Inclusive τ Hadronic Decay Rate in a Renormalon-free Gluon Condensate Scheme

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arXiv:2008.00578
arXiv:2105.11222
(with Christoph Regner)
and arXiv:2202.10957
(with Miguel Benitez-Rathgeb, Diogo Boito and Matthias Jamin)

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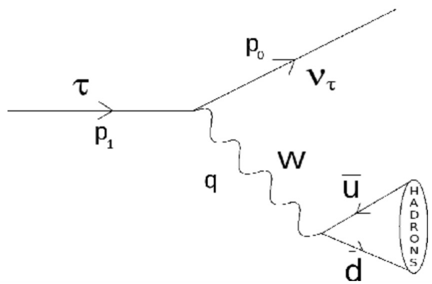
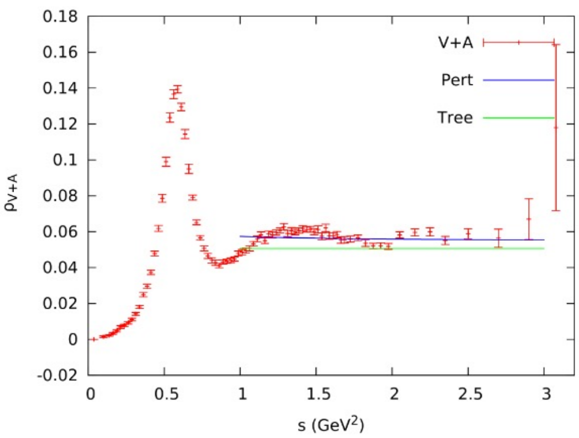
Hadronic τ Spectral Function Moments

ALEPH: τ hadronic width (HFLAV 2019)

$$R_\tau \equiv \frac{\Gamma[\tau^- \to \text{hadrons}\, \nu_\tau(\gamma)]}{\Gamma[\tau^- \to e^- \overline{\nu}_e \nu_\tau(\gamma)]} = 3.6355 \pm 0.0081$$



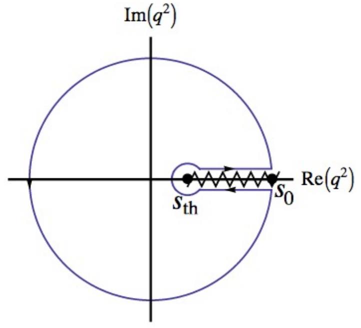
Inclusive hadronic mass spectrum



$$(p^\mu p^\nu - g^{\mu\nu} p^2)\, \Pi(p^2) \equiv i \int dx\, e^{ipx} \langle \Omega | T\{ j^\mu_{v/av,jk}(x)\, j^\nu_{v/av,jk}(0)^\dagger \} \Omega \rangle$$

Braaten, Narison, Pich, Le Diberder, ... 90‘s

$$A^\omega_{V/A}(s_0) \equiv \int_{s_{th}}^{s_0} \frac{ds}{s_0}\, \omega(s)\, \mathrm{Im}\, \Pi_{V/A}(s) = \frac{i}{2} \oint_{|s|=s_0} \frac{ds}{s_0}\, \omega(s)\, \Pi_{V/A}(s)$$



Hadronic τ Spectral Function Moments

Theory: Operator product expansion

Adler function: $$\frac{1}{4\pi^2}\Big(1+D(s)\Big) \equiv -s\frac{d\Pi(s)}{ds}$$

Braaten, Narison, Pich, Le Diberder, ... 90's

$$A_{W_i}(s_0) = \frac{N_c}{2}|V_{ud}|^2\left[\delta^{\rm tree}_{W_i} + \delta^{(0)}_{W_i}(s_0) + \sum_{d\geq 2}\delta^{(d)}_{W_i}(s_0) + \delta^{\rm DV}_{W_i}(s_0)\right]$$

$$W_i(x) = \sum_{n=0}^{m} a_n\, x^n$$

pQCD ↑ OPE ↑ Duality violation ↑

$$\hat{D}(s) = \sum_{n=1}^{\infty} c_{n,1}\left(\frac{\alpha_s(-s)}{\pi}\right)^n$$ ← Perturbative

OPE non-pert. corrections

$$\hat{D}^{\rm OPE}(s) = \frac{C(\alpha_s(-s))}{(-s)^2}\langle\alpha_s G^2\rangle + \sum_{p=3}^{\infty}\frac{1}{(-s)^p}\Big[C_0(\alpha_s(-s))\langle\mathcal{O}_{2p,\gamma_1}\rangle + C_1(\alpha_s(-s))\langle\mathcal{O}_{2p,\gamma_2}\rangle + \ldots\Big]$$

$$\delta^{(0)}_{W_i}(s_0) = \frac{1}{2\pi i}\oint_{|s|=s_0}\frac{{\rm d}s}{s}\,W_i\!\left(\tfrac{s}{s_0}\right)\hat{D}(s)$$

$$\delta^{(d)}_{W_i}(s_0) \sim \frac{1}{2\pi i}\oint_{|s|=s_0}\frac{ds}{s}\,W_i\!\left(\frac{s}{s_0}\right)\frac{\Lambda^d_{\rm QCD}}{s^{d/2}}$$

FOPT-CIPT Discrepancy Problem

$$\hat{D}(s) = \sum_{n=1}^{\infty} c_{n,1} \left(\frac{\alpha_s(-s)}{\pi}\right)^n ,$$

$$= \sum_{n=1}^{\infty} \left(\frac{\alpha_s(s_0)}{\pi}\right)^n \sum_{k=1}^{n+1} k\, c_{n,k} \ln^{k-1}\left(\frac{-s}{s_0}\right)$$

