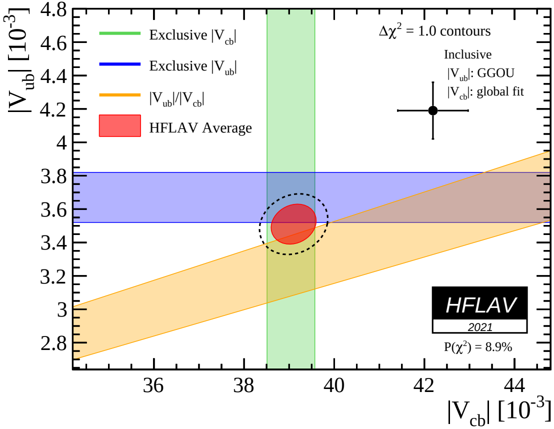
Third order QCD corrections for inclusive semileptonic B decays

Loops and Legs in Quantum Field Theory – Ettal – 30 Apr. 2022

Matteo Fael | with K. Schönwald and M. Steinhauser

Why V_{cb}?

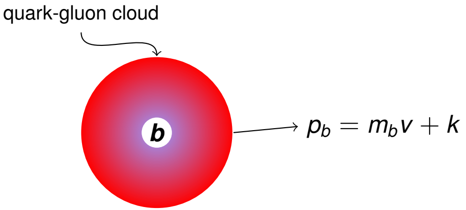
- $|V_{ub}|$ and $|V_{cb}|$ are SM input parameters.
- $|V_{cb}|$ uncertainties dominates in
  - $K \to \pi\nu\bar{\nu}$, $K_L \to \mu\mu \simeq |V_{cb}|^4$
  - $\mathcal{B}(B_s \to \mu^+\mu^-) \simeq |V_{tb}V_{ts}^*|^2 \simeq |V_{cb}|^2(1 + O(\lambda))$
  - $\epsilon_K \simeq |V_{cb}|^4$
- Test of the SM flavour picture.



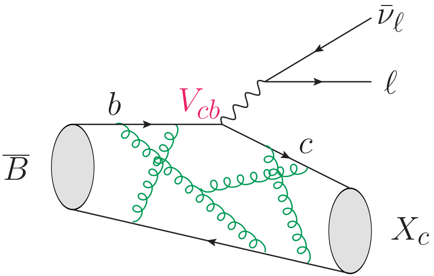
The Heavy-Quark Expansion

$$\Gamma_{\rm sl} = \Gamma_{\rm free} + \Gamma_{\mu_\pi} \frac{\mu_\pi^2}{m_b^2} + \Gamma_{\mu_G} \frac{\mu_G^2}{m_b^2} + \Gamma_{\rho_D} \frac{\rho_D^3}{m_b^3} + \Gamma_{\rho_{LS}} \frac{\rho_{LS}^3}{m_b^3} + \dots$$

Reviews:
Benson, Bigi, Mannel, Uraltsev, Nucl.Phys. B665 (2003) 367;
Dingfelder, Mannel, Rev.Mod.Phys. 88 (2016) 035008.



- The HQE parameters: $\mu_\pi, \mu_G, \rho_D, \rho_{LS} \sim \langle B| \; \mathcal{O}_i^{\bar b b} \; |B\rangle$.
- HQE parameters are **fitted from kinematic moments**.
- Γ_i are computed in **perturbative QCD**.
- Theoretical uncertainties impact on $|V_{cb}|$.



Third order corrections for $B \to X_c \ell \nu_\ell$

- Total semileptonic rate

MF, Schönwald, Steinhauser, PRD 104 (2021) 016003, JHEP 10 (2020) 087

- Relation between $\overline{\mathrm{MS}}$ mass and the kinetic mass

MF, Schönwald, Steinhauser, PRL 125 (2020) 052003, PRD 103 (2021) 014005

- First glance to the spectral moments.

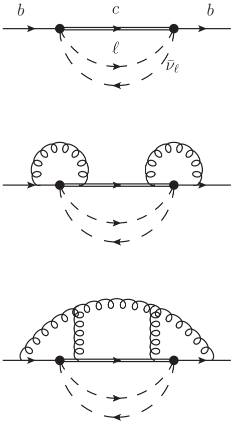
MF, Schönwald, Steinhauser, in preparation

Higher QCD corrections to $b \to X_c \ell \bar{\nu}_\ell$

$$\Gamma_{\text{free}} = \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 \left[X_0(\rho) + C_F \sum_{n\geq 1} \left(\frac{\alpha_s}{\pi}\right)^n X_n(\rho) \right]$$

- Exact dependence on $\rho = m_c/m_b$ only at $O(\alpha_s)$.
  Nir, Phys.Lett.B 221 (1989) 184
- Numerical approach
  Melnikov, PLB 666 (2008) 336
- **Exploit $m_c < m_b$ for asymptotic expansion**
  Czarnecki, Pak, PRD 78 (2008) 114015
  - Optical theorem
  - Method of Regions

  Beneke, Smirnov, NPB 522 (1998) 321.



