Recent progress on two-loop massless pentabox integrals with one off-shell leg

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in collaboration with
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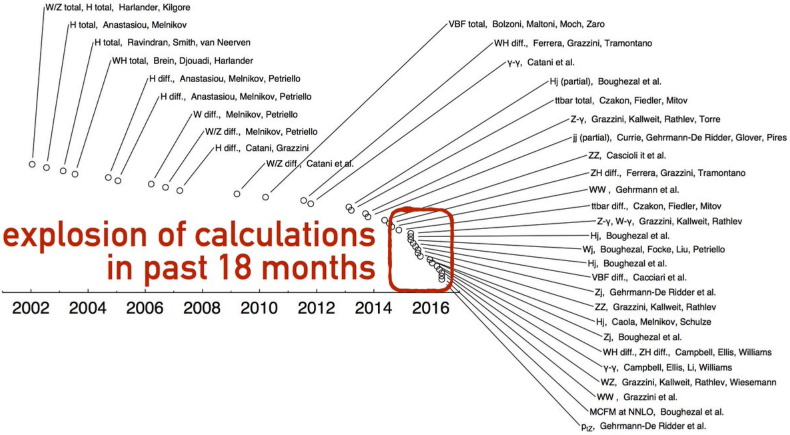
Loops and Legs in Quantum Field Theory, Ettal, April 26, 2022

Towards higher precision: NNLO and beyond

NNLO Revolution

Gavin Salam → click to link



[talk by Stefan Kallweit]

NNLO QCD: $pp \to \gamma\gamma\gamma + X$

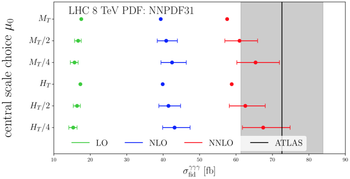


Figure 1. Predictions for the fiducial cross-section in LO (green), NLO (blue) and NNLO (red) QCD versus ATLAS data (black). Shown are predictions for six scale choices. The error bars on the theory predictions reflect scale variation only. For two of the scales only the central predictions are shown.

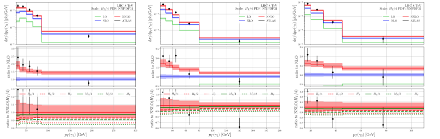


Figure 2. p_T distribution of the hardest photon γ_1 (left), γ_2 (center) and the softest one γ_3 (right). Top plot shows the absolute distribution at NNLO (red), NLO (blue) and LO (green) versus ATLAS data (black). Middle plot shows same distributions but normalized to the NLO. Bottom plot shows central NNLO predictions for 6 different scale choices (only the central scale is shown) with respect to the default choice $\mu_0 = H_T/4$. The bands represent the 7-point scale variations about the corresponding central scales.

H. A. Chawdhry, M. L. Czakon, A. Mitov and R. Poncelet, JHEP **2002** (2020) 057

NNLO QCD: $pp \to \gamma\gamma\gamma + X$

fiducial setup for $pp \to \gamma\gamma\gamma + X$; used in the ATLAS 8 TeV analysis of Ref. [37]
$p_{T,\gamma_1} \geq 27\,\text{GeV}$, $p_{T,\gamma_2} \geq 22\,\text{GeV}$, $p_{T,\gamma_3} \geq 15\,\text{GeV}$, $0 \leq \lvert\eta_\gamma\rvert \leq 1.37$ or $1.56 \leq \lvert\eta_\gamma\rvert \leq 2.37$,
$\Delta R_{\gamma\gamma} \geq 0.45$, $m_{\gamma\gamma\gamma} \geq 50\,\text{GeV}$, Frixione isolation with $n = 1$, $\delta_0 = 0.4$, and $E_T^{\text{ref}} = 10\,\text{GeV}$.

Table 1: Definition of phase space cuts.

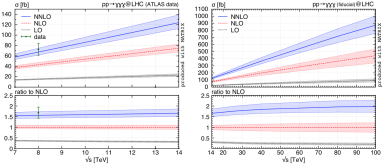


Figure 4: Fiducial cross sections for $pp \to \gamma\gamma\gamma + X$ as a function of the centre-of-mass energy at LO (black dotted), at NLO (red dashed), and at NNLO (blue, solid) The green data point at 8 TeV corresponds to the cross section measured by ATLAS in Ref. [37].