Change of renormalization scale

$$c_{0,1} = c_{1,1} = 1\,, \quad c_{2,1} = 1.640$$

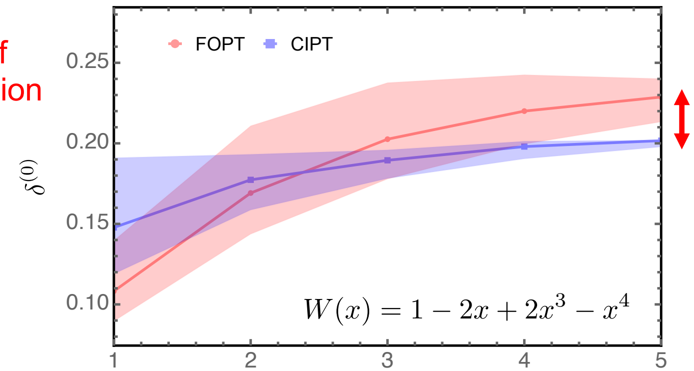
$$c_{3.1} = 6.371$$

$$c_{4,1} = 49.076$$

4-loop: Gorishni etal., Surguladze etal. 1991

5-loop: Baikov etal. 2008

FOPT CIPT



- CIPT resums powers of π with respect to FOPT
- CIPT leads in general to smaller moments than FOPT
- OPE and DV corrections assumed to be universal
- Strong coupling from CIPT larger than from FOPT

Contour-improved perturbation theory (CIPT):

$$\delta^{(0),\mathrm{CIPT}}_{W_i}(s_0) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} c_{n,1} \oint_{|x|=1} \frac{\mathrm{d}x}{x} W_i(x) \left(\frac{\alpha_s(-x s_0)}{\pi}\right)^n$$

Fixed-order perturbation theory (FOPT):

$$x = \frac{s}{s_0}$$

$$\delta^{(0),\mathrm{FOPT}}_{W_i}(s_0) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \left(\frac{\alpha_s(s_0)}{\pi}\right)^n \sum_{k=1}^{n+1} k\, c_{n,k} \oint_{|x|=1} \frac{\mathrm{d}x}{x} W_i(x) \ln^{k-1}(-x)$$

Outline

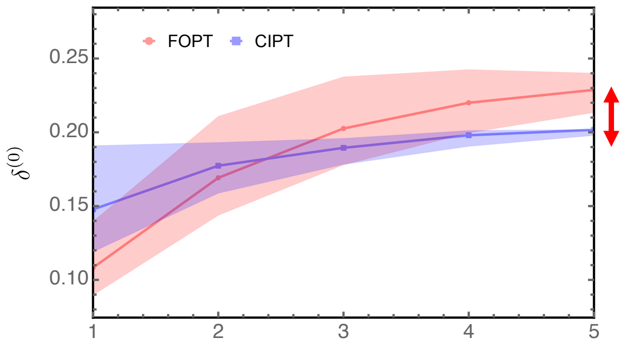
- Introduction
- FOPT and CIPT Borel representations are different:
  Asymptotic Separation Δ
  → FOPT and CIPT expansions describe different quantities
- Properties of the original CIPT expansion:
  CIPT expansion not consistent with the standard association
  OPE power corrections ↔ IR renormalons
- Reconciling CIPT and FOPT:
  renormalon-free gluon condensate scheme
- Preliminary results and outlook

Interesting Observations: Total Decay Rate

$$W_\tau(x) = 1 - 2x + 2x^3 - x^4 \qquad \delta^{(d)}_{W_i}(s_0) \sim \frac{1}{2\pi i}\oint_{|s|=s_0}\frac{ds}{s}\left(\frac{s}{s_0}\right)^m \frac{\Lambda^4_{\rm QCD}}{s^2} = \frac{\Lambda^4_{\rm QCD}}{s_0^2}\,\delta_{m2}$$

→ Sensitivity to leading O($\Lambda^4_{\rm QCD}$) gluon condensate strongly suppressed

Moment's perturbation series:



- Discrepancy between CIPT and FOPT scales as $\sim \frac{\Lambda^4_{\rm QCD}}{s_0^2}$

  → Accidental or indication of a quartic IR sensitivity?

  → Contradiction to standard OPE

  → How can there be O($\Lambda^4_{\rm QCD}$) sensitivity left ?

- CIPT is not an expansion in powers of α_s at a definite renormalization scale. It is impossible to switch between the CIPT and FOPT moment series terms through a change of scheme of the strong coupling and a reexpansion of the series due to the **contour integration** !!

→ Worth to reconsider CIPT and FOPT from scratch: OPE ↔ IR renormalons

Renormalon Calculus: Euclidean Adler Function

Perturbative series in QCD are not convergent, but asymptotic in expansion variable $\alpha_s(s_0)$. $\rightarrow$ $\hat{D}(-s_0) \sim \sum_{n=1}^{\infty} n!\,(\frac{\alpha_s(s_0)}{\pi})^n$

`t Hooft; David; Müller; ... Beneke; ...

Borel calculus:

$$\hat{D}(s) = \sum_{n=1}^{\infty} c_{n,1}\left(\frac{\alpha_s(s_0)}{\pi}\right)^n \quad \Longrightarrow \quad B[\hat{D}](u) = \sum_{n=1}^{\infty} \frac{c_{n,1}}{\Gamma(n)}\, u^{n-1}$$

Borel sum: $$\hat{D}_{\rm Borel}(-s_0) = {\rm PV}\int_0^{\infty} {\rm d}u\; B[\hat{D}](u)\, e^{-\frac{4\pi u}{\beta_0 \alpha_s(s_0)}}$$

Association: IR renormalon poles/cuts $\Leftrightarrow$ (standard) OPE Corrections

$$\hat{D}^{\rm OPE}(s) \;=\; \frac{C(\alpha_s(-s))}{(-s)^2}\langle\alpha_s G^2\rangle + \sum_{p=3}^{\infty}\frac{1}{(-s)^p}\Big[C_0(\alpha_s(-s))\langle\mathcal{O}_{2p,\gamma_1}\rangle + C_1(\alpha_s(-s))\langle\mathcal{O}_{2p,\gamma_2}\rangle + \ldots\Big]$$