Second order corrections

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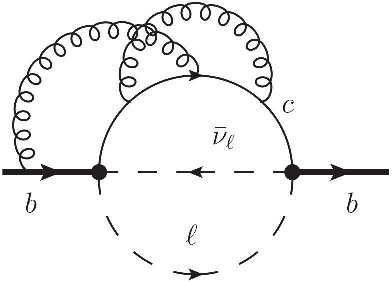
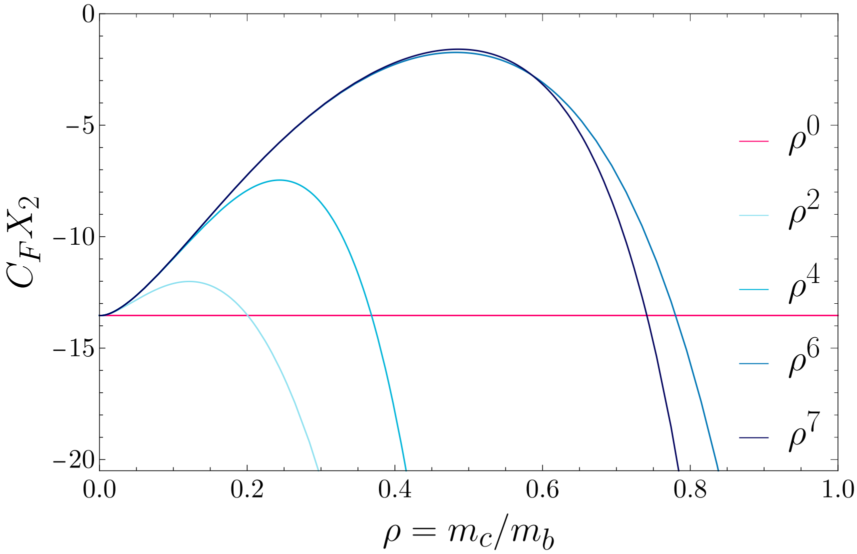
Expand around the massless limit $\rho \to \mathbf{0}$:

$$\Gamma_{\text{free}} = \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 \left[X_0 + C_F \frac{\alpha_s}{\pi} X_1 + C_F \left(\frac{\alpha_s}{\pi}\right)^2 X_2 + \dots \right]$$



Czarnecki, Pak, PRD 78 (2008) 114015; PRL 100 (2008) 241807.

- Loop momenta **hard** (m_b) or **soft** (m_c).
- **11 different regions**.
- Four-loop diagrams all hard region.
- **33 four-loop master integrals**.
- Expansion depth: $O(\rho^7)$ ($\rho = m_c/m_b$).



based on Czarnecki, Pak, PRD 78 (2008) 114015; PRL 100 (2008) 241807.

Towards $O(\alpha_s^3)$ corrections

	α_s^2		α_s^3
n. diagrams	62	$\rightarrow$	1450
n. loops	4	$\rightarrow$	5
regions	11	$\rightarrow$	O(20)
expansion depth	7	$\rightarrow$	?
master integrals	33	$\rightarrow$	?

The heavy daughter limit

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Dowling, Piclum, Czarnecki, PRD 78 (2008) 074024

- The most natural expansion parameter sometime is not the best one.

$$\frac{m_c}{m_b} \sim 0.3 \qquad 1 - \frac{m_c}{m_b} \sim 0.7$$

- Heavy daughter limit $m_c \sim m_b$:

$$\delta = 1 - \rho = 1 - \frac{m_c}{m_b} \ll 1$$

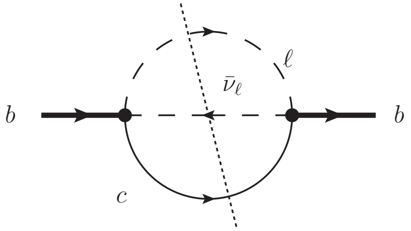
- Leading power in δ:

$$\Gamma_{\mathrm{sl}} \overset{m_c \to m_b}{\simeq} \frac{G_F^2}{192\pi^3}(m_b - m_c)^5 = \frac{G_F^2 m_b^5}{192\pi^3}\delta^5$$

Total rate in the heavy daughter limit

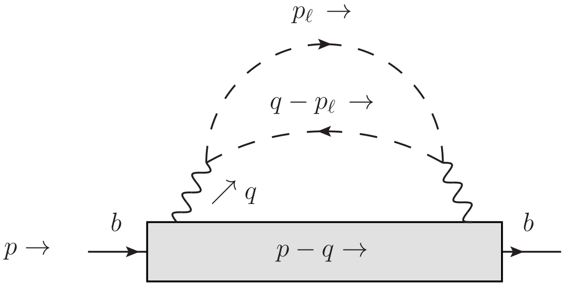
$$\Gamma_{\rm sl} = \frac{G_F^2 m_b^5}{192\pi^3}|V_{cb}|^2 \left[\frac{64}{5}\delta^5 - \frac{96}{5}\delta^6 + \frac{288}{35}\delta^7 + \dots\right]$$

where $\delta = 1 - m_c/m_b$.

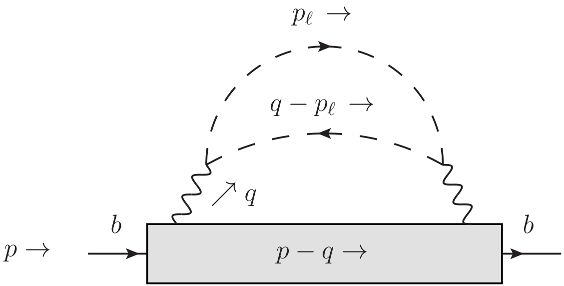


- Loop momenta are
  - **hard** $\sim m_b$
  - **ultrasoft** $\sim m_b - m_c = m_b\delta$
- Cut propagators must carry **ultrasoft** momentum.
- **Considerably smaller number of regions.**

$$\frac{1}{(p_b-k)^2 - m_c^2} \to \frac{1}{2p_b \cdot k - 2m_b\delta}$$



$$\int \frac{d^d q \, d^d k}{(q^2)^{n_1} (k^2)^{n_2} (2p \cdot k - 2p \cdot q + 2\delta)^{n_3}}$$



$$\int \frac{d^d q\, d^d k}{(q^2)^{n_1}(k^2)^{n_2}(2p\cdot k - 2p\cdot q + 2\delta)^{n_3}} \overset{k\to k(-2p\cdot q+2\delta)}{=} \int \frac{d^d q}{(q^2)^{n_1}(-2p\cdot q + 2\delta)^{-d+2n_2+n_3}} \times \int \frac{d^d k}{(k^2)^{n_2}(2p\cdot k+1)^{n_3}}.$$

