Massive Quark Form Factors

Loops and Legs in Quantum Field Theory, 2022 (LL2022)

Kay Schönwald | Ettal, April 26, 2022

TTP KARLSRUHE

[based on: Fael, Lange, Schönwald, Steinhauser arxiv:2202.05276
Egner, Fael, Lange, Schönwald, Steinhauser arxiv:2203.11231]

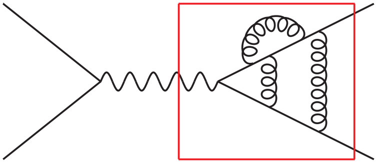
Institute for Theoretical Particle Physics

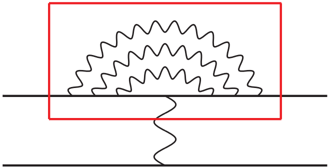
TRR 257 - Particle Physics Phenomenology after the Higgs Discovery

Outline

Motivation

- Form factors are basic building blocks for many physical observables:
  - $t\,\bar{t}$ production at hadron and $e^+ e^-$ colliders
  - $\mu\, e$ scattering
  - Higgs production and decay
  - ...
- Form factors exhibit an universal infrared behavior.

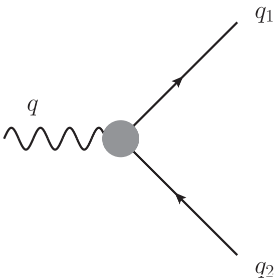




The Process

$$X(q) \to Q(q_1) + Q(q_2)$$

$$q_1^2 = q_2^2 = m^2, \quad q^2 = s = \hat{s} \cdot m^2$$



vector : $j_\mu^v = \overline{\psi}\gamma_\mu\psi \qquad \Gamma_\mu^v = F_1^v(s)\gamma_\mu - \frac{i}{2m}F_2^v(s)\sigma_{\mu\nu}q^\nu$

axial-vector : $j_\mu^a = \overline{\psi}\gamma_\mu\gamma_5\psi \qquad \Gamma_\mu^a = F_1^a(s)\gamma_\mu\gamma_5 - \frac{1}{2m}F_2^a(s)q_\mu\gamma_5$

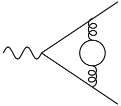
scalar : $j^s = m\overline{\psi}\psi \qquad \Gamma^s = mF^s(s)$

pseudo-scalar : $j^p = im\overline{\psi}\gamma_5\psi \qquad \Gamma^p = imF^p(s)\gamma_5$

Previous Calculations

NNLO

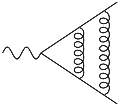
$F_l^{(2)}$ fermionic corrections [Hoang, Teubner '97]

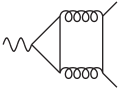


$F_l^{(2)}$ [Bernreuther, Bonciani, Gehrmann, Heinesch, Leineweber, Mastrolia, Remiddi '04-'06]

$+\mathcal{O}(\epsilon)$ [Gluza, Mitov, Moch, Riemann '09]

$+\mathcal{O}(\epsilon^2)$ [Ahmed, Henn, Steinhauser '17; Ablinger, Behring, Blümlein, Falcioni, Freitas, Marquard, Rana, Schneider '17]



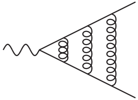


NNNLO

$F_l^{(3)}$ large-N_c [Henn, Smirnov, Smirnov, Steinhauser '16-'18; Ablinger, Marquard, Rana, Schneider '18]

n_l [Lee, Smirnov, Smirnov, Steinhauser '18]