Leading Gluon Condensate: $$B(u) \sim \frac{1}{(2-u)^{1+4\hat{b}_1}} + \ldots \quad \Longleftrightarrow \quad \frac{\langle\bar{G}^2\rangle}{s^2}$$

FOPT vs. CIPT Borel Representation (large-β_0)

FOPT expansion: → Expansion parameter: $\alpha_s(s_0)$

$$\hat{D}(s) \;=\; \sum_{n=1}^{\infty} \underbrace{\left(\tfrac{\alpha_s(s_0)}{\pi}\right)^n}_{\text{expansion variable}} \underbrace{\sum_{k=1}^{n} k\, c_{n,k}\, \ln^{k-1}(-x)}_{\text{coefficient}}$$

Borel sum:

$$\mathrm{PV}\int_0^\infty du \left[B[\hat{D}](u)\, e^{-u\ln(-x)}\right] e^{-\frac{4\pi u}{\beta_0 \alpha_s(s_0)}}$$

$$e^{-u\ln(-x)}\, e^{-\frac{4\pi u}{\beta_0 \alpha_s(s_0)}} \;=\; e^{-\frac{4\pi u}{\beta_0 \alpha_s(-s)}}$$

➡ FOPT Borel representation = “true“ Borel representation

$$\delta^{(0),\mathrm{FOPT}}_{W_i,\mathrm{Borel}}(s_0) = \mathrm{PV}\int_0^\infty \mathrm{d}u\, \frac{1}{2\pi i} \oint_{|x|=1} \frac{\mathrm{d}x}{x}\, W_i(x)\, B[\hat{D}](u)\, e^{-\frac{4\pi u}{\beta_0 \alpha_s(-x s_0)}}$$

FOPT vs. CIPT Borel Representation

CIPT expansion: $\rightarrow$ No obvious expansion parameter !

$$\delta^{(0),\mathrm{CIPT}}_{W_i}(s_0) = \frac{1}{2\pi i}\sum_{n=1}^{\infty} c_{n,1} \oint_{|x|=1} \frac{\mathrm{d}x}{x} W_i(x) \left(\frac{\alpha_s(-xs_0)}{\pi}\right)^n = \frac{1}{2\pi i}\sum_{n=1}^{\infty} \underbrace{\left(\frac{\alpha_s(s_0)}{\pi}\right)^n}_{\text{expansion variable}} \underbrace{c_{n,1} \oint_{|x|=1} \frac{\mathrm{d}x}{x} W_i(x) \left(\frac{\alpha_s(-xs_0)}{\alpha_s(s_0)}\right)^n}_{\text{coefficient}},$$

➡ CIPT Borel representation: NEW !

Regner, Hoang arXiv:2008.00578

$$\delta^{(0),\mathrm{CIPT}}_{W_i,\mathrm{Borel}}(s_0) = \int_0^\infty \mathrm{d}\bar{u}\, \frac{1}{2\pi i} \oint_{\mathcal{C}_x} \frac{\mathrm{d}x}{x} W_i(x) \left(\frac{\alpha_s(-xs_0)}{\alpha_s(s_0)}\right) B[\hat{D}]\left(\frac{\alpha_s(-xs_0)}{\alpha_s(s_0)}\bar{u}\right) e^{-\frac{4\pi\bar{u}}{\beta_0\alpha_s(s_0)}}$$

Contour needs to be deformed from |x|=1

„Asymptotic Separation“

$$\Delta_W(s_0) \equiv \delta^{(0),\mathrm{CIPT}}_{W,\mathrm{Borel}}(s_0) - \delta^{(0),\mathrm{FOPT}}_{W,\mathrm{Borel}}(s_0)$$

$$\Delta_W(s_0) \sim \frac{\Lambda^d_{\mathrm{QCD}}}{s_0^{d/2}} \quad \text{for } \mathcal{O}(\Lambda^d_{\mathrm{QCD}}) \text{ IR renormalon contained in } \hat{D}$$

Character of the Asymptotic Separation

FOPT Borel representation

$$\delta^{(0),\mathrm{FOPT}}_{W_i,\mathrm{Borel}}(s_0) = \mathrm{PV}\int_0^\infty \mathrm{d}u \,\frac{1}{2\pi i}\oint_{|x|=1}\frac{\mathrm{d}x}{x}\, W_i(x)\, B[\hat{D}](u)\, e^{-\frac{4\pi u}{\beta_0 \alpha_s(-x s_0)}}$$

CIPT Borel representation

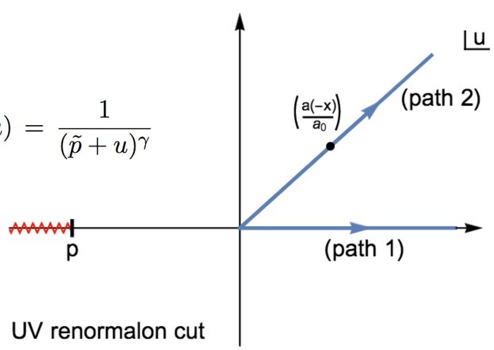
$$\delta^{(0),\mathrm{CIPT}}_{W_i,\mathrm{Borel}}(s_0) = \int_0^\infty \mathrm{d}\bar{u}\,\frac{1}{2\pi i}\oint_{\mathcal{C}_x}\frac{\mathrm{d}x}{x}\, W_i(x)\left(\frac{\alpha_s(-x s_0)}{\alpha_s(s_0)}\right) B[\hat{D}]\left(\frac{\alpha_s(-x s_0)}{\alpha_s(s_0)}\bar{u}\right) e^{-\frac{4\pi \bar{u}}{\beta_0 \alpha_s(s_0)}}$$

