \tag{8}$$
$$\text{t-channel}: 1 < x,\ s_{12} < 0,\ y \leq -1 \tag{9}$$
$$\text{u-channel}: 1 < x,\ s_{12} < 0,\ y \geq 0. \tag{10}$$

Canonical SDE

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$$\partial_x \mathbf{g} = \epsilon \left(\sum_{i=1}^{4} \frac{\mathbf{M}_i}{x - l_i} \right) \mathbf{g} \tag{11}$$

- l_i: contain all kinematic dependence.
- $\mathbf{M}_i$: residue matrices corresponding to each pole l_i, consisting of rational numbers.

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- l_i: contain all kinematic dependence.
- $\mathbf{M}_i$: residue matrices corresponding to each pole l_i, consisting of rational numbers.
- Alphabet:

$$\left\{ x,\ x-1,\ x-\frac{1}{1+y},\ x+\frac{1}{y} \right\}. \tag{12}$$

- **Same alphabet with F1 family and with the two-loop case!**

General solution up to weight six

$$
\begin{aligned}
\mathbf{g} = {} & \epsilon^0 \mathbf{b}_0^{(0)} + \epsilon \left(\sum \mathcal{G}_i \mathbf{M}_i \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)} \right) + \epsilon^2 \left(\sum \mathcal{G}_{ij} \mathbf{M}_i \mathbf{M}_j \mathbf{b}_0^{(0)} + \sum \mathcal{G}_i \mathbf{M}_i \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)} \right) + \dots \\
& + \epsilon^6 \left(\mathbf{b}_0^{(6)} + \sum \mathcal{G}_{ijklmn} \mathbf{M}_i \mathbf{M}_j \mathbf{M}_k \mathbf{M}_l \mathbf{M}_m \mathbf{M}_n \mathbf{b}_0^{(0)} + \sum \mathcal{G}_{ijklm} \mathbf{M}_i \mathbf{M}_j \mathbf{M}_k \mathbf{M}_l \mathbf{M}_m \mathbf{b}_0^{(1)} \right. \\
& \left. + \sum \mathcal{G}_{ijkl} \mathbf{M}_i \mathbf{M}_j \mathbf{M}_k \mathbf{M}_l \mathbf{b}_0^{(2)} + \sum \mathcal{G}_{ijk} \mathbf{M}_i \mathbf{M}_j \mathrm{M}_k \mathbf{b}_0^{(3)} + \sum \mathcal{G}_{ij} \mathbf{M}_i \mathbf{M}_j \mathbf{b}_0^{(4)} + \sum \mathcal{G}_i \mathbf{M}_i \mathbf{b}_0^{(5)} \right)
\end{aligned} \tag{13}
$$

- $\mathcal{G}_{ab\dots} := \mathcal{G}(l_a, l_b, \dots; x)$.
- $\mathbf{b}_0^{(i)}$: boundary terms involving rational numbers and $\{\zeta(i),\ \log(-s_{12}),\ \log(y)\}$.

Boundary terms

- Master equation:

$$\mathbf{Rb} = \lim_{x\to 0} \mathbf{TG}\Big|_{\mathcal{O}\left(x^{0+a_j\epsilon}\right)} \tag{14}$$

- Residue matrix for $l_1 = 0 \to \mathbf{M}_1 = \mathbf{SDS}^{-1}$.
- $\mathbf{R} = \mathbf{S}\mathrm{e}^{\epsilon\mathbf{D}\log(x)}\mathbf{S}^{-1}$.
- $\mathbf{b} = \sum_{i=0}^{6} \epsilon^i \, b_0^{(i)}$.
- IBP reduction: $\mathbf{g} = \mathbf{TG}$.
- Expansion-by-regions using `asy`: $G_i \underset{x\to 0}{=} \sum_j x^{b_j+a_j\epsilon} G_i^{(b_j+a_j\epsilon)}$.