S. Kallweit, V. Sotnikov and M. Wiesemann, Phys. Lett. B **812** (2021) 136013

NNLO QCD: $pp \to 3\text{jets} + X$

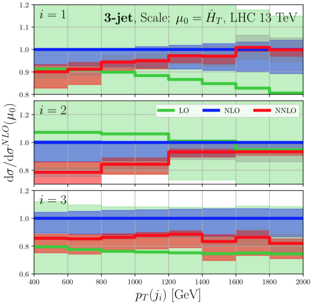


FIG. 1: The three panels show the ith leading jet transverse momentum $p_T(j_i)$ for $i = 1, 2, 3$ for the production of (at least) three jets. LO (green), NLO (blue) and NNLO (red) are shown for the central scale (solid line). 7-point scale variation is shown as a coloured band. The grey band corresponds to the uncertainty from Monte Carlo integration.

M. Czakon, A. Mitov and R. Poncelet, Phys. Rev. Lett. **127** (2021) no.15, 152001 [arXiv:2106.05331 [hep-ph]].

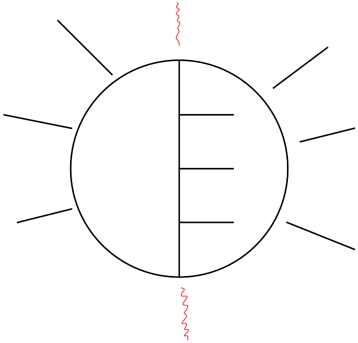
X. Chen, T. Gehrmann, N. Glover, A. Huss and M. Marcoli, [arXiv:2203.13531 [hep-ph]].

NNLO ~~R~~EVOLUTION

- H. A. Chawdhry, M. Czakon, A. Mitov and R. Poncelet, "Two-loop leading-colour QCD helicity amplitudes for two-photon plus jet production at the LHC," arXiv:2103.04319 [hep-ph].
- H. A. Chawdhry, M. Czakon, A. Mitov and R. Poncelet, "Two-loop leading-color helicity amplitudes for three-photon production at the LHC," arXiv:2012.13553 [hep-ph].
- S. Abreu, F. Febres Cordero, H. Ita, B. Page and V. Sotnikov, "Leading-Color Two-Loop QCD Corrections for Three-Jet Production at Hadron Colliders," arXiv:2102.13609 [hep-ph].
- S. Abreu, B. Page, E. Pascual and V. Sotnikov, "Leading-Color Two-Loop QCD Corrections for Three-Photon Production at Hadron Colliders," JHEP **2101** (2021) 078
- S. Abreu *et al.*, "Caravel: A C++ Framework for the Computation of Multi-Loop Amplitudes with Numerical Unitarity," arXiv:2009.11957 [hep-ph].
- B. Agarwal, F. Buccioni, A. von Manteuffel and L. Tancredi, "Two-loop helicity amplitudes for diphoton plus jet production in full color," arXiv:2105.04585 [hep-ph].
- B. Agarwal, F. Buccioni, A. von Manteuffel and L. Tancredi, "Two-loop leading colour QCD corrections to $q\bar{q} \to \gamma\gamma g$ and $qg \to \gamma\gamma q$," JHEP **2104** (2021) 201
- S. Badger, H. B. Hartanto and S. Zoia, "Two-loop QCD corrections to $Wb\bar{b}$ production at hadron colliders," arXiv:2102.02516 [hep-ph].
- H. B. Hartanto, S. Badger, C. Brynnum-Hansen and T. Peraro, "A numerical evaluation of planar two-loop helicity amplitudes for a W-boson plus four partons," JHEP **1909** (2019) 119

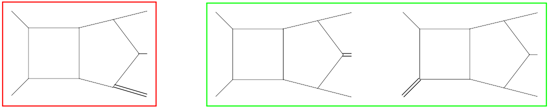
$+$ rational terms, IBP reduction, computation of Master Integrals, etc.

Two-loop graph

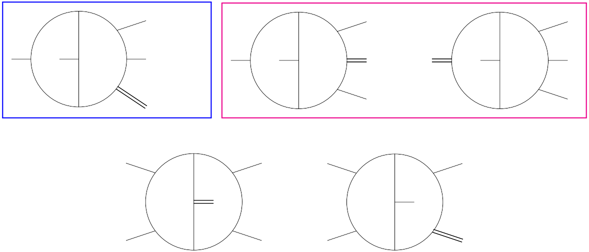


5-POINT 2-LOOP - ONE LEG OFF-SHELL: ALL FAMILIES

C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **04** (2016), 078 [arXiv:1511.09404 [hep-ph]].
C. G. Papadopoulos and C. Wever, JHEP **2002** (2020) 112
S. Abreu, H. Ita, F. Moriello, B. Page, W. Tschernow and M. Zeng, JHEP **2011** (2020) 117
D. D. Canko, C. G. Papadopoulos and N. Syrrakos, JHEP **2101** (2021) 199
S. Abreu, H. Ita, B. Page and W. Tschernow, JHEP **03** (2022), 182 [arXiv:2107.14180 [hep-ph]].
A. Kardos, C. G. Papadopoulos, A. V. Smirnov, N. Syrrakos and C. Wever, [arXiv:2201.07509 [hep-ph]].