- Related through complex-valued change of variables

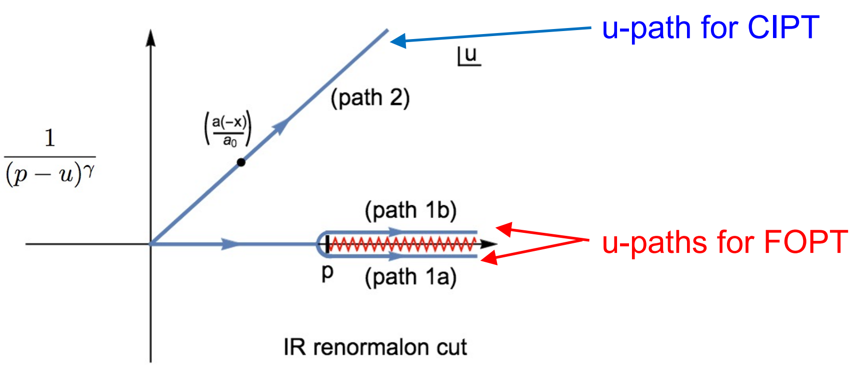
$$u = \frac{\alpha_s(-x s_0)}{\alpha_s(s_0)}\bar{u}$$

- Equivalent in perturbation theory (u-Taylor series)
- Agree at Euclidean point x = -1
- Difference in presence of IR renormalon cuts

$$B^{\mathrm{UV}}_{\hat{D},-\tilde{p},\gamma}(u) = \frac{1}{(\tilde{p}+u)^\gamma}$$



$$B^{\mathrm{IR}}_{\hat{D},p,\gamma}(u) = \frac{1}{(p-u)^\gamma}$$



UV renormalons:

FOPT and CIPT Borel representations equivalent because closing up paths 1 and 2 does not contain cuts

IR renormalons: finite difference !

FOPT and CIPT Borel representations inequivalent

- FOPT: PV prescription needs to be imposed
- CIPT: automatically well-defined by complex-valued α_s
- Difference because closing paths 1a/1b and 2 always contains cuts

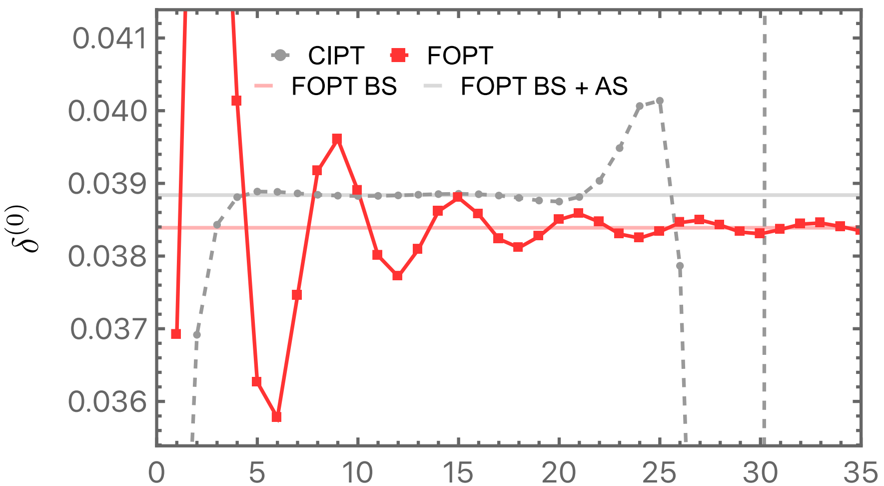
Brief Numerical Analysis

Single renormalon model:

$$B(u) \sim \frac{1}{(2-u)^{1+4\hat{b}_1}} + \dots \quad \Longleftrightarrow \quad \frac{\langle \bar{G}^2 \rangle}{s^2}$$

$$W(x) = 1$$

$\bar{c}^{(1)}_{4,0} = 0,\ R = 0.8 m_\tau,\ W(x) = 1$



$s_0 = m_\tau^2$

Pure O(Λ^4_{QCD}) renormalon in Adler function

→ Gluon condensate corrections vanishes

→ Per. series should be convergent

- CIPT series is **divergent** !
  FOPT series convergent.

  This fact was overlooked in the past

- CIPT not compatible with standard OPE !

  CIPT Borel representation should not be considered as ”true”, but it correctly characterizes the CIPT expansion

- Excellent description of CIPT-FOPT discrepancy by asymptotic separation Δ

- Δ can have any sign (dep. choice of W(x)).

- Moments with small asymptotic separation can be identified.

Renormalon-Free GC Scheme

Conclusion from the asymptotic separation:

Benitez-Rathgeb, Boito, Jamin, AHH: 2202.10957

- Asymptotic separation vanishes if IR renormalons are absent
- CIPT and FOPT should become consistent for IR-subtracted perturbation theory

Idea: “short-distance“ scheme for the gluon condensate

$$\langle \bar{G}^2 \rangle^{\overline{\mathrm{MS}},(n)}$$

Original $\overline{\mathrm{MS}}$ GC contains pure O(Λ^4_{QCD}) renormalon (scale invariant)

$$\langle \bar{G}^2 \rangle^{\overline{\mathrm{MS}},(n)} \equiv \underbrace{\langle G^2 \rangle (R^2)}_{\text{renormalon-free, R-dependent}} - R^4 \overbrace{\sum_{\ell=1}^{n} \underbrace{N_g}_{\text{renormalon norm (approximately known)}} r_\ell^{(4,0)} \bar{a}^\ell (R^2)}^{\text{Expand perturbatively with Adler function}} ,$$

Expand perturbatively with Adler function

renormalon-free
R-dependent

renormalon norm
(approximately known)

IR factorization scale R

$$r_\ell^{(4,0)} = \left(\frac{1}{2}\right)^{\ell + 4\hat{b}_1} \frac{\Gamma(\ell + 4\hat{b}_1)}{\Gamma(1 + 4\hat{b}_1)}$$

$$\bar{a}(R^2) = \frac{\beta_0\, \bar{\alpha}_s(R^2)}{4\pi}$$

C-scheme (C=0)

Boito, Jamin, Miravitllas 2016

Renormalon-Free GC Scheme

Benitez-Rathgeb, Boito, Jamin, AHH: 2202.10957

$$\langle G^2\rangle(R^2) - \langle G^2\rangle(R'^2)$$

Renormalon-free (convergent series)

$$\frac{\mathrm{d}}{\mathrm{d}\ln R^2}\,\langle G^2\rangle(R^2) \;=\; \frac{N_g}{2^{4\hat{b}_1}}\,\frac{R^4\,\bar{a}(R^2)}{1-2\hat{b}_1\bar{a}(R^2)}$$

(R-evolution equation)

Convergent series!