Boundary terms for family F3

- Boundaries for all 4 top-sector basis elements are fixed in terms of lower-sector boundaries, e.g.

$$
\begin{aligned}
b_{166} = & -\frac{1531 b_1}{4752} - \frac{128 b_2}{297} + \frac{47 b_4}{33} - \frac{1891 b_5}{396} + \frac{74 b_{10}}{9} + \frac{20 b_{11}}{3} + \frac{7 b_{12}}{3} - \frac{127 b_{13}}{36} \\
& - \frac{415 b_{15}}{264} + \frac{13 b_{16}}{8} + \frac{10 b_{17}}{3} - \frac{47 b_{18}}{36} - 2 b_{19} + \frac{5 b_{20}}{6} - \frac{21 b_{22}}{16} - \frac{11 b_{23}}{6} + \frac{5 b_{24}}{12} \\
& - \frac{35 b_{25}}{132} - \frac{6 b_{26}}{11} + \frac{16 b_{29}}{3} + \frac{32 b_{30}}{9} - \frac{10 b_{31}}{3} + \frac{581 b_{35}}{132} + \frac{29 b_{36}}{18} - \frac{197 b_{38}}{33} \\
& + \frac{3 b_{43}}{2} - \frac{14 b_{49}}{3} + 7 b_{52} - 5 b_{53} - \frac{89 b_{54}}{12} + \frac{13 b_{57}}{3} - \frac{8 b_{60}}{3} - \frac{b_{61}}{6} + 2 b_{62} - \frac{7 b_{77}}{33} \\
& - \frac{b_{81}}{6} + 3 b_{83} - \frac{b_{84}}{2} - \frac{13 b_{87}}{6} + \frac{7 b_{88}}{12} - \frac{2 b_{89}}{3} + \frac{5 b_{97}}{6} - \frac{b_{108}}{3} - \frac{2 b_{123}}{3} - b_{130} \\
& + \frac{2 b_{137}}{3} + 2 b_{144} - \frac{4 b_{152}}{3} - \frac{2 b_{159}}{3}.
\end{aligned}
$$

- We can fix the boundaries for 109 basis elements with similar relations.

Boundary terms for family F3

- Assuming that families F1 and F2 are solved, leaves the following boundaries to be computed:

$$\left\{ b_{108}, b_{123}, b_{135}, b_{144}, b_{157}, b_{159} \right\}. \tag{15}$$

- From master equation for boundaries:

$$\begin{aligned} b_{108} = & -2b_{19} + \frac{3b_{21}}{4} + s_{12}^2 \epsilon^5 G^{(-2\epsilon)}_{111101012000000} \\ & + 4s_{12}\epsilon^4 G^{(-\epsilon)}_{1022010110-10000} - 3s_{12}^2\epsilon^4 G^{(-\epsilon)}_{112201001000000} \\ & + 6s_{12}\epsilon^5 G^{(0)}_{011101012000000}. \end{aligned} \tag{16}$$

- Direct integration for up to seven Feynman parameters.

Boundary terms for family F3

- Five undetermined boundary terms $\left\{b_{123}, b_{135}, b_{144}, b_{157}, b_{159}\right\}$.
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- Exploit $x \rightarrow 1$ limit:
  1. Construct solution using ansatz for the undetermined boundary terms, i.e. $b_i = \sum_{k=0}^{6} a(i,k)\epsilon^k$.
  2. Take $x \rightarrow 1$ limit of the solution: $\tilde{\mathbf{g}} = \tilde{\mathbf{R}}_0 \ \mathbf{g}_{reg}|_{x=1}$.
  3. Map the $x \rightarrow 1$ limit of F3, i.e. the massless tennis-court, to the known solution of the same family from **Henn, Smirnov, Smirnov, JHEP07(2013)128**.

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E.g.

$$\tilde{g}_{123} = -\frac{f_{29}^{\mathrm{E}}}{4}, \;\; \tilde{g}_{159} = -\frac{f_{37}^{\mathrm{E}}}{8}.$$

Analytic continuation

- Tools: `HyperInt`, `PolyLogTools`, `GiNaC`.

Regions	Indices	Argument	Indices	Argument
Euclidean	$\{0, 1, -1/y, 1/(1+y)\}$	x	$-$	$-$
s-channel	$\{0, 1, -1/y, 1/(1+y)\}$	x	$-$	$-$
t-channel	$\{0, 1, -y, 1+y\}$	$1/x$	$\{0, 1\}$	$-1/y$
u-channel	$\{0, 1, -y, 1+y\}$	$1/x$	$\{0, -1\}$	y

Table: Structure of GPLs appearing in each of the 4 kinematic regions.

R	$W=1$	$W=2$	$W=3$	$W=4$	$W=5$	$W=6$	**Total**	**Timings (sec)**
E	4	14	50	124	367	734	1293	39.0225769
s	4	14	50	124	367	734	1293	39.2172529
t	6	18	58	155	419	603	1259	62.0567800
u	5	16	54	147	403	572	1197	55.1049640

Table: Number of GPLs per weight and region, and timings for the numerical evaluation of the total GPLs.