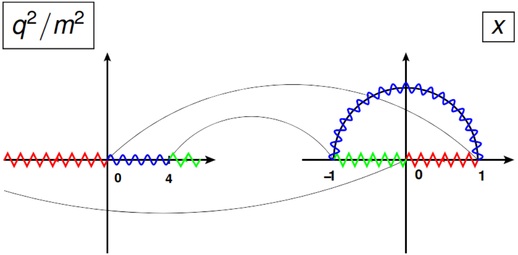
n_h (partially) [Blümlein, Marquard, Rana, Schneider '19]



this talk: full (numerical) results for non-singlet diagrams at NNNLO

Previous Calculations

$$q^2 = s = -\frac{(1-x)^2}{x}$$



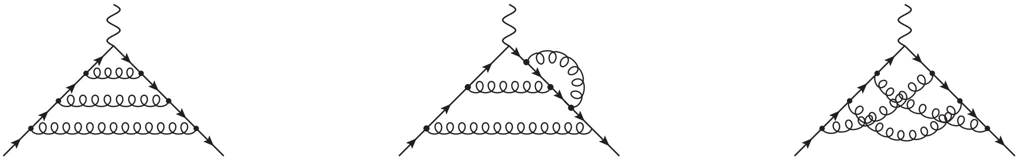
- The large-N_c and n_l contributions at NNNLO can be written as iterated integrals over the letters:

$$\frac{1}{x}, \frac{1}{1+x}, \frac{1}{1-x}, \quad \frac{1}{1-x+x^2}, \frac{x}{1-x+x^2}$$

- The n_h terms already contain structures which go beyond iterated integrals.

$\Rightarrow$ We aim at the full solution through analytic series expansions and numerical matching.

Technical Details



- Generate diagrams with `QGRAF`. [Nogueira '93]
- Use `FORM` [Ruijl, Ueda, Vermaseren '17] for Lorentz Dirac and color algebra. [Ritbergen, Schellekens, Vermaseren '98]
- Map the output to predefined integral families with `q2e/exp`. [Harlnder, Seidensticker, Steinhauser '97-'99]
- Reduce the scalar integrals to masters with `Kira`. [Klappert, Lange, Maierhöfer, Usovitsch, Uwer '17,'20]
  - We ensure a good basis where denominators factorize in ϵ and $\hat{s}$ with `ImproveMasters.m`. [Smirnov, Smirnov '20]
- Establish differential equations in variable $\hat{s}$ using `LiteRed`. [Lee '12,'14]

	non-singlet
diagrams	271
families	34
masters	422

Algorithm to Solve Master Integrals

- Establish a system of differential equations for the master integrals in the variable $\hat{s}$.
- Compute an expansion around $\hat{s} = 0$ by:
  - Inserting an ansatz for the master integrals into the differential equation.

$$M_n(\epsilon, \hat{s} = 0) = \sum_{i=-3}^{\infty} \sum_{j=0}^{j_{\max}} c_{ij}^{(n)} \epsilon^i \hat{s}^j$$

  - Compare coefficients in ϵ and $\hat{s}$ to establish a linear system of equations for the $c_{ij}^{(n)}$.
  - Solve the linear system in terms of a small number of boundary constants using `Kira` with `FireFly`.

[Klappert, Klein, Lange '19,'20]

- Compute boundary values for $\hat{s} = 0$ and obtain an analytic expansion.
- Build a general expansion around a new point, e.g. $\hat{s} = \hat{s}_0$, by modifying the ansatz and repeating the steps above.
- Match both expansions numerically at a point where both expansions converge, e.g. $\hat{s}_0/2$.
- Repeat the procedure for the next point.

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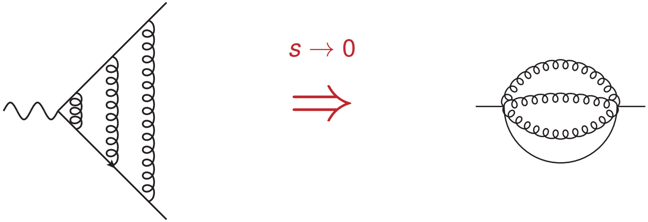
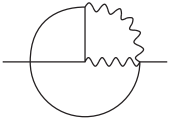
Calculation of Boundary Conditions



- For $s = 0$ the master integrals reduce to 3-loop on-shell propagators:
  - These integrals are well studied in the literature. [Laporta, Remiddi '96; Melnikov, Ritbergen '00; Lee, Smirnov '10]
- The reduction introduces high inverse powers in ϵ, which require some integrals up to weight 9.
- We calculate the needed terms with `SummerTime.m` [Lee, Mingulov '15] and `PSLQ` [Ferguson, Bailey '92].