We can define an R-independent „short-distance“ GC:

treated like a tree-level term
(Do not expand !)

$$\langle G^2\rangle(R^2) \;\equiv\; \langle G^2\rangle^{\mathrm{RF}} + N_g\,\bar{c}_0(R^2)\,.$$

→ R-invariance of scheme at infinite truncation order

$$\bar{c}_0(R^2) \equiv R^4\;\;\mathrm{PV}\int_0^\infty \frac{\mathrm{d}u\; e^{-\frac{u}{\bar{a}_R}}}{(2-u)^{1+4\hat{b}_1}} = -\frac{R^4\,e^{-\frac{2}{\bar{a}(R^2)}}}{(\bar{a}(R^2))^{4\hat{b}_1}}\,\mathrm{Re}\left[e^{4\pi\hat{b}_1 i}\,\Gamma\Big(-4\hat{b}_1, -\frac{2}{\bar{a}(R^2)}\Big)\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln R^2}\,\langle G^2\rangle^{\mathrm{RF}} \;=\; 0$$

Scale-invariant “short-distance“ scheme for the gluon condensate

→ „true“ Borel sum value unchanged (i.e. N_g-independent) !   („minimal scheme“)

CIPT and FOPT: RF GC Scheme

Benitez-Rathgeb, Boito, Jamin, AHH: 2202.10957

Single renormalon model:

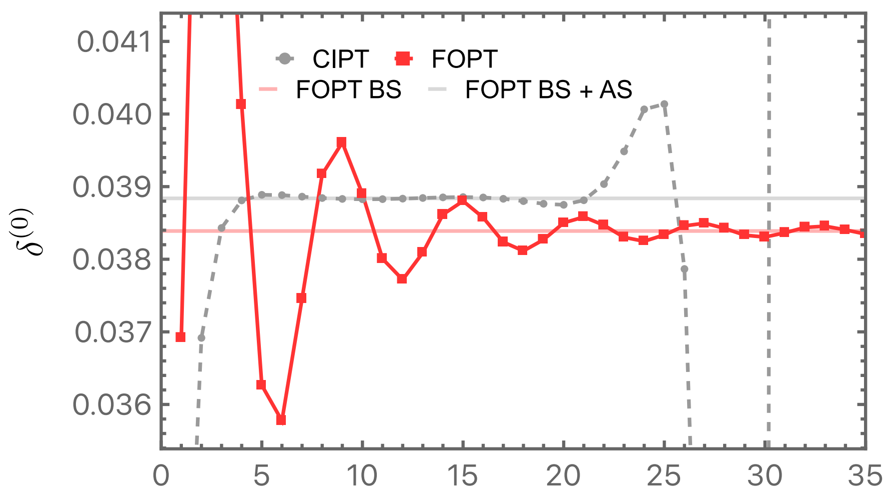
Pure O(Λ^4_{QCD}) renormalon in Adler function

$$B(u) \sim \frac{1}{(2-u)^{1+4\hat{b}_1}} + \ldots \quad \Longleftrightarrow \quad \frac{\langle \bar{G}^2 \rangle}{s^2}$$

→ Gluon condensate corrections vanishes

$$W(x) = 1$$

$\bar{c}^{(1)}_{4,0} = 0,\ R = 0.8 m_\tau,\ W(x) = 1$



$$s_0 = m_\tau^2$$

- FOPT same as in the original GC scheme
- CIPT^{RS} series is convergent
- CIPT^{RS} consistent with FOPT !
- CIPT^{RS} compatible with standard OPE !
- CIPT^{RS} Borel sum = FOPT Borel sum
- CIPT^{RS} converges much faster than FOPT

CIPT and FOPT: RF GC Scheme

Benitez-Rathgeb, Boito, Jamin, AHH: 2202.10957

Single renormalon model:

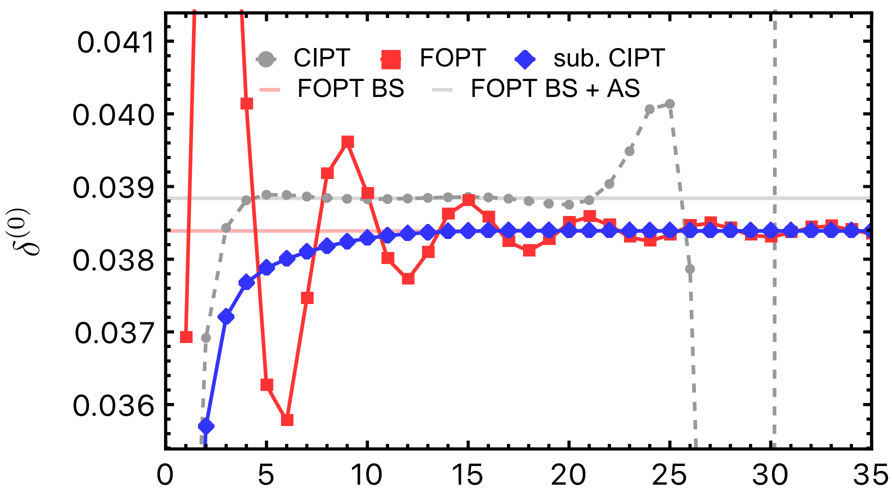
$$B(u) \sim \frac{1}{(2-u)^{1+4\hat{b}_1}} + \ldots \quad \Longleftrightarrow \quad \frac{\langle \bar{G}^2 \rangle}{s^2}$$