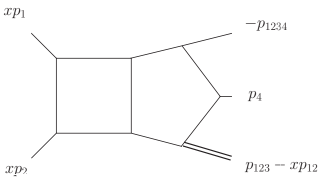


The three planar pentaboxes of the families P_1 (left), P_2 (middle) and P_3 (right) with one external massive leg.



The five non-planar families with one external massive leg.

Pentabox - one leg off-shell: P1



$$q_1 \to p_{123} - xp_{12}, \; q_2 \to p_4, \; q_3 \to -p_{1234}, \; q_4 \to xp_1$$

SDE parametrisation: n off-shell legs $\to$ $n-1$ off-shell legs $+$ the x variable.

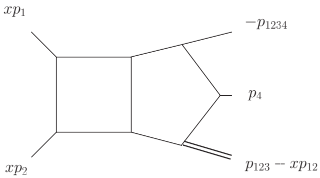
C. G. Papadopoulos, "Simplified differential equations approach for Master Integrals," JHEP **1407** (2014) 088

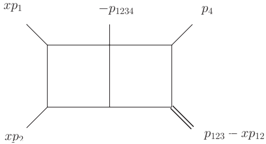
- $p_i, \; i = 1 \ldots 5$, satisfy $\sum_1^5 p_i = 0$, with $p_i^2 = 0, \; i = 1 \ldots 5, \; p_{i \ldots j} := p_i + \ldots + p_j$. The set of independent invariants: $\{S_{12}, S_{23}, S_{34}, S_{45}, S_{51}, x\}$, with $S_{ij} := (p_i + p_j)^2$.

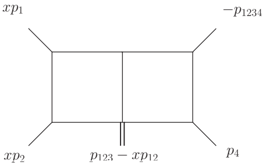
$$q_1^2 = (1-x)(S_{45} - S_{12}x), \; s_{12} = (S_{34} - S_{12}(1-x))\,x, \; s_{23} = S_{45}, \; s_{34} = S_{51}x,$$

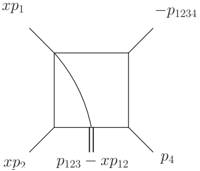
$$s_{45} = S_{12}x^2, \; s_{15} = S_{45} + (S_{23} - S_{45})x$$

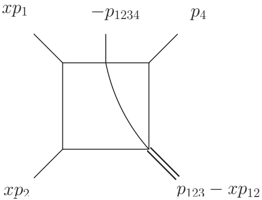
Pentabox - one leg off-shell: P1

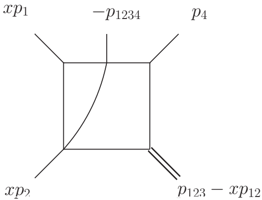


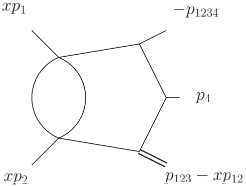












4-POINT UP TO TWO LEGS OFF-SHELL

J. M. Henn, K. Melnikov and V. A. Smirnov, JHEP **1405** (2014) 090
T. Gehrmann, A. von Manteuffel, L. Tancredi and E. Weihs, JHEP **06** (2014), 032
F. Caola, J. M. Henn, K. Melnikov and V. A. Smirnov, JHEP **1409** (2014) 043
C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **1501** (2015) 072
T. Gehrmann, A. von Manteuffel and L. Tancredi, JHEP **09** (2015), 128

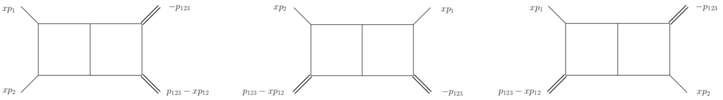


Figure 3. The parametrization of external momenta for the three planar double boxes of the families P_{12} (left), P_{13} (middle) and P_{23} (right) contributing to pair production at the LHC. All external momenta are incoming.