Calculation of Boundary Conditions

E.g. extension of G_{66} (given up to and including $\mathcal{O}(\epsilon^3)$ in [Lee, Smirnov '10]):



$$
= \dots + \epsilon^4 \Big(-4704 s_6 - 9120 s_{7a} - 9120 s_{7b} - 547 s_{8a} + 9120 s_6 \ln(2) + 28 \ln^4(2) + \frac{112 \ln^5(2)}{3} - \frac{808}{45} \ln^6(2)
$$

$$
- \frac{347}{9} \ln^8(2) + 672 \mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{5552}{3} \ln^4(2) \mathrm{Li}_4\left(\frac{1}{2}\right) - 22208 \mathrm{Li}_4\left(\frac{1}{2}\right)^2 - 4480 \mathrm{Li}_5\left(\frac{1}{2}\right) - 12928 \mathrm{Li}_6\left(\frac{1}{2}\right) + \dots \Big)
$$

$$
+ \epsilon^5 \Big(14400 s_6 - \frac{377568 s_{7a}}{7} - \frac{93984 s_{7b}}{7} - 2735 s_{8a} + 7572912 s_{9a} - 3804464 s_{9b} - \frac{5092568 s_{9c}}{3} - 136256 s_{9d}
$$

$$
+ 681280 s_{9e} + 272512 s_{9f} + \frac{377568}{7} s_6 \ln(2) - \frac{32465121}{20} s_{8a} \ln(2) - 10185136 s_{8b} \ln(2) + 136256 s_{7b} \ln^2(2) + \dots \Big)
$$

$$
+ \mathcal{O}(\epsilon^6)
$$

Series Expansions

- Special points:

$s = 0$	$s = 4m^2$	$s = \pm\infty$
$x = 1$	$x = -1$	$x = 0$
static limit	2-particle threshold	high energy limit

- Every expansion point needs a different ansatz:

$$M_n(\epsilon, \hat{s} = 0) = \sum_{i=-3}^{\infty} \sum_{j=0}^{j_{\max}} c_{ij}^{(n)} \, \epsilon^i \, \hat{s}^j$$

Series Expansions

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- Every expansion point needs a different ansatz:

$$M_n(\epsilon, \hat{s} = 4) = \sum_{i=-3}^{\infty} \sum_{j=-s_{\min}}^{j_{\max}} \sum_{k=0}^{i+3} c_{ijk}^{(n)} \epsilon^i \left[\sqrt{4 - \hat{s}}\right]^j \ln^k \left(\sqrt{4 - \hat{s}}\right)$$

Series Expansions

- Special points:

$s = 0$	$s = 4m^2$	$s = \pm\infty$
$x = 1$	$x = -1$	$x = 0$
static limit	2-particle threshold	high energy limit

- Every expansion point needs a different ansatz:

$$M_n(\epsilon, \hat{s} \to \pm\infty) = \sum_{i=-3}^{\infty} \sum_{j=-s_{\min}}^{j_{\max}} \sum_{k=0}^{\boxed{i+6}} c_{ijk}^{(n)} \, \epsilon^i \, \hat{s}^{-j} \, \ln^k(\hat{s})$$

Series Expansions

- Special points:

$s = 0$	$s = 4m^2$	$s = \pm\infty$	$s = 16m^2$
$x = 1$	$x = -1$	$x = 0$	$x = 4\sqrt{3} - 7$
static limit	2-particle threshold	high energy limit	4-particle threshold

- Every expansion point needs a different ansatz:

$$M_n(\epsilon, \hat{s} = 16) = \sum_{i=-3}^{\infty} \sum_{j=-s_{\min}}^{j_{\max}} \sum_{k=0}^{i+3} c_{ijk}^{(n)} \, \epsilon^i \left[\sqrt{16 - \hat{s}}\right]^j \ln^k\left(\sqrt{16 - \hat{s}}\right)$$