$$W(x) = 1$$

Pure $O(\Lambda^4_{QCD})$ renormalon in Adler function

→ Gluon condensate corrections vanishes

$\bar{c}^{(1)}_{4,0} = 0, \; R = 0.8 m_\tau, \; W(x) = 1$



$s_0 = m_\tau^2$

- FOPT same as in the original GC scheme
- CIPT[RS] series is convergent
- CIPT[RS] consistent with FOPT !
- CIPT[RS] compatible with standard OPE !
- CIPT[RS] Borel sum = FOPT Borel sum
- CIPT[RS] converges much faster than FOPT

CIPT and FOPT: RF GC Scheme

Realistic Multi renormalon model: GC, $O(\Lambda^4_{QCD}, \Lambda^6_{QCD})$ + UV renormalons in Adler function

Beneke, Jamin 2008

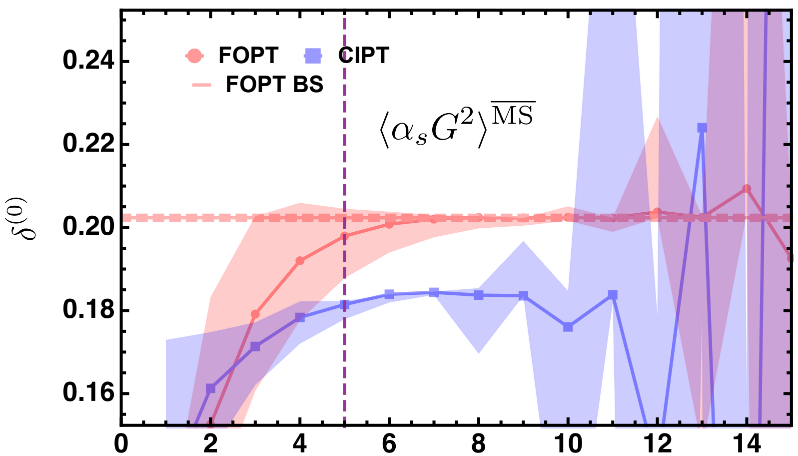
$$W(x) = 1 - 2x + 2x^3 - x^4$$

→ GC suppressed

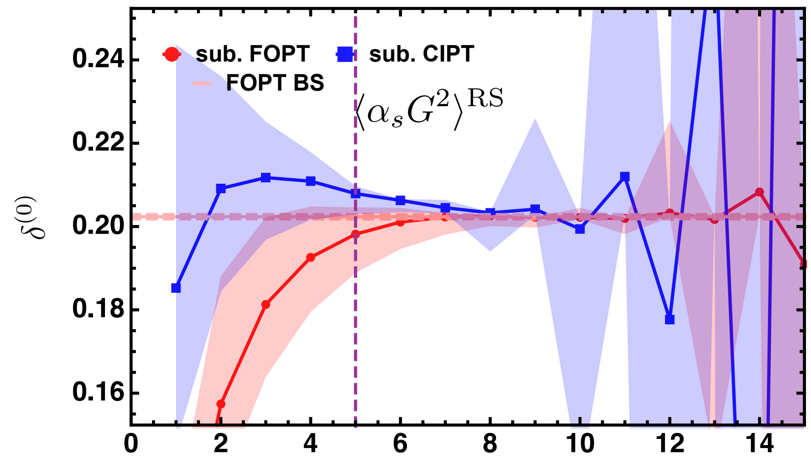
$$s_0 = m_\tau^2\,, \quad \frac{1}{2} \leq \xi \leq 2\,, \quad N_g = 0.64$$

Benitez-Rathgeb, Boito, Jamin, AHH: 2202.10957

$\bar{c}^{(1)}_{4,0} = -22/81,\ W(x) = (1-x)^3(1+x)$



$\bar{c}^{(1)}_{4,0} = -22/81,\ R = 0.8 m_\tau,\ W(x) = (1-x)^3(1+x)$



New RF GC Scheme !

- Discrepancy between CIPT and FOPT removed
- CIPT becomes consistent with FOPT (which is only slightly modified)
- Higher precision for α_s determinations from hadronic tau decays achievable
- Additional uncertainty from uncertainties in N_g

CIPT and FOPT: RF GC Scheme

Realistic Multi renormalon model:

GC, $O(\Lambda^4_{QCD}, \Lambda^6_{QCD})$ + UV renormalons in Adler function

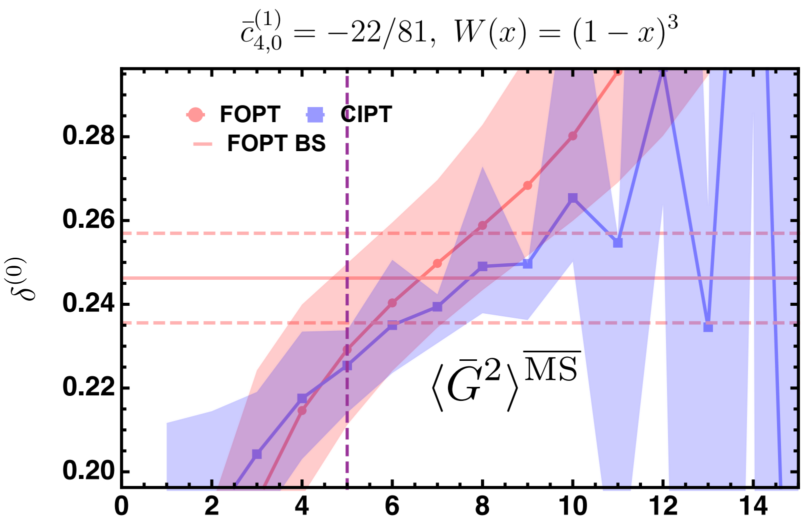
Beneke, Jamin 2008

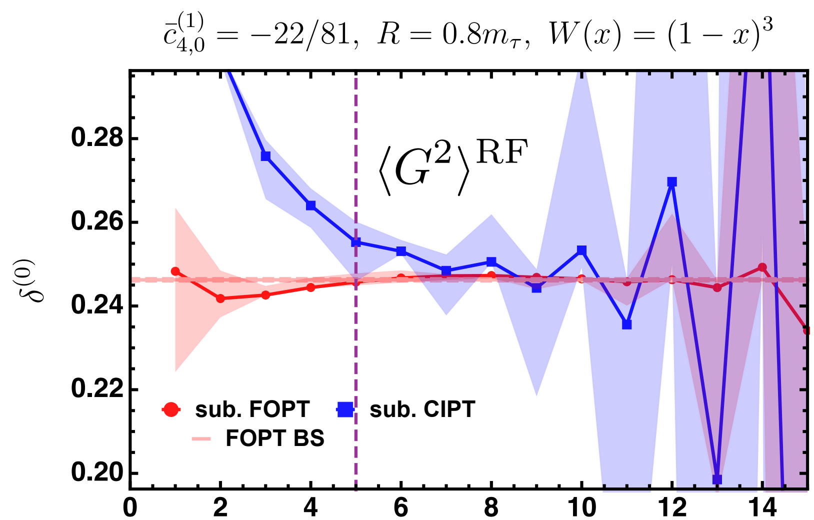
$$W(x) = (1-x)^3$$

→ GC enhanced

$$s_0 = m_\tau^2\,, \quad \frac{1}{2} \le \xi \le 2\,, \quad N_g = 0.64$$

Benitez-Rathgeb, Boito, Jamin, AHH: 2202.10957





New RF GC Scheme !

- FOPT and CIPT expansions both get improved substantially
- Spectral function moments with high sensitivity to the GC can now be used for high-precision determinations of the strong coupling and the GC

CIPT and FOPT: RS GC Scheme

Impact on α_s determination:

Full-fledged analysis: w.i.p.

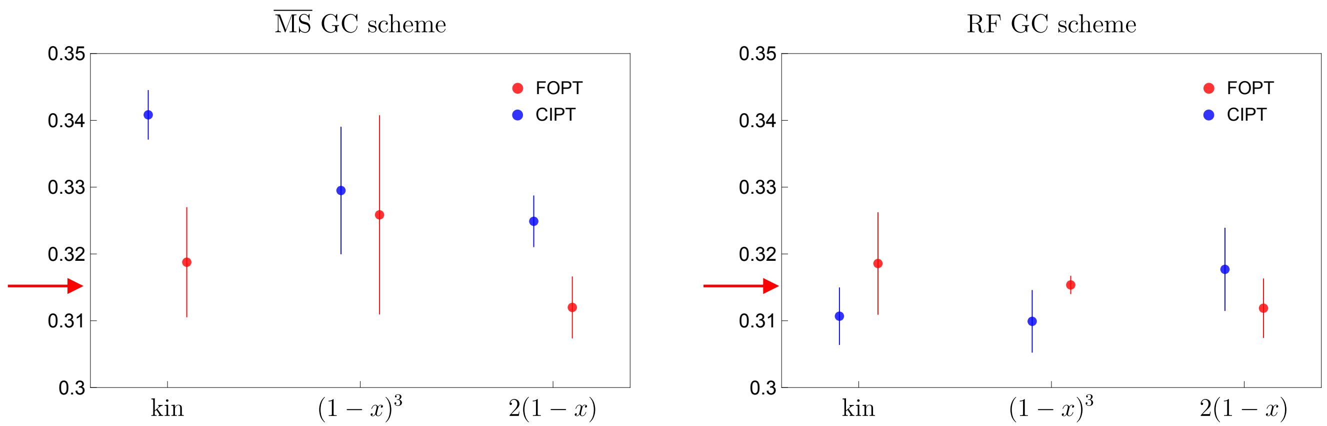
→ For now: use the the Beneke-Jamin model as „fake-data“

$$s_0 = m_\tau^2\,, \quad \frac{1}{2} \le \xi \le 2\,, \quad N_g = 0.64\,, \quad R = (0.8 \pm 0.1) m_\tau$$

Benitez-Rathgeb, Boito, Jamin, AHH: 2202.10957

Input value:

$\alpha_s(m_\tau) = 0.315$



- FOPT-CIPT for GC suppressed moments remedied
- Spectral function moments with high sensitivity to the GC can now be used for high-precision determinations of the strong coupling and the GC
- Uncertainty on GC renormalon norm N_g still to be addressed.

Summary and Conclusions

- CIPT Borel representation different from FOPT Borel representation in the presence of IR renormalons. $\longrightarrow$ **Asymptotic Separation**

  $\longrightarrow$ **CIPT expansion NOT consistent with standard OPE approach**

- Problems of CIPT resolved largely in gluon condensate (GC) scheme.

- We have devised such a GC scheme in the most minimalistic and transparent way. (Additional uncertainty from N_g (GC renormalon norm), and factorization scale R.)

- New GC scheme: Disparity between CIPT and FOPT reconciled

- New GC scheme: Moments with high sensitivity to the GC can be used for high precision analyses

$\longrightarrow$ **The known $O(\alpha_s^{4,5})$ corrections are consistent with the assumption of a sizeable GC renormalon norm $N_g \approx 0.64$**

$\longrightarrow$ **All studies support that the CIPT-FOPT discrepancy is of IR origin**

$\longrightarrow$ **Excellent prospects for new high-precision determinations of α_s and $\langle G^2\rangle^{\mathrm{RF}}$**

$\longrightarrow$ **6-loop corrections highly welcome to further increase precision!**

(1) CIPT Borel Sum Contour Integration

The contour integration for the CIPT Borel representation must be deformed away from |x| = 1.