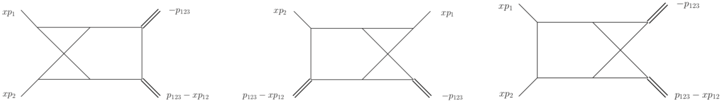


Figure 4. The parametrization of external momenta for the three non-planar double boxes of the families N_{12} (left), N_{13} (middle) and N_{34} (right) contributing to pair production at the LHC. All external momenta are incoming.

As well as planar and nonplanar double box with one off-shell leg expressed in UT basis.

PENTABOX - ONE LEG OFF-SHELL: P1-3

$$G^{P_1}_{a_1\cdots a_{11}} := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+q_1)^{2a_2}(k_1+q_{12})^{2a_3}(k_1+q_{123})^{2a_4}}$$
$$\times\frac{1}{k_2^{2a_5}(k_2+q_{123})^{2a_6}(k_2+q_{1234})^{2a_7}(k_1-k_2)^{2a_8}(k_1+q_{1234})^{2a_9}(k_2+q_1)^{2a_{10}}(k_2+q_{12})^{2a_{11}}},$$

$$G^{P_2}_{a_1\cdots a_{11}} := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1-q_{1234})^{2a_2}(k_1-q_{234})^{2a_3}(k_1-q_{34})^{2a_4}}$$
$$\times\frac{1}{k_2^{2a_5}(k_2-q_{34})^{2a_6}(k_2-q_4)^{2a_7}(k_1-k_2)^{2a_8}(k_2-q_{1234})^{2a_9}(k_2-q_{234})^{2a_{10}}(k_1-q_4)^{2a_{11}}},$$

$$G^{P_3}_{a_1\cdots a_{11}} := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+q_2)^{2a_2}(k_1+q_{23})^{2a_3}(k_1+q_{234})^{2a_4}}$$
$$\times\frac{1}{k_2^{2a_5}(k_2+q_{234})^{2a_6}(k_2-q_1)^{2a_7}(k_1-k_2)^{2a_8}(k_1-q_1)^{2a_9}(k_2+q_2)^{2a_{10}}(k_2+q_{23})^{2a_{11}}},$$

where $q_{i\ldots j} := q_i + \ldots + q_j$.

Pentabox - one leg off-shell: P1

J. M. Henn, Phys. Rev. Lett. **110** (2013) 251601
S. Abreu, H. Ita, F. Moriello, B. Page, W. Tschernow and M. Zeng, JHEP **2011** (2020) 117
D. D. Canko, C. G. Papadopoulos and N. Syrrakos, JHEP **2101** (2021) 199

$$d\vec{g} = \epsilon \sum_a d\log\left(W_a\right) \tilde{M}_a \vec{g}$$

- Also from direct differentiation of MI wrt to x. Just g in terms of FI.

$$\boxed{\frac{d\vec{g}}{dx} = \epsilon \sum_b \frac{1}{x - \ell_b} M_b \vec{g}}$$

- ℓ_b, are independent of x, some depending only on the reduced invariants, $\{S_{12}, S_{23}, S_{34}, S_{45}, S_{51}\}$. M_b are independent of the invariants.
- number of letters smaller than in AIMPTZ representation
- Main contribution for us from AIMPTZ: the canonical basis (+ numerics)

J. M. Henn, Phys. Rev. Lett. **110** (2013) 251601
S. Abreu, H. Ita, F. Moriello, B. Page, W. Tschernow and M. Zeng, JHEP **2011** (2020) 117
D. D. Canko, C. G. Papadopoulos and N. Syrrakos, JHEP **2101** (2021) 199

$$d\vec{g} = \epsilon \sum_a d\log\left(W_a\right) \tilde{M}_a \vec{g}$$

$$\frac{d\log\left(W_a\right)}{dx}$$

- Also from direct differentiation of MI wrt to x. Just g in terms of FI.

$$\boxed{\frac{d\vec{g}}{dx} = \epsilon \sum_b \frac{1}{x-\ell_b} M_b \vec{g}}$$

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PENTABOX - ONE LEG OFF-SHELL: P1

J. M. Henn, Phys. Rev. Lett. **110** (2013) 251601
S. Abreu, H. Ita, F. Moriello, B. Page, W. Tschernow and M. Zeng, JHEP **2011** (2020) 117
D. D. Canko, C. G. Papadopoulos and N. Syrrakos, JHEP **2101** (2021) 199

$$d\vec{g} = \epsilon \sum_a d\log\left(W_a\right) \tilde{M}_a \vec{g}$$

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- number of letters smaller than in AIMPTZ representation
- Main contribution for us from AIMPTZ: the canonical basis (+ numerics)