Factorization in the $m_c \simeq m_b$ limit

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5 loop calculation $\longrightarrow$ 3 loop calculation!

order	regions
α_s	u, h
α_s^2	**uu**, **hh**, hu
α_s^3	**uuu**, **hhh**, h**uu**, u**hh**

Regions in red need IBP reduction

- Tensor integral of a massive one-loop two-point function:

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^{\mu_1} \dots k^{\mu_N}}{(-k^2)^{n_1}(-k^2 + 2p\cdot k)^{n_2}}$$

- Tensor integral of a one-loop ultra-soft two-point function:

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^{\mu_1} \dots k^{\mu_N}}{(-k^2)^{n_1}(-2p\cdot k + \delta)^{n_2}}$$

Computational Challenges

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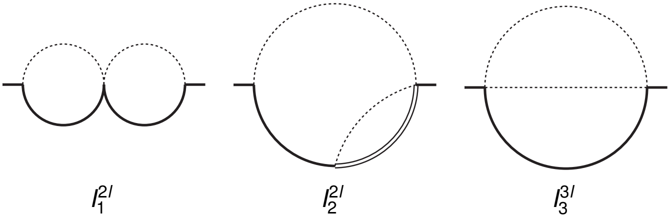
- Check loop momentum routing with `asy`.
  Pak, Smirnov, EPJC 71 (2011) 1626
- Several subtleties with `FORM`
  - Propagators expansion up to **10th - 12th order**.
  - Avoid uncontrolled grow of **intermediate expressions size**.
  - Intermediate `FORM` expressions up to $O(100)$ GB.
- Partial fraction and map to a minimal set of families with `LIMIT`
  Herren, PhD thesis 2020
- Private version of `FIRE6.4` combined with `LiteRed`.
  Smirnov, Chuharev, hep-ph/1901.07808; Lee, hep-ph/1212.2685.
- Master Integrals
  - $O(\alpha_s^2)$: 3 (uu) and 3 (hh).
  - $O(\alpha_s^3)$: **20 (uuu)** and **19 (hhh)**.
    Melnikov, van Ritbergen, Nucl.Phys.B 591 (2000) 515;
    MF, Schönwald, Steinhauser, Phys.Rev.Lett. 125 (2020) 5.

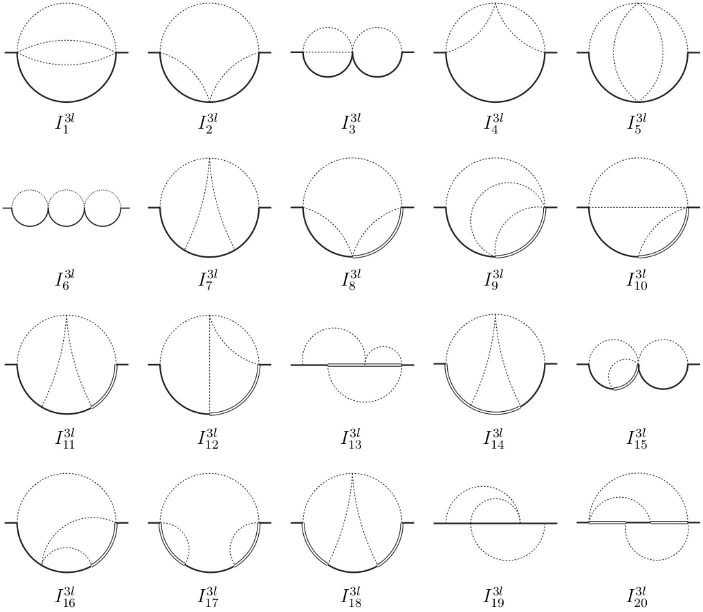


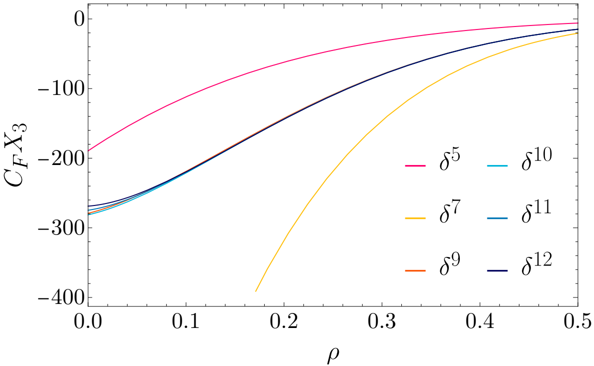
- Feynman diagrams with Eikonal-massive propagators

$$I_2^{2l} = \int d^d k_1 d^d k_2 \frac{1}{k_1^2 (k_1 - k_2)^2 (2k_1 \cdot p - y)(2k_2 \cdot p)}$$

- Mellin-Barnes: `MB package`
  Czakon, Comput. Phys. Commun. 175 (2006) 559;
  Smirnov^2, EPJC 62 (2009) 445.
- PSLQ
- Analytic summation of residues

  HarmincSums
  www3.risc.jku.at/research/combinat/software/HarmonicSums/
- DEQs in auxiliary variable
  Kotikov, PLB 254 (1991), 158
  Gehrmann, Remiddi, NPB 580 (2000) 485
  Henn, PRL 110 (2013), 251601.





$$C_F X_3(0.28) = -91.2 \pm 0.4\,(0.4\%)$$

MF, Schönwald, Steinhauser, Phys.Rev.D 104 (2021) 1, 016003

- Expansion up to δ^{12}.
- Abelian limit up to δ^9 confirmed by

Czakon, Czarnecki, Dowling, Phys.Rev.D 103 (2021) L111301

The kinetic scheme

The total rate in the on-shell scheme:

$$\Gamma_{\rm sl} = \frac{G_F^2 |V_{cb}|^2 (m_b^{\rm OS})^5}{192\pi^3} f(0.28) \left[1 - 1.72 \left(\frac{\alpha_s}{\pi}\right) - 13.1 \left(\frac{\alpha_s}{\pi}\right)^2 - 163.3 \left(\frac{\alpha_s}{\pi}\right)^3\right] + O\left(\frac{1}{m_b^2}\right)$$

See: Bigi, Shifman, Uraltsev, Vainshtein PRD 50 (1994) 2234; Beneke, Braun, NPB 426 (1994) 301;
Ball, Beneke, Braun, PRD 52 (1995) 3929; Melnikov, van Ritbergen, PLB 482 (2000) 99.

