Beyond a single elliptic curve

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April 27, 2022

Motivation

- The method of differential equations is a popular method to compute Feynman integrals.

  (Kotikov '90, Remiddi '97, Gehrmann and Remiddi '99)

- We may **systematically** derive a differential equation for any Feynman integral.

  (`Reduze`, `Fire`, `Kira`)

- The essential step is to transform the differential equation into a particular nice form (ε-form).

  (Henn '13)

  In fact, the problem of computing Feynman integrals is reduced to finding an appropriate transformation for the differential equation.

Notation

I	$= (I_1, ..., I_{N_F})$	Master integrals
$\mathbf{N_F}$	$= N_{\mathrm{Fibre}}$	Number of master integrals
x	$= (x_1, ..., x_{N_B})$	Kinematic variables
$\mathbf{N_B}$	$= N_{\mathrm{Base}}$	Number of kinematic variables
ω	$= (\omega_1, ..., \omega_{N_L})$	Differential one-forms/letters
$\mathbf{N_L}$	$= N_{\mathrm{Letters}}$	Number of letters

The method of differential equations

We want to calculate

$$I = (I_1, ..., I_{N_F})$$

1. *Find a differential equation with respect to the kinematic variables for the Feynman integrals.*

$$[d + A(\varepsilon, x)] I = 0.$$

2. *Transform the differential equation into a simple form.*

$$[d + \varepsilon A(x)] I = 0.$$

3. *Solve the latter differential equation with appropriate boundary conditions.*

Simple differential equations

The system of differential equations is particular simple, if A is of the form

$$A = \varepsilon \sum_{j=1}^{N_L} C_j \, \omega_j,$$

where

- C_j is a $N_F \times N_F$-matrix, whose entries are numbers,
- the **only dependence on ε** is **given by** the **explicit prefactor**,
- the differential one-forms ω_j are closed and have **only simple poles**.

Open question

Question:

Can any family of Feynman integrals be transformed to an ε-form?

Supporting evidence:

- Many examples of Feynman integrals evaluating to multiple polylogarithms
- Feynman integrals depending on a single elliptic curve (sunrise, kite, ...)
- This talk: A Feynman integral depending on two elliptic curves

Motivation

Question:

If yes, what are the differential one-forms ω_j appearing in the differential equation?

Known so far:

- No elliptic curve: dlog-forms
- One elliptic curve and one kinematic variable:
  - modular forms
- One elliptic curve and several kinematic variables:
  - differential forms related to the Kronecker function (aka elliptic polylogarithms)
  - modular forms

Motivation

- If the differential equation is not in ε-form, we might have

$$\frac{dx}{x-1}, \quad \frac{dx}{(x-1)^2}, \quad \frac{dx}{(x-1)^3}.$$

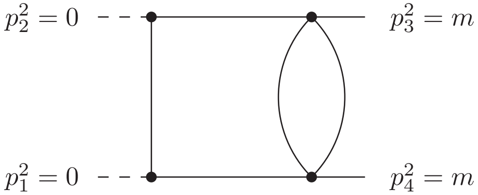
- If the differential equation is in ε-form, this reduces to

$$\frac{dx}{x-1}.$$

Section 1

The Feynman integral

The Feynman integral



Solid lines correspond to propagators with a mass m.

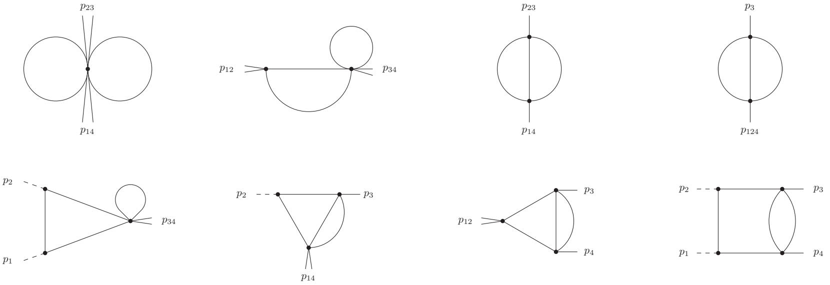
$$s = (p_1 + p_2)^2 \qquad t = (p_2 + p_3)^2$$

The Feynman integral

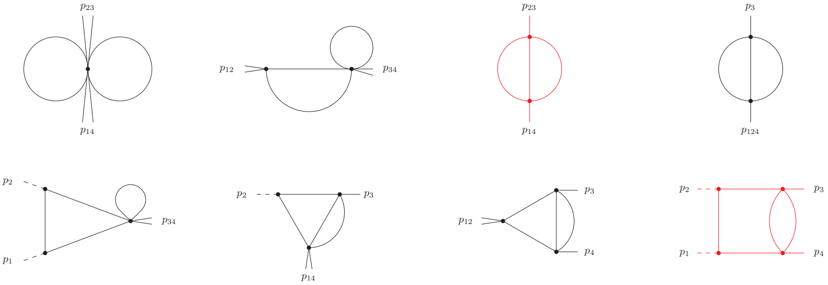
- 12 master integrals
- 2 kinematic variables

$$\frac{s}{m^2}, \qquad \frac{t}{m^2}$$

Master topologies



Master topologies



Section 2

The elliptic curves

Maximal cuts

Sunrise

MaxCut $I_{100100100}(2) \approx$

$$\frac{m^2}{\pi^2} \int\limits_{\mathcal{C}} \frac{du}{\sqrt{(u-t)(u-t+4m^2)(u^2+2m^2u-4m^2t+m^4)}}$$