Series Expansions

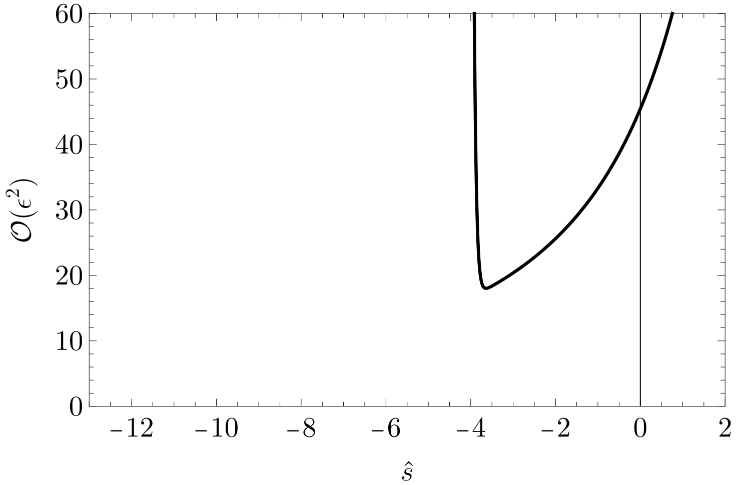
- Special points:

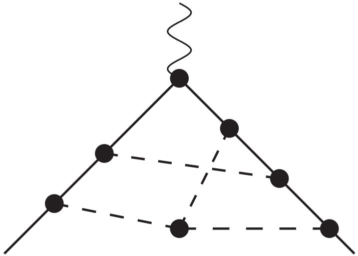
$s = 0$	$s = 4m^2$	$s = \pm\infty$	$s = 16m^2$
$x = 1$	$x = -1$	$x = 0$	$x = 4\sqrt{3} - 7$
static limit	2-particle threshold	high energy limit	4-particle threshold

- Every expansion point needs a different ansatz.
- We construct expansions with $j_{\max} = 50$ around:

$$\hat{s} = \{-\infty, -32, -28, -24, -16, -12, -8, -4, 0, 1, 2, 5/2, 3, 7/2, 4, 9/2, 5, 6, 7, 8, 10, 12, 14, 15, 16, 17, 19, 22, 28, 40\}$$

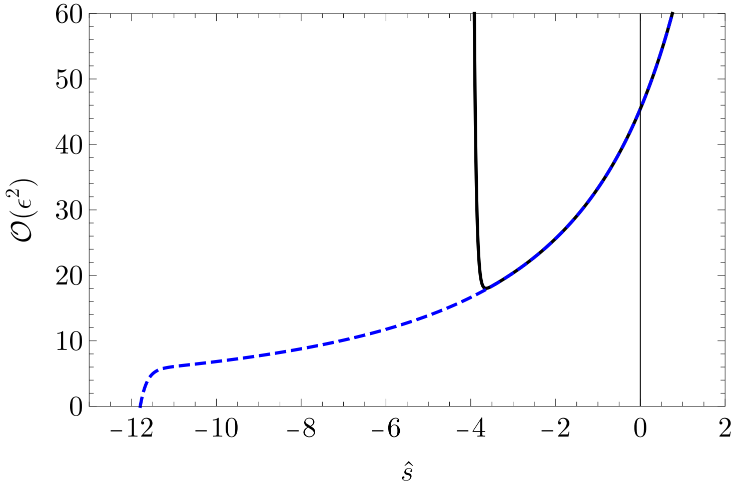
Example

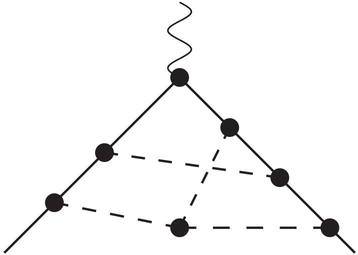




- Expansion around $\hat{s} = 0$.

Example





- Expansion around $\hat{s} = 0$.
- Expansion around $\hat{s} = -4$, matched at $\hat{s} = -2$.