In the $\overline{\rm MS}$ scheme

$$\Gamma_{\rm sl} = \frac{G_F^2 |V_{cb}|^2 \overline{m}_b^5}{192\pi^3} f(0.28) \left[1 + 3.07 \left(\frac{\alpha_s}{\pi}\right) + 13.3 \left(\frac{\alpha_s}{\pi}\right)^2 + \ 62.7 \left(\frac{\alpha_s}{\pi}\right)^3 \ldots\right] + O\left(\frac{1}{m_b^2}\right)$$

Meson-quark mass relation

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$$m_b = M_B - \overline{\Lambda} - \frac{\mu_\pi^2}{2m_b} + \dots$$

- $\overline{\Lambda}$: the B-meson binding energy.
- μ_π: the kinetic energy induced by the residual motion of the heavy quark.

The relevant parameter in $\Gamma_{\rm sl}$ is $\boldsymbol{m_b^5}$**, not** $\boldsymbol{M_B^5}$:

$$\Gamma_{\rm sl} \simeq \frac{G_F^2 |V_{cb}|^5}{192\pi^3} (M_B - \overline{\Lambda})^5$$

Bigi, Shifman, Uraltsev, Vainshtein, PRD 56 (1997) 4017

The kinetic mass

$$m_b^{\mathrm{kin}}(\mu) = m_b^{\mathrm{OS}} - [\overline{\Lambda}(\mu)]_{\mathrm{pert}} - \frac{[\mu_\pi^2(\mu)]_{\mathrm{pert}}}{2m_b^{\mathrm{kin}}(\mu)} - \ldots$$

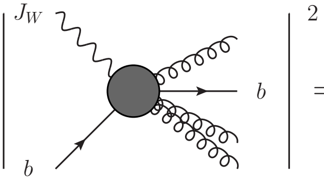
Bigi, Shifman, Uraltsev, Vainshtein, PRD 56 (1997) 4017.
see also: Czarnecki, Melnikov, Uraltsev, PRL 80 (1998) 3189;
Gambino, JHEP 09 (2011) 055;

- In pQCD, we can make a short-distance mass definition by identifying:

$$m_b(\mu) \to m_b^{\mathrm{kin}}(\mu) \qquad \overline{M}_B \to m_b^{\mathrm{OS}}$$

$$\overline{\Lambda}(\mu) \to [\overline{\Lambda}(\mu)]_{\mathrm{pert}} \qquad [\mu_\pi^2(\mu)] \to [\mu_\pi^2(\mu)]_{\mathrm{pert}}$$

The Kinetic Mass as a Threshold Mass



$$= W(\omega, \vec{v})$$

- Heavy quark scattering close to **one-particle Threshold limit**

$$y = s - m_b^2 \simeq 2 m_b \omega \ll m_b^2$$

- Factorization in the Small Velocity limit $q^2 = m_b^2 \vec{v}^2 \ll m_b^2$

$$W(\omega, \vec{v}) \simeq H \cdot U(\omega, \vec{v})$$

The Small Velocity Sum Rules

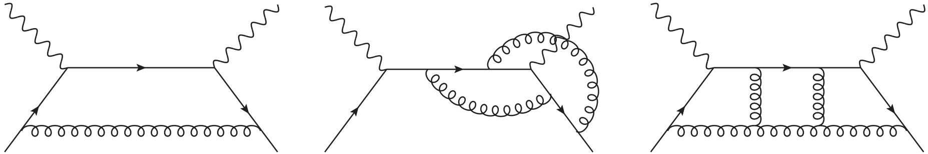
$$[\overline{\Lambda}(\mu)]_{\text{pert}} = \lim_{\vec{v}\to 0} \lim_{m_b\to\infty} \frac{2}{\vec{v}^{\,2}} \frac{\int_0^{\mu} \mathrm{d}\omega\, \omega\, W(\omega, \vec{v})}{\int_0^{\mu} \mathrm{d}\omega\, W(\omega, \vec{v})}$$

$$[\mu_\pi^2(\mu)]_{\text{pert}} = \lim_{\vec{v}\to 0} \lim_{m_b\to\infty} \frac{3}{\vec{v}^{\,2}} \frac{\int_0^{\mu} \mathrm{d}\omega\, \omega^2\, W(\omega, \vec{v})}{\int_0^{\mu} \mathrm{d}\omega\, W(\omega, \vec{v})}$$

Bigi, Shifman, Uraltsev, Vainshtein, PRD 56 (1997) 4017.

Ingredients for $m_b^{\mathrm{OS}} - m_b^{\mathrm{kin}}$ at $O(\alpha_s^3)$

- $W(\omega, \vec{v})$ up to $O(\alpha_s^3)$
- Discontinuity of forward scattering amplitudes:



$$W(\omega, \vec{v}) = W_{\mathrm{el}}(\vec{v}\,)\,\delta(\omega) + \frac{\vec{v}^2}{\omega} W_{\mathrm{real}}(\omega)\,\theta(\omega) + \mathcal{O}\left(v^4, \frac{\omega}{m_b}\right)$$

- **Threshold expansion:** $y = s - m_b^2$.

Beneke, Smirnov, NPB 522 (1998) 321; Smirnov Springer Tracts Mod. Phys. 250 (2010)

Heavy daughter limit $\leftrightarrow$ one-particle threshold limit.