Pentabox - one leg off-shell: P1-3

$$\frac{d\mathbf{g}}{dx} = \sum_a \frac{1}{x-\ell_a}\mathbf{M}_a\mathbf{g}$$

$$\begin{aligned}
\mathbf{g} = {} & \epsilon^0 \mathbf{b}_0^{(0)} + \epsilon\left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)}\right) \\
& + \epsilon^2\left(\sum \mathcal{G}_{ab}\mathbf{M}_a\mathbf{M}_b\mathbf{b}_0^{(0)} + \sum \mathcal{G}_a\mathbf{M}_a\mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)}\right) \\
& + \epsilon^3\left(\sum \mathcal{G}_{abc}\mathbf{M}_a\mathbf{M}_b\mathbf{M}_c\mathbf{b}_0^{(0)} + \sum \mathcal{G}_{ab}\mathbf{M}_a\mathbf{M}_b\mathbf{b}_0^{(1)} + \sum \mathcal{G}_a\mathbf{M}_a\mathbf{b}_0^{(2)} + \mathbf{b}_0^{(3)}\right) \\
& + \epsilon^4\Bigg(\sum \mathcal{G}_{abcd}\mathbf{M}_a\mathbf{M}_b\mathbf{M}_c\mathbf{M}_d\mathbf{b}_0^{(0)} + \sum \mathcal{G}_{abc}\mathbf{M}_a\mathbf{M}_b\mathbf{M}_c\mathbf{b}_0^{(1)} \\
& + \sum \mathcal{G}_{ab}\mathbf{M}_a\mathbf{M}_b\mathbf{b}_0^{(2)} + \sum \mathcal{G}_a\mathbf{M}_a\mathbf{b}_0^{(3)} + \mathbf{b}_0^{(4)}\Bigg) + \ldots
\end{aligned}$$

$$\mathcal{G}_{ab\ldots} := \mathcal{G}(\ell_a, \ell_b, \ldots; x)$$

D. D. Canko, C. G. Papadopoulos and N. Syrrakos, arXiv:2009.13917 [hep-ph]. `Results.txt`

PENTABOX - ONE LEG OFF-SHELL: BOUNDARY CONDITIONS

- starting from the full equation

$$\frac{d\vec{g}}{dx} = \epsilon \frac{1}{x} M_0 \vec{g} + \mathcal{O}(x^0)$$

- using all letters W_a, with the solution ($\mathbf{b} := \sum_{i=0}^{4} \epsilon^i \mathbf{b}_0^{(i)}$)

$$\mathbf{g}_0 = \mathbf{S} e^{\epsilon \log(x) \mathbf{D}} \mathbf{S}^{-1} \mathbf{b}$$

- $\mathbf{S}$ and $\mathbf{D}$ are obtained through Jordan decomposition of the $\mathbf{M}_0$
- Resummed: $\mathbf{R}_0 = \mathbf{S} e^{\epsilon \log(x) \mathbf{D}} \mathbf{S}^{-1}$
- What we know about:

$$\mathbf{R}_0 = \sum_i x^{n_i \varepsilon} \mathbf{R}_{0i} + \sum_j \varepsilon x^{n_j \varepsilon} \log(x) \, \mathbf{R}_{0j0}$$

Pentabox - one leg off-shell: Boundary conditions

- IBP reduction in terms of Master Integrals

$$\mathbf{g} = \mathbf{TG}.$$

D. D. Canko, C. G. Papadopoulos and N. Syrrakos, arXiv:2009.13917 [hep-ph]. `Masters.m`

- Expansion by regions. [no logarithmic terms]

$$G_i \underset{x \to 0}{=} \sum_j x^{b_j + a_j \varepsilon} G_i^{(j)}$$

- Linear equations:

$$\mathbf{g}_0 := \mathbf{R}_0 \mathbf{b} = \lim_{x \to 0} \mathbf{TG}\Big|_{\mathcal{O}\left(x^{0+a_j\varepsilon}\right)}$$

- Matrix $\mathbf{T}$ is horrible-looking depending on x, ε and S_{ij}. But

$$\mathbf{R}_0\mathbf{b} \to \varepsilon, x, \mathit{Rationals} \otimes \mathit{polyLogs} \qquad G_i^{(j)} \to \mathit{Simple}\left[S_{ij}\right] \otimes \mathit{polyLogs}$$

so we can afford IBP reduction with only x, ε symbolic: i.e. `FIRE6` or `Kira2`.