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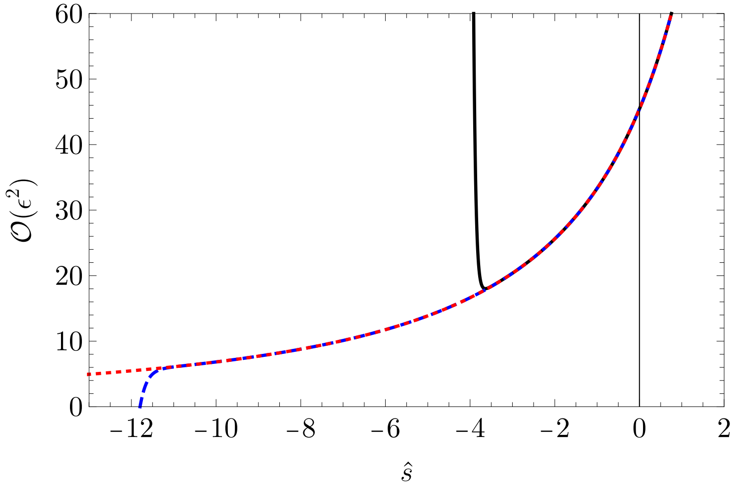

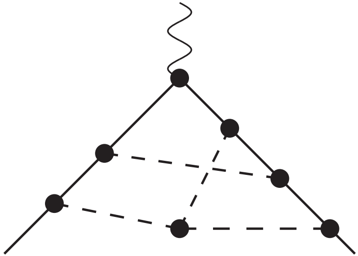

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- Expansion around $\hat{s} = -8$, matched at $\hat{s} = -6$.

Algorithm to Solve Master Integrals

There are other approaches based on expansions:

- `SolveCoupledSystems.m` [Blümlein, Schneider ’17]
- `DESS.m` [Lee, Smirnov, Smirnov ’18]
- `DiffExp.m` [Hidding ’20]
- ...

Our approach ...

- ... does not require a special form of differential equation.
- ... provides approximation in whole kinematic range.
- ... is applied to physical quantity. [Fael, Lange, KS, Steinhauser ’21]

Renormalization and Infrared Structure

UV renormalization

- On-shell renormalization of mass Z_m^{OS}, wave function Z_2^{OS}, and (if needed) the currents.
  [Chetyrkin, Steinhauser '99; Melnikov, Ritbergen '00]

IR subtraction

- Structure of the infrared poles is given by the cusp anomalous dimension Γ_{cusp}.
  [Grozin, Henn, Korchemski, Marquard '14]
- Define finite form factors $F = Z_{\mathrm{IR}} F^{\mathrm{finite}}$ with the UV renormalized form factor F and

$$Z_{\mathrm{IR}} = 1 - \frac{\alpha_s}{\pi}\frac{1}{2\epsilon}\Gamma_{\mathrm{cusp}}^{(1)} - \left(\frac{\alpha_s}{\pi}\right)^2\left(\frac{\cdots}{\epsilon^2} + \frac{1}{4\epsilon}\Gamma_{\mathrm{cusp}}^{(2)}\right) - \left(\frac{\alpha_s}{\pi}\right)^3\left(\frac{\cdots}{\epsilon^3} + \frac{\cdots}{\epsilon^2} + \frac{1}{6\epsilon}\Gamma_{\mathrm{cusp}}^{(3)}\right)$$

- $\Gamma_{\mathrm{cusp}} = \Gamma_{\mathrm{cusp}}(x)$ depends on kinematics.
- Γ_{cusp} is universal for all currents.

Results – Analytic $\hat{s} = 0$ Expansion

Analytic expansion for $\hat{s} = 0$:

$$F_1^V(\hat{s}=0) = \Bigg\{ C_F^3 \Big(-15 a_4 - \frac{17\pi^2\zeta_3}{24} - \frac{18367\zeta_3}{1728} + \frac{25\zeta_5}{8} - \frac{5 l_2^4}{8} - \frac{19}{40}\pi^2 l_2^2 + \frac{4957\pi^2 l_2}{720} + \frac{3037\pi^4}{25920} - \frac{24463\pi^2}{7776} + \frac{13135}{20736} \Big) + C_A C_F^2 \Big(\frac{19 a_4}{2} - \frac{\pi^2\zeta_3}{9} + \frac{17725\zeta_3}{3456} - \frac{55\zeta_5}{32} + \frac{19 l_2^4}{48} - \frac{97}{720}\pi^2 l_2^2 + \frac{29\pi^2 l_2}{240} - \frac{347\pi^4}{17280} - \frac{4829\pi^2}{10368} + \frac{707}{288} \Big) + C_A^2 C_F \Big(-a_4 + \frac{7\pi^2\zeta_3}{96} + \frac{4045\zeta_3}{5184} - \frac{5\zeta_5}{64} - \frac{l_2^4}{24} + \frac{67}{360}\pi^2 l_2^2 - \frac{5131\pi^2 l_2}{2880} + \frac{67\pi^4}{8640} + \frac{172285\pi^2}{186624} - \frac{7876}{2187} \Big) \Bigg\} \hat{s} + \text{fermionic corrections} + \mathcal{O}(\hat{s}^2)$$