The Kinetic Mass

$$\frac{m^{\rm kin}}{m^{\rm OS}} = 1 - \frac{\alpha_s^{(n_l)}}{\pi} C_F\left(\frac{4}{3}\frac{\mu}{m^{\rm OS}} + \frac{1}{2}\frac{\mu^2}{(m^{\rm OS})^2}\right) + \left(\frac{\alpha_s^{(n_l)}}{\pi}\right)^2 C_F\left\{\frac{\mu}{m^{\rm OS}}\left[C_A\left(-\frac{215}{27} + \frac{2\pi^2}{9} + \frac{22}{9}l_\mu\right) + n_l T_F\left(\frac{64}{27} - \frac{8}{9}l_\mu\right)\right]\right.$$
$$\left. + \frac{\mu^2}{(m^{\rm OS})^2}\left[C_A\left(-\frac{91}{36} + \frac{\pi^2}{12} + \frac{11}{12}l_\mu\right) + n_l T_F\left(\frac{13}{18} - \frac{1}{3}l_\mu\right)\right]\right\} + \left(\frac{\alpha_s^{(n_l)}}{\pi}\right)^3 C_F\left\{\frac{\mu}{m^{\rm OS}}\left[C_A^2\left(-\frac{130867}{1944}\right.\right.\right.$$
$$+ \frac{511\pi^2}{162} + \frac{19\zeta_3}{2} - \frac{\pi^4}{18} + \left(\frac{2518}{81} - \frac{22\pi^2}{27}\right)l_\mu - \frac{121}{27}l_\mu^2\Big) + C_A n_l T_F\left(\frac{19453}{486} - \frac{104\pi^2}{81} - 2\zeta_3\right.$$
$$+ \left(-\frac{1654}{81} + \frac{8\pi^2}{27}\right)l_\mu + \frac{88}{27}l_\mu^2\Big) + C_F n_l T_F\left(\frac{11}{4} - \frac{4\zeta_3}{3} - \frac{2}{3}l_\mu\right) + n_l^2 T_F^2\left(-\frac{1292}{243} + \frac{8\pi^2}{81} + \frac{256}{81}l_\mu - \frac{16}{27}l_\mu^2\right)\Bigg]$$
$$+ \frac{\mu^2}{(m^{\rm OS})^2}\left[C_A^2\left(-\frac{96295}{5184} + \frac{445\pi^2}{432} + \frac{57\zeta_3}{16} - \frac{\pi^4}{48} + \left(\frac{2155}{216} - \frac{11\pi^2}{36}\right)l_\mu - \frac{121}{72}l_\mu^2\right) + C_A n_l T_F\left(\frac{13699}{1296} - \frac{23\pi^2}{54}\right.\right.$$
$$\left.\left.\left. - \frac{3\zeta_3}{4} + \left(-\frac{695}{108} + \frac{\pi^2}{9}\right)l_\mu + \frac{11}{9}l_\mu^2\right) + C_F n_l T_F\left(\frac{29}{32} - \frac{\zeta_3}{2} - \frac{1}{4}l_\mu\right) + n_l^2 T_F^2\left(-\frac{209}{162} + \frac{\pi^2}{27} + \frac{26}{27}l_\mu - \frac{2}{9}l_\mu^2\right)\right]\right\}, \quad (4)$$

MF, Schönwald, Steinhauser, PRL 125 (2020) 052003

- The mass relation is written in terms of $\alpha_s^{(n_l)}$.
- n_l = number of **massless quarks**, $l_\mu = \log(2\mu/\mu_s)$.

Implications for $\overline{B} \to X_c \ell \bar{\nu}_\ell$

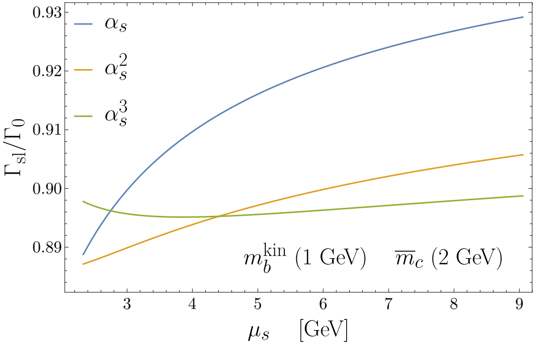
$$\Gamma_{\rm sl} = \frac{G_F^2 m_b^5 |V_{cb}|^2}{192\pi^3} f(\rho) \left[1 + \sum_n Y_n \left(\frac{\alpha_s}{\pi}\right)^n \right]$$

with $\alpha_s \equiv \alpha_s^{(4)}(m_b)$.

n=1 Jezabek, Kühn, Jezabek, Kuhn, NPB 314 (1989) 1

n=2 Melnikov, PLB 666 (2008) 336; Pak, Czarnecki, PRD 78 (2008) 114015.

n=3 Fael, Schönwald, Steinhauser, hep-ph/2011.13654

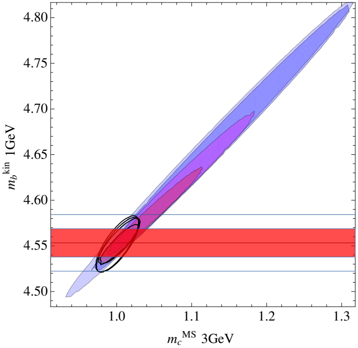


$$m_b^{\rm OS} : m_c^{\rm OS} \quad 1 - 1.78\left(\frac{\alpha_s}{\pi}\right) - 13.1\left(\frac{\alpha_s}{\pi}\right)^2 - 163.3\left(\frac{\alpha_s}{\pi}\right)^3$$

$$m_b^{\rm kin}(1\ \text{GeV}) : \overline{m}_c(2\ \text{GeV}) \quad 1 - 1.24\left(\frac{\alpha_s}{\pi}\right) - 3.65\left(\frac{\alpha_s}{\pi}\right)^2 - 1.0\left(\frac{\alpha_s}{\pi}\right)^3$$

$$m_b^{\rm 1S} : m_c \text{ via HQET} \quad 1 - 1.38\left(\frac{\alpha_s}{\pi}\right) - 6.32\left(\frac{\alpha_s}{\pi}\right)^2 - 33.1\left(\frac{\alpha_s}{\pi}\right)^3$$