Validation

- Numerical checks using `FIESTA4` and `pySecDec`

$$
\begin{aligned}
\text{Euclidean}:&\ s_{12} \to -7,\ y \to 3/7,\ x \to 1/4 \\
\text{s-channel}:&\ s_{12} \to 2,\ y \to -1/2,\ x \to 1/4 \\
\text{t-channel}:&\ s_{12} \to -2,\ y \to -3/2,\ x \to 5/3 \\
\text{u-channel}:&\ s_{12} \to -2,\ y \to 3/2,\ x \to 5/3.
\end{aligned}
$$

- Analytic check at $x \to 1$ with known results[6] in the Euclidean region.

E.g.

$$
\tilde{g}^{\mathrm{F3}}_{163} = \frac{f^{\mathrm{E}}_{39}}{8},\ \tilde{g}^{\mathrm{F3}}_{164} = -\frac{f^{\mathrm{E}}_{40}}{8},\ \tilde{g}^{\mathrm{F3}}_{166} = -\frac{f^{\mathrm{E}}_{41}}{8} \tag{17}
$$

$$
\tilde{g}^{\mathrm{F3}}_{165} = \sum_i c_i f^{\mathrm{E}}_i,\ \{i = 1, 2, 4, 10, 12, 13, 14, 15, 16, 19, 23, 24, 41\} \tag{18}
$$

[6] Henn, Smirnov, Smirnov, JHEP07(2013)128.

Table of Contents

Conclusions

- Analytic results for all three planar families in all physical regions of phase space in terms of real-valued GPLs (max. W=6).
- Same alphabet for all 3-loop planar integrals, identical to their two-loop planar and non-planar counterparts.
- Publicly available tools (`KIRA2`, `FIRE6`, `HyperInt`, `PolyLogTools`, `GiNaC`, `FIESTA4`, `pySecDec`) sufficient for performing the calculation and cross-checking our results.
- Future steps:
  1. Non-planar integral families.
  2. Amplitudes, e.g. $Z/W/H + jet$ production.

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Thank you for your attention!

Table of Contents

The $x \to 1$ limit

Scale reduction

The $x \to 1$ limit yields the momentum configuration for a system with one less external mass.

- The introduction of the $x-$parameterisation effectively captures the off-shellness of one external particle. By taking the limit $x \to 1$ we can obtain the solution to a family of FI with one scale less.
- Assuming that we have an integral family with m external massive particles whose solution we have expressed in terms of GPLs, we can obtain the solution for a family with $m - 1$ external masses through the $x \to 1$ limit of the former.

Scale reduction

- Exploit the shuffle properties of GPLs to write solution as an expansion in terms of $\log(1-x)$:

$$\mathbf{g} = \sum_{n \geq 0} \epsilon^n \sum_{i=0}^{n} \frac{1}{i!} \mathbf{c}_i^{(n)} \log^i(1-x) \tag{19}$$

- Regular part of (19) at $x = 1$:

$$\mathbf{g}_{reg} = \sum_{n \geq 0} \epsilon^n \mathbf{c}_0^{(n)} \tag{20}$$

- Truncated part of (19):

$$\mathbf{g}_{trunc} = \mathbf{g}_{reg}(x = 1) \tag{21}$$

Scale reduction

- Utilise the residue matrix that corresponds to the letter $\{1\}$, $\mathbf{M}_2$, and define the *resummation matrix* $\tilde{\mathbf{R}}$:

$$\tilde{\mathbf{R}} = \tilde{\mathbf{S}} e^{\epsilon \tilde{\mathbf{D}} \log(1-x)} \tilde{\mathbf{S}}^{-1} \tag{22}$$

- $\tilde{\mathbf{S}}, \tilde{\mathbf{D}}$ are constructed through the Jordan decomposition of $\mathbf{M}_2$.
- The *resummation matrix* $\tilde{\mathbf{R}}$ has terms of $(1-x)^{a_i \epsilon}$, with a_i being the eigenvalues of $\mathbf{M}_2$.
- Set all terms $(1-x)^{a_i \epsilon}$ equal to zero and define the purely numerical matrix $\tilde{\mathbf{R}}_0$.

Final formula for $x \to 1$ limit:

$$\tilde{\mathbf{g}} = \tilde{\mathbf{R}}_0 \mathbf{g}_{trunc} \tag{23}$$