with $l_2 = \ln(2)$, $a_4 = \mathrm{Li}_4(1/2)$ and $C_A = 3$, $C_F = 4/3$ for QCD.

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Results – Pole Cancellation

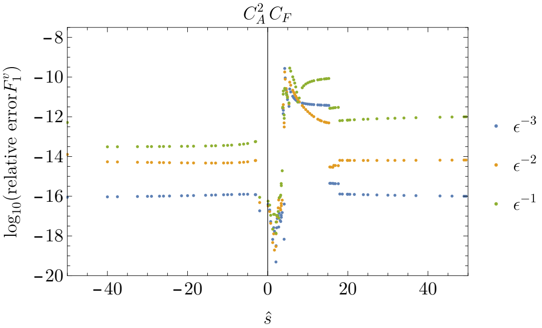
- Except for $s = 0$ the results of the expansions are not analytic.
- We can use the pole cancellation to estimate the precision.

$\Rightarrow$ We find at least 8 significant digits, although some regions are much more precise.

- To estimate the number of significant digits we use:

$$\log_{10}\left(\left|\frac{\text{expansion} - \text{analytic}}{\text{analytic}}\right|\right)$$

- The analytic expressions for the poles are expressed by Harmonic Polylogarithms which can be evaluated with `ginac`. [Vollinga, Weinzierl '05]



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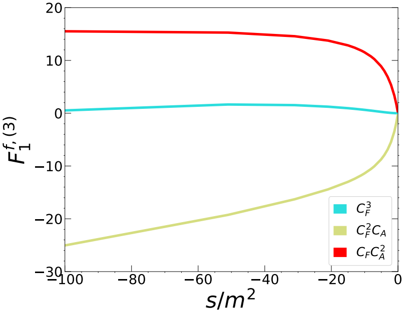

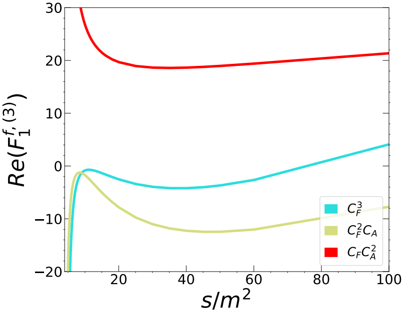

Results – High Energy Limit

- For $s \to \infty$ there is the prediction: [Liu, Penin, Zerf ’17]

$$F_1^{f,(3)} = -\frac{C_F^3}{384} l_s^6 - \frac{m^2}{s}\left(\frac{C_F^3}{240} - \frac{C_F^2 C_A}{960} - \frac{C_F C_A^2}{1920}\right) l_s^6 + \ldots, \quad \text{with } l_s = \ln\left(\frac{m^2}{-s}\right)$$

- We obtain:

$$\begin{aligned}
F_1^{f,(3)}\Big|_{s\to-\infty} &= 4.7318 C_F^3 - 20.762 C_F^2 C_A + 8.3501 C_F C_A^2 + \Big[3.4586 C_F^3 - 4.0082 C_F^2 C_A - 6.3561 C_F C_A^2\Big] l_s \\
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Results – Matching Coefficients

- For $Q\overline{Q}$ production close to threshold it is advantageous to calculate the cross section in non-relativistic QCD (NRQCD).
- The naive expansion around the threshold of the form factor

$$x = \sqrt{4 - \hat{s}} = 0$$

defines the matching coefficients between QCD and NRQCD. [Pineda '11]