Precise $\overline{m}_b - m_b^{\mathrm{kin}}$ conversion up to $O(\alpha_s^3)$



Gambino, Schwanda, PRD 89 (2014) 014022
Horizontal error bands superimposed by MF

- Mass relation implemented in `(C)RunDec` and `REvolver`
  Herren, Steinhauser, Comput.Phys.Commun.224, 333 (2018)
  Hoang, Lepenik, Mateu, Comput.Phys.Commun. 270 108145 (2022)
- Input from FLAG19:
  - $\overline{m}_b(\overline{m}_b) = 4.198(12)$ GeV
  - $\overline{m}_c(3\text{ GeV}) = 0.988(7)$ GeV

$$m_b^{\mathrm{kin}}(1\text{ GeV}) = 4.198 + 0.261 + 0.079 + 0.027 = 4.564\text{ GeV}$$

- Conversion-uncertainty $\delta m_b = 15$ MeV (half $O(\alpha_s^3)$ correction)
- Uncertainty at $O(\alpha_s^2)$ was $\delta m_b = 40$ MeV

$$m_b^{\mathrm{kin}}(1\text{ GeV}) = 4.565\,(15)_{\mathrm{th}}(13)_{\mathrm{lat}}\text{ GeV} = 4.565\,(20)\text{ GeV}$$

Improvement in $|V_{cb}|$

- Fit BR and moments from B factories.
- No new data added since 2014.
- Global fit strategy from 2014
  Gambino, Schwanda, Phys.Rev.D 89 (2014) 014022
  Alberti, Gambino, Healey, Nandi, Phys.Rev.Lett. 114 (2015) 6, 061802
- **New**: $O(\alpha_s^3)$ for Γ_{free} and $\overline{m}_b - m_b^{\text{kin}}$

- NLO correction to ρ_D in Γ_{sl}.
  Mannel, Pivovarov, PRD 100 (2019) 093001
- Precise input from FLAG + $\overline{m}_b - m_b^{\text{kin}}$ at $O(\alpha_s^3)$.

$$\overline{m}_c(3\text{ GeV}) = 0.988\,(7)\text{ GeV}$$
$$m_b^{\text{kin}}(1\text{ GeV}) = 4.565\,(19)\text{ GeV}$$

$$|V_{cb}| = 42.16\,(30)_{\text{th}}(32)_{\text{exp}}(25)_{\Gamma} \times 10^{-3} = 42.16\,(51) \times 10^{-3}$$

Bordone, Capdevila, Gambino, hep-ph/2107.00604

34% error improvement compared to 2014!

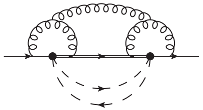
Moments of kinematic distributions

$$M^n[w] = \int_{\text{veto}} \mathrm{d}\Phi \, w^n(p_\ell, q, v_B) \frac{d\Gamma}{dq_0 \, dq^2 \, dE_\ell}$$

- Extend computation strategy of Γ_{sl} to **moments without cuts**.

Observable	$w(p_\ell, q, v_B)$
Semileptonic rate	1
Electron energy moments	$p_\ell \cdot v_B$
Hadronic invariant mass	$(M_B v_B - q)^2$
Leptonic invariant mass	q^2

$$w^n(p_\ell, q, v) \times$$



- Third order corrections relevant for q^2 and M_X^2 moments.

MF, Schönwald, Steinhauser, in preparation.

Outlook & Conclusions

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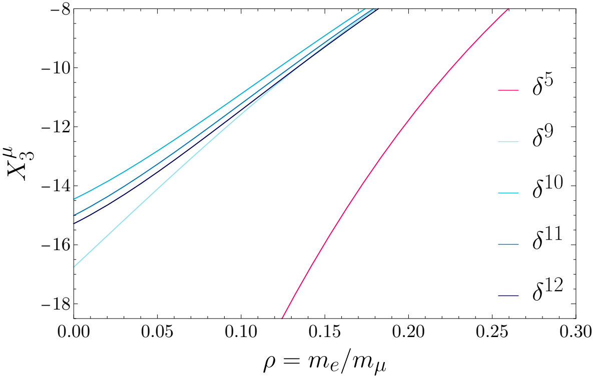
- Inclusive observables in B decays greatly benefit from developments in multi-loop methods.
- New $O(\alpha_s^3)$ corrections
  - $\Gamma(B \to X_c \ell \nu)$.
  - $m^{\text{OS}} - m^{\text{kin}}$ relation.
- 34% error improvement in inclusive $|V_{cb}|$.
- $|V_{cb}|$ puzzle still remains.

What is still needed?

- Short term:
  - NNLO corrections to q^2 moments with cuts and A_{FB} asymmetries.
  - Fully differential NLO corrections to ρ_D (Gambino, Nandi et al.)
  - Kinetic scheme at higher order in $1/m_b$ (and $O(\alpha_s^4)$?).
- Long term:
  - N^{3}LO and NNLO $\times 1/m_b^n$ corrections with cuts for selected observables might be doable.
  - Improve prediction for $b \to u \ell \nu$. Charm mass effects.