Sector 79

MaxCut $I_{111200100}(4) \approx$

$$\frac{m^4}{4\pi^3 s} \int\limits_{\mathcal{C}} \frac{du}{\sqrt{(u-t)(u-t+4m^2)\left(u^2+2m^2u-4m^2t+m^4-\frac{4m^2(m^2-t)^2}{s}\right)}}$$

The elliptic curves

- Curve (a):

$$v^2 = (u-t)\left(u-t+4m^2\right)\left(u^2+2m^2u-4m^2t+m^4\right)$$

- Curve (b):

$$v^2 = (u-t)\left(u-t+4m^2\right)\left(u^2+2m^2u-4m^2t+m^4-\frac{4m^2\left(m^2-t\right)^2}{s}\right)$$

- For generic (s,t), the two curves are **neither isomorphic nor isogenic**.
- For $s=\infty$ the two curves are identical.

Section 3

Variables

Kinematic variables

The integral depends on two kinematic variables. We may either use

- $\left(\frac{s}{m^2}, \frac{t}{m^2}\right)$
- (x, y) with $\frac{s}{m^2} = -\frac{(1-x)^2}{x}$, $\frac{t}{m^2} = y$
- $\left(\tau^{(a)}, \tau^{(b)}\right)$
- $\left(\bar{q}^{(a)}, \bar{q}^{(b)}\right)$ with $\bar{q}^{(a)} = e^{2\pi i \tau^{(a)}}$, $\bar{q}^{(b)} = e^{2\pi i \tau^{(b)}}$
- $\left(z^{(b)}, \tau^{(b)}\right)$
- $\left(\bar{w}^{(b)}, \bar{q}^{(b)}\right)$ with $\bar{w}^{(b)} = e^{2\pi i z^{(b)}}$

Definition of the variables

- Choose two independent periods $\psi_1^{(a)}, \psi_2^{(a)}$ for curve (a). Then the modular parameter $\tau^{(a)}$ is given by

$$\tau^{(a)} = \frac{\psi_2^{(a)}}{\psi_1^{(a)}}$$

- Similar fur curve (b):

$$\tau^{(b)} = \frac{\psi_2^{(b)}}{\psi_1^{(b)}}$$

- $z^{(b)}$ corresponds to a marked point on curve (b).

Section 4

The differential equation

Starting point

$$\left[d + A^{(0)}(x) + \varepsilon A^{(1)}(x) \right] I^{\mathrm{pre}} = 0$$

with

$$A^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A^{(0)}_{11,10} & 0 & 0 \\ 0 & 0 & A^{(0)}_{12,3} & 0 & A^{(0)}_{12,5} & A^{(0)}_{12,6} & A^{(0)}_{12,7} & A^{(0)}_{12,8} & 0 & A^{(0)}_{12,10} & A^{(0)}_{12,11} & 0 \end{pmatrix}$$

L. Adams, E. Chaubey, S.W, 2018

The differential equation

- Redefine the master integrals to put the differential equation into an ε-form.
- $A^{(0)}$ is strictly lower triangular, therefore this **can be done by integration**. Example:

$$I_{11} = I_{11}^{\mathrm{pre}} + F_{11,10} I_{10}^{\mathrm{pre}}, \qquad F_{11,10} = \int A_{11,10}^{(0)}$$

- Need only integrals of $A_{11,10}^{(0)}$ and $A_{12,3}^{(0)}$, the rest can be related algebraically to these.

The differential equation

After the redefinition, the **differential equation is in ε-form**:

$$[d + \varepsilon A(x)]\, I \quad = \quad 0$$

Master integrals in the top sector:

$$\begin{aligned}
I_{10} &= \varepsilon^3 \frac{(1-x)^2}{x} \frac{\pi}{\psi_1^{(b)}} I_{111200100}, \\
I_{11} &= \varepsilon^3 (1-2\varepsilon) \frac{(1-x)^2}{x} I_{111100100} + F_{11,10} I_{10}, \\
I_{12} &= \frac{\left(\psi_1^{(b)}\right)^2}{2\pi i \varepsilon W_y^{(b)}} \frac{\partial}{\partial y} I_{10} + F_{12,11} \left(I_{11} - \frac{2}{3} I_8 - \frac{4}{3} I_7 - \frac{2}{3} I_6 + \frac{1}{9} I_5 \right) + F_{12,10} I_{10} + F_{12,3} I_3.
\end{aligned}$$