Pentabox - one leg off-shell: Boundary conditions

- No regions in the top-sector are needed.
- To obtain expressions for regions, $G_i^{(j)}$, in Feynman parameter space, we use `FIESTA asyexpand`, for $x \to 0$ limit (SDE).
- In most cases integration is straightforward and the resulting ${}_2F_1$ hypergeometric functions are expanded with `HypExp`.
- In few cases we use Mellin-Barnes techniques using the `MB`, `MBSums` and `XSummer` along with the in-house (A. Kardos) package `Gsuite`.
- Boundary terms only depends on 12 Goncharov

$$G\left[0,1,-\frac{\text{S12}-\text{S34}}{\text{S51}}\right], G\left[1,-\frac{\text{S12}-\text{S34}}{\text{S51}}\right], G\left[0,0,1,-\frac{\text{S12}-\text{S34}}{\text{S51}}\right], G\left[0,1,1,-\frac{\text{S12}-\text{S34}}{\text{S51}}\right],$$
$$G\left[1,0,1,-\frac{\text{S12}-\text{S34}}{\text{S51}}\right], G\left[0,0,0,1,-\frac{\text{S12}-\text{S34}}{\text{S51}}\right], G\left[0,0,1,1,-\frac{\text{S12}-\text{S34}}{\text{S51}}\right],$$
$$G\left[0,1,0,1,-\frac{\text{S12}-\text{S34}}{\text{S51}}\right], G\left[0,1,1,1,-\frac{\text{S12}-\text{S34}}{\text{S51}}\right], G\left[1,0,0,1,-\frac{\text{S12}-\text{S34}}{\text{S51}}\right], G\left[1,0,1,1,-\frac{\text{S12}-\text{S34}}{\text{S51}}\right],$$
$$G\left[1,1,0,1,-\frac{\text{S12}-\text{S34}}{\text{S51}}\right]$$

- and 4 Logarithms $\{\text{Log}[-\text{S12}], \text{Log}[-\text{S45}], \text{Log}[\text{S12}-\text{S34}], \text{Log}[-\text{S51}]\}$.

D. D. Canko, C. G. Papadopoulos and N. Syrrakos, arXiv:2009.13917 [hep-ph] `Boundaries.m`

Pentabox - one leg off-shell: Kinematical Regions

- Euclidean region:

$$\left\{ \text{S12} \to -2, \text{S23} \to -3, \text{S34} \to -5, \text{S45} \to -7, \text{S51} \to -11, x \to \frac{1}{4} \right\}$$

no letter l in the region $[0, x]$, all boundary terms real. [very fast `GiNaC`]

Family	W=1	W=2	W=3	W=4
P_1 (g_{72})	17 (14)	116 (95)	690 (551)	2740 (2066)
P_2 (g_{73})	25 (14)	170 (140)	1330 (1061)	4950 (3734)
P_3 (g_{84})	22 (12)	132 (90)	1196 (692)	4566 (2488)

TABLE: Number of GP entering in the solution, as explained in the text.

- with timings, running the `GiNaC` Interactive Shell `ginsh`, given by 1.9, 3.3, and 2 seconds for P_1, P_2 and P_3 respectively and for a precision of 32 significant digits
- A very different canonical basis, several elements start at ϵ^4.

Pentabox - one leg off-shell: Kinematical Regions

- One-scale integrals - closed form

$$
\begin{aligned}
(-s_{34})^{-\epsilon} &= (-S_{51})^{-\epsilon} x^{-\epsilon} \\
(-s_{45})^{-\epsilon} &= (-S_{12})^{-\epsilon} x^{-2\epsilon} \\
(-s_{15})^{-\epsilon} &= (-S_{45})^{-\epsilon} \left(1 - \frac{S_{45} - S_{23}}{S_{45}} x\right)^{-\epsilon} \\
(-p_{1s})^{-\epsilon} &= (1-x)^{-\epsilon} (-S_{45})^{-\epsilon} \left(1 - \frac{S_{12}}{S_{45}} x\right)^{-\epsilon} \\
(-s_{12})^{-\epsilon} &= x^{-\epsilon} (S_{12} - S_{34})^{-\epsilon} \left(1 - \frac{S_{12}}{S_{12} - S_{34}} x\right)^{-\epsilon},
\end{aligned}
$$

- One-scale integrals - expanded form

$$
\begin{aligned}
&\mathrm{Log}[-\mathrm{p