Spare

Implications for Muon Decay



$$X_3^\mu = -15.3 \pm 2.3 \quad (15\%)$$

MF, Schönwald, Steinhauser, Phys.Rev.D 104 (2021) 1, 016003

Previous estimate: $X_3^\mu \simeq -20$

Ferroglia, Ossola, Sirlin, Nucl.Phys.B 560 (1999) 23

Final Error Budget

$$\frac{\hbar}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^2} F(\rho)\,(1+\Delta q)$$

$$\Delta q = \left(-4\,234\,530\big|_{\hat\alpha} + 36\,332\big|_{\hat\alpha^2} - 196\big|_{\hat\alpha^3} \pm 29\big|_{\delta\hat\alpha^3} \pm 5\big|_{\delta\mathrm{had}}\right) \times 10^{-9}$$

Behrends, Finkelstein, Sirlin Phys.Rev.101 (1956) 866;
van Ritbergen, Stuart Phys.Rev.Lett. 82 (1999) 488 $(m_e = 0)$
Czarnecki, Pak, Phys.Rev.Lett. 100 (2008) 241807 $(m_e \ll m_\mu)$

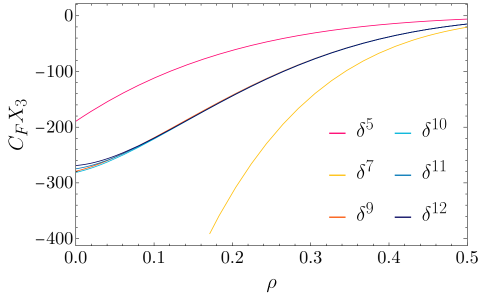
- $\delta(1+\Delta q)$ = 0.029 ppm
- $\frac{1}{2}\frac{\delta\tau_\mu}{\tau_\mu}$ = 0.5 ppm
- $\frac{5}{2}\frac{\delta m_\mu}{m_\mu}$ = 0.05 ppm

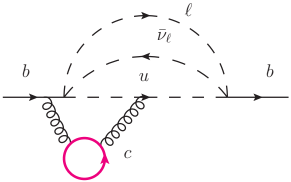
- at $O(\alpha^2)$: $\delta(1+\Delta q)$ = 0.17 ppm
  van Ritbergen, Stuart, Nucl.Phys.B 564 (2000) 343
  Sirlin, Ferroglia, Rev.Mod.Phys. 85 (2013) 1
- Precise $\overline{\mathrm{MS}}$-on shell conversion of α up to 4 loops

  Baikov, Chetyrkin, Kuhn, Sturm, Nucl.Phys.B 867 (2013) 182

		tree	α_s	α_s^2	α_s^3	
1		✓	✓	✓	✓	Jezabek, Kuhn, NPB 314 (1989) 1; Gambino et al., NPB 719 (2005) 77; Melnikov, PLB 666 (2008) 336; Pak, Czarnecki, PRD 78 (2008) 114015. **MF, Schönwald, Steinhauser, PRD 104 (2021) 016003**;
$1/m_b^2$	μ_π	✓	✓			Becher, Boos, Lunghi, JHEP 0712 (2007) 062; Alberti, Ewerth, Gambino, Nandi, NPB 870 (2013) 16
	μ_G	✓	✓			Alberti, Gambino, Nandi, JHEP 1401 (2014) 147; Mannel, Pivovarov, Rosenthal, PRD 92 (2015) 054025.
$1/m_b^3$	ρ_D	✓	✓			Mannel, Pivovarov, PRD100 (2019) 093001; Mannel, Moreno, Pivovarov, PRD 105 (2022) 054033.
	ρ_{LS}	✓	✓			
$1/m_b^{4,5}$		✓				Dassinger, Mannel, Turczyk, JHEP 0703 (2007) 087, JHEP 1011 (2010) 109 MF, Mannel, Vos, JHEP 02 (2019) 177, JHEP 12 (2019) 067.
$\overline{m}_b - m_b^{\rm kin}$			✓	✓	✓	Bigi et al, PRD 56 (1997) 4017; Czarnecki, Melnikov, Uraltsev, PRL 80 (1998) 3189. **MF, Steinhauser, Schönwald, PRL 125 (2020) 5, PRD 103 (2021) 014005.**

A new puzzle in V_{ub}?





- $\Gamma_{\rm sl}(B \to X_u \ell \nu_\ell)$ from $\rho \to 0$ limit: $C_F X_3^u = -269 \pm 26$.

$$\Gamma_{b\to u}(m_b^{\rm kin}) = \Gamma_0 \left[1 - 0.020\big|_{\alpha_s} + 0.014\big|_{\alpha_s^2} + 0.031\big|_{\alpha_s^3} \right]$$

$$\Gamma_{b\to u}^{\rm no\ charm}(m_b^{\rm kin}) = \Gamma_0 \left[1 - 0.020\big|_{\alpha_s} + 0.012\big|_{\alpha_s^2} + 0.016\big|_{\alpha_s^3} \right]$$

$$\Gamma_{b\to u}(m_b^{\rm 1S}) = \Gamma_0 \left[1 - 0.116\epsilon - 0.032\epsilon^2 + 0.002\epsilon^3 \right]$$

- Relevant for $|V_{ub}|$ in GGOU scheme

Gambino, Giordano, Ossola, Uraltsev, JHEP 10 (2007) 058

- Large m_c effects at $O(\alpha_s^3)$?
- Kinetic mass not good for $b \to u$?

First glance to α_s^3 corrections for the moments

- Centralized moments (no cuts):

$$\left\langle M^n[w] \right\rangle = \frac{M^n[w]}{M^0[w]} \quad \to \quad \left\langle (M[w] - \langle M[w] \rangle)^n \right\rangle$$

- QCD corrections up to $O(\alpha_s^3)$ at leading order in the HQE.
- Tree level contribution to $O(1/m_b^2)$ and $O(1/m_b^3)$.
- Results in the kinetic scheme: m_b^{kin} and $\overline{m}_c(3\ \mathrm{GeV})$.
- We quote:
  - Higher QCD corrections flagged by "α_s^n".
  - Power correction up to $1/m_b^3$ flagged by "pw"
  - Uncertainties in α_s^n from finite δ expansion.
  - Uncertainties from HQE parameters.