- At threshold the momenta can have different scalings: [Beneke, Smirnov '98]
  - hard (h): $k_0 \sim m,\ k_i \sim m$
  - potential (p): $k_0 \sim x^2 \cdot m,\ k_i \sim x \cdot m$
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$I_a \sim x^{-0\epsilon} \cdot$ Taylor expansion $(h-h-h)$

$I_b \sim x^{-2\epsilon} \cdot$ Taylor expansion $(h-h-p), (h-h-s), \ldots$

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$I_e \sim x^{-8\epsilon} \cdot$ Taylor expansion $(h-u-u), (u-p-p), \ldots$

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- The previous calculation relied heavily on sector decomposition and numerical integration with FIESTA. [Marquard, Piclum, Seidel, Steinhauser '14]
- We can improve the precision significantly:

$$c_v^{(3)} = C_F^3 c_{FFF} + C_F C_A^2 c_{FFA} + C_F C_A^2 c_{FAA} + \text{fermionic and singlet contributions}$$

$$\begin{aligned} c^v_{FFF} &= 36.55(0.53) &\rightarrow& \quad 36.49486246 \\ c^v_{FFA} &= -188.10(0.83) &\rightarrow& \quad -188.0778417 \\ c^v_{FAA} &= -97.81(0.38) &\rightarrow& \quad -97.73497327 \end{aligned}$$

- We calculated the matching coefficients for all four currents.

Conclusions and Outlook

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- We have calculated the non-singlet contributions to the massive quark form factors at NNNLO.
- We applied a semianalytic method by constructing series expansions and numerical matching.
- We can reproduce known results in the literature, e.g.
  - the large N_c limit.
  - expansion terms in the static, high energy and threshold expansion.
- We estimate the precision to 8 significant digits over the whole real axis.
- The method is promising to tackle other one-scale problems.

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- Calculate the contributions of the singlet diagrams.
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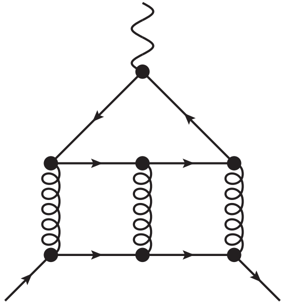

Backup

Moebius Transformations

- The radius of convergence is at most the distance to the closest singularity.
- We can extend the radius of convergence by changing to a new expansion variable.
- If we want to expand around the point x_k with the closest singularities at x_{k-1} and x_{k+1}, we can use:

$$y_k = \frac{(x - x_k)(x_{k+1} - x_{k-1})}{(x - x_{k+1})(x_{k-1} - x_k) + (x - x_{k-1})(x_{k+1} - x_k)}$$

- The variable change maps $\{x_{k-1}, x_k, x_{k+1}\} \to \{-1, 0, 1\}$.

Results – Threshold Expansion

- Close to threshold it is interesting to consider:

$$\sigma(e^+e^- \to Q\bar{Q}) = \sigma_0 \beta \underbrace{\left(\left|F_1^v + F_2^v\right|^2 + \frac{\left|(1-\beta^2)F_1^v + F_2^v\right|^2}{2(1-\beta^2)} \right)}_{=3/2\ \Delta}$$

with $\beta = \sqrt{1 - 4m^2/s}$.

- Real radiation is supressed by β^3.
- We find (with $l_{2\beta} = \ln(2\beta)$):

$$\Delta^{(3)} = C_F^3\Big[-\frac{32.470}{\beta^2} + \frac{1}{\beta}\big(14.998 - 32.470 l_{2\beta}\big)\Big] + C_A^2 C_F \frac{1}{\beta}\big[16.586 l_{2\beta}^2 - 22.572 l_{2\beta} + 42.936\big]$$
$$+ C_A C_F^2\Big[\frac{1}{\beta^2}\big(-29.764 l_{2\beta} - 7.770339\big) + \frac{1}{\beta}\big(-12.516 l_{2\beta} - 11.435\big)\Big]$$
$$+ \mathcal{O}(\beta^0) + \text{fermionic contributions}$$