Section 5

The entries

The entries

$$A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A_{2,1} & A_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A_{3,3} & A_{3,4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A_{4,1} & 0 & A_{4,3} & A_{4,4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A_{6,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A_{7,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A_{8,2} & 0 & 0 & A_{8,5} & 0 & 0 & A_{8,8} & A_{8,9} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_{9,5} & 0 & 0 & A_{9,8} & A_{9,9} & 0 & 0 & 0 \\
0 & 0 & A_{10,3} & 0 & A_{10,5} & A_{10,6} & A_{10,7} & A_{10,8} & 0 & A_{10,10} & A_{10,11} & A_{10,12} \\
0 & A_{11,2} & A_{11,3} & 0 & A_{11,5} & A_{11,6} & A_{11,7} & A_{11,8} & A_{11,9} & A_{11,10} & A_{11,11} & A_{11,12} \\
0 & A_{12,2} & A_{12,3} & A_{12,4} & A_{12,5} & A_{12,6} & A_{12,7} & A_{12,8} & A_{12,9} & A_{12,10} & A_{12,11} & A_{12,12}
\end{pmatrix}$$

The white entries: dlog-forms

Only dependent on s (or x):

$$\frac{dx}{x}, \quad \frac{dx}{x-1}, \quad \frac{dx}{x+1}$$

The yellow entries: Modular forms

Only dependent on t (or y or $\tau^{(a)}$) and hence **only dependent on curve** (a):

Modular forms of $\Gamma_1(6)$:

$$\omega = f_k(\tau^{(a)}) \cdot 2\pi i d\tau^{(a)}$$

The red entries: One-forms on $\mathcal{M}_{1,2}$

Only dependent on curve (b), but dependent on two variables (s,t) (or $(z^{(b)}, \tau^{(b)})$):

1. *From modular forms:*

$$\omega_k^{\mathrm{modular}} = 2\pi i \, f_k(\tau) d\tau$$

2. *From the Kronecker function:*

$$\omega_k^{\mathrm{Kronecker}} = (2\pi i)^{2-k} \left[g^{(k-1)}(z,\tau)\, dz + (k-1)\, g^{(k)}(z,\tau) \frac{d\tau}{2\pi i} \right],$$

$$z = \alpha z^{(b)} + \beta, \qquad \tau = \tau^{(b)}.$$

The Kronecker function

Define the **first Jacobi theta function** $\theta_1(z, \bar{q})$ by

$$\theta_1(z, \bar{q}) = -i \sum_{n=-\infty}^{\infty} (-1)^n \bar{q}^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} e^{i\pi(2n+1)z}.$$

The **Kronecker function** $F(z, \alpha, \tau)$:

$$F(z, \alpha, \tau) = \theta_1'(0, \bar{q}) \frac{\theta_1(z+\alpha, \bar{q})}{\theta_1(z, \bar{q})\,\theta_1(\alpha, \bar{q})} = \frac{1}{\alpha} \sum_{k=0}^{\infty} \mathbf{g}^{(\mathbf{k})}(\mathbf{z}, \tau)\alpha^k$$

We are interested in the coefficients $g^{(k)}(z, \tau)$ of the Kronecker function.

What is $z^{(b)}$?

- We have

$$\omega_1^{\mathrm{Kronecker}} = 2\pi i dz^{(b)}$$

- May **get $z^{(b)}$ from integrating an entry of modular weight 1** with respect to curve (b):

$$z^{(b)} = \alpha + \beta \int A_{10,11}$$

up to two constants α and β.

The orange entries: The mixed entries

- The mixed entries **depend on curve** (a) **and curve** (b).
- For curve (a) we have coordinate $\tau^{(a)}$,
  for curve (b) we have two coordinates $(z^{(b)}, \tau^{(b)})$.
- The Feynman integral **depends only on two kinematic variables**.
- May express $\tau^{(a)}$ as a function of $(z^{(b)}, \tau^{(b)})$
  or $z^{(b)}$ as a function of $(\tau^{(a)}, \tau^{(b)})$.

The orange entries: The mixed entries

For practical purposes: Expansions:

$$
\begin{aligned}
A_{10,3} &= d\Omega_{10,3}, \\
\Omega_{10,3} &= \frac{1}{4}\ln\bar{q}^{(b)} - \frac{1}{2}\ln\left(\bar{q}^{(a)} - \bar{q}^{(b)}\right) \\
&+ \left[5 + 27\bar{q}^{(b)} - 82\left(\bar{q}^{(b)}\right)^2 - 2310\left(\bar{q}^{(b)}\right)^3 + \dots\right]\bar{q}^{(a)} \\
&- \frac{1}{2}\left[65 + 212\bar{q}^{(b)} - 11043\left(\bar{q}^{(b)}\right)^2 + \dots\right]\left(\bar{q}^{(a)}\right)^2 \\
&+ \frac{1}{3}\left[689 - 4968\bar{q}^{(b)} + \dots\right]\left(\bar{q}^{(a)}\right)^3 + \dots
\end{aligned}
$$

Section 6

Conclusions

Conclusions

- Study of a two-loop four-point function.
- This Feynman integral depends on two elliptic curves.
- The differential equation can be transformed into an ε-form.
- Most of the differential one-forms in the differential equation have appeared before.