(Leaves FOPT Borel sum unchanged!)

Do the contour-integral first:

$$\delta^{(0),\mathrm{CIPT}}_{\{(-x)^m,p,\gamma\},\mathrm{Borel}}(s_0) = \int_0^\infty \mathrm{d}\bar{u}\,\frac{1}{2\pi i}\oint_{\mathcal{C}_x}\frac{\mathrm{d}x}{x}\,(-x)^m\left(\frac{a(-x)}{a_0}\right)\frac{e^{-\frac{\bar{u}}{a_0}}}{\left(p-\frac{a(-x)}{a_0}\bar{u}\right)^\gamma}$$

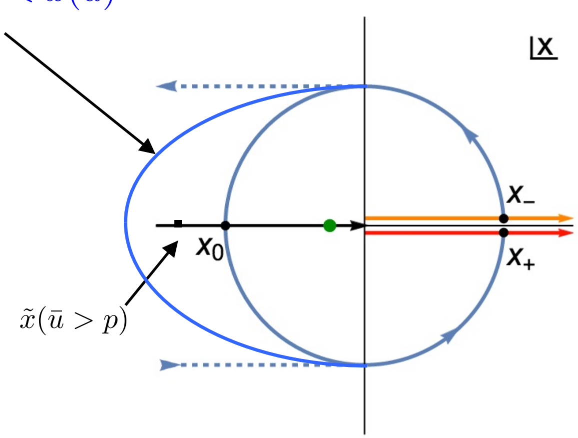
$$= \int_0^\infty \mathrm{d}\bar{u}\,e^{-\frac{\bar{u}}{a_0}}\,\tilde{C}(p,\gamma,m,s_0;\bar{u})\,.$$

pole in x-plane at (arge-β_0)

$$\tilde{x}(\bar{u}) = -e^{(\bar{u}-p)/pa_0} = -\left(\frac{\Lambda^2_{\mathrm{QCD}}}{s_0}\right)^{\frac{p-\bar{u}}{p}}$$

$$< -1 \quad \text{for} \quad \bar{u} > p$$

Contour must always cross real axis for $x < \tilde{x}(\bar{u})$



$$\tilde{x}(0) = -\left(\frac{\Lambda^2_{\mathrm{QCD}}}{s_0}\right) \quad \text{(Landau pole)}$$

$$\tilde{x}(\bar{u}\to\infty) \to -\infty$$

(2) CIPT Borel Sum Contour Integration

The contour integration for the CIPT Borel representation must be deformed away from $|x| = 1$.

(Leaves FOPT Borel sum unchanged!)

Do the Borel-u-integral first:

$$\Delta(m,p,\gamma,s_0) \equiv \delta^{(0),\mathrm{CIPT}}_{\{(-x)^m,p,\gamma\},\mathrm{Borel}}(s_0) - \delta^{(0),\mathrm{FOPT}}_{\{(-x)^m,p,\gamma\},\mathrm{Borel}}(s_0)$$

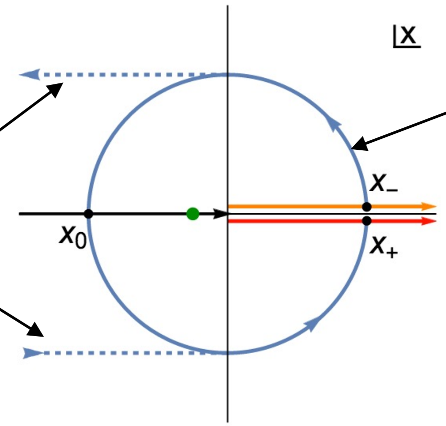
$$= \frac{1}{2\Gamma(\gamma)} \oint_{\mathcal{C}_x} \frac{\mathrm{d}x}{x} (-x)^m \, \mathrm{sig}[\mathrm{Im}[a(-x)]] \, (a(-x))^{1-\gamma} \, e^{-\frac{p}{a(-x)}} \, .$$

„Asymptotic Separation“

Cut along the negative real s-axis!

Power-suppressed $\sim \left(\frac{\Lambda^2_{\mathrm{QCD}}}{s}\right)^p$

Remaining contour integration must be deformed (to negative real infinity in the x-plane)



Contour along the unit circle

Deformed contour for the asymptotic separation

(3) Computation of the CIPT Borel Sum

An analytic continuation is mandatory to compute the CIPT Borel sum for m>p

$$W(x) \sim x^m$$

$$B(u) \sim \frac{1}{(p-u)^\gamma}$$

$$\Delta(m,p,\gamma,s_0) \equiv \delta^{(0),\mathrm{CIPT}}_{\{(-x)^m,p,\gamma\},\mathrm{Borel}}(s_0) - \delta^{(0),\mathrm{FOPT}}_{\{(-x)^m,p,\gamma\},\mathrm{Borel}}(s_0)$$

$$= \frac{1}{2\Gamma(\gamma)} \oint_{\mathcal{C}_x} \frac{\mathrm{d}x}{x} (-x)^m \, \mathrm{sig}[\mathrm{Im}[a(-x)]] \, (a(-x))^{1-\gamma} \, e^{-\frac{p}{a(-x)}} \, .$$

$$\sim x^{-p}$$

Properties of the asymptotic separation:

- Renormalization scheme invariant
- **Much larger than canonical FOPT Borel sum ambiguity estimate if the Borel function has a sizeable gluon condensate cut**
- Fully analytic results
- **Properties of CIPT Borel representation imply that OPE corrections for CIPT do not have the common standard form C x <condensate> / s^p**