Bordone, Capevila, Gambino, PLB 822 (2021) 136679

q^2 moments: $q_1 = \langle q^2 \rangle, \quad q_{n\geq 2} = \langle (q^2 - \langle q^2 \rangle)^n \rangle$

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MF, Schönwald, Steinhauser, in preparation

$$\hat{q}_1 = 0.232947\Big[1 - 0.0106345_{\alpha_s} - 0.008736(15)_{\alpha_s^2} - 0.00505(13)_{\alpha_s^3} - 0.0875(97)_{\rm pw}\Big],$$

$$\hat{q}_2 = 0.0235256\Big[1 - 0.035937_{\alpha_s} - 0.0217035(20)_{\alpha_s^2} - 0.01118(17)_{\alpha_s^3} - 0.237(27)_{\rm pw}\Big],$$

$$\hat{q}_3 = 0.0014511\Big[1 - 0.0700381_{\alpha_s} - 0.035693(73)_{\alpha_s^2} - 0.01909(12)_{\alpha_s^3} - 0.726(94)_{\rm pw}\Big],$$

$$\hat{q}_4 = 0.00120161\Big[1 - 0.0585199_{\alpha_s} - 0.042276(11)_{\alpha_s^2} - 0.02411(20)_{\alpha_s^3} - 0.631(77)_{\rm pw}\Big].$$

$$q_1(q^2 > 3\ \mathrm{GeV}^2) = 6.23\,(8)\ \mathrm{GeV}^2 \qquad (1.3\%)$$

$$q_2(q^2 > 3\ \mathrm{GeV}^2) = 4.44\,(15)\ \mathrm{GeV}^4 \qquad (3.1\%)$$

$$q_3(q^2 > 3\ \mathrm{GeV}^2) = 4.13\,(68)\ \mathrm{GeV}^6 \qquad (16\%)$$

$$q_3(q^2 > 3\ \mathrm{GeV}^2) = 46.6\,(5.6)\ \mathrm{GeV}^8 \qquad (12\%)$$

Belle, PRD 104 (2021) 112011

Electron energy: $\ell_1 = \langle E_\ell \rangle, \quad \ell_{n\geq 2} = \langle (E_\ell - \langle E_\ell \rangle)^n \rangle$

MF, Schönwald, Steinhauser, in preparation

$$\hat{\ell}_1 = 0.315615\Big[1 - 0.0101064_{\alpha_s} - 0.005082(17)_{\alpha_s^2} - 0.00227(13)_{\alpha_s^3} - 0.0192(31)_{\rm pw}\Big],$$

$$\hat{\ell}_2 = 0.00900585\Big[1 - 0.01992_{\alpha_s} - 0.006152(41)_{\alpha_s^2} + 0.0002(21)_{\alpha_s^3} + 0.017(11)_{\rm pw}\Big],$$

$$\hat{\ell}_3 = -0.000464269\Big[1 - 0.0639319_{\alpha_s} - 0.035673(10)_{\alpha_s^2} - 0.0142(46)_{\alpha_s^3} - 0.175(22)_{\rm pw}\Big],$$

$$\hat{\ell}_4 = 0.00020743\Big[1 - 0.028854_{\alpha_s} - 0.00717(23)_{\alpha_s^2} - 0.(0.25)_{\alpha_s^3} + 0.(0.021)_{\rm pw}\Big].$$

$$\ell_1(E_\ell > 0.4\ \mathrm{GeV}) = 1393.92(6.73)(3.02)\,\mathrm{MeV} \qquad (0.5\%)$$

$$\ell_2(E_\ell > 0.4\ \mathrm{GeV}) = 168.77(3.68)(1.53) \times 10^{-3}\ \mathrm{GeV}^2 \qquad (2.3\%)$$

$$\ell_3(E_\ell > 0.4\ \mathrm{GeV}) = -21.04(1.93)(0.66) \times 10^{-3}\ \mathrm{GeV}^3 \qquad (9.6\%)$$

$$\ell_4(E_\ell > 0.4\ \mathrm{GeV}) = 64.153(1.813)(0.935) \times 10^{-3}\ \mathrm{GeV}^4 \qquad (3.2\%)$$

Belle, PRD 75 (2007) 032001

Hadronic mass: $h_1 = \langle M_X^2 \rangle, \quad h_{n\geq 2} = \langle (M_X^2 - \langle M_X^2 \rangle)^n \rangle$

MF, Schönwald, Steinhauser, in preparation

$$
\begin{aligned}
\hat{h}_1 &= 0.00899843\Big[+23.4975 \quad +1+0.4223(15)_{\alpha_s^2} \quad +0.147(11)_{\alpha_s^3} \quad +0.04(20)_{\rm pw}\Big],\\
\hat{h}_2 &= 0.000745468\Big[+0.87352 \quad +1+0.4505(74)_{\alpha_s^2} \quad +0.34(43)_{\alpha_s^3} \quad +3.33(59)_{\rm pw}\Big],\\
\hat{h}_3 &= 0.0000915954\Big[-0.0729568 \quad +1+0.165(62)_{\alpha_s^2} \quad +2.29(55)_{\alpha_s^3} \quad +7.3(1.1)_{\rm pw}\Big],\\
\hat{h}_4 &= 0.000091207\Big[+0.0100938 \quad +1+0.51(17)_{\alpha_s^2} \quad +1(145)_{\alpha_s^3} \quad +0.380(52)_{\rm pw}\Big].
\end{aligned}
$$

$$
\begin{aligned}
h_1 &= 4.541\,(101)\ \text{GeV}^2 && (2\%)\\
h_2 &= 1.56\,(0.18)\,(0.16)\ \text{GeV}^4 && (15\%)\\
h_3 &= 4.05\,(0.74)\,(0.32)\ \text{GeV}^6 && (20\%)\\
h_4 &= 21.1\,(4.5)\,(2.1)\ \text{GeV}^8 && (23\%)
\end{aligned}
$$

DELPHI, EPJ C 45 (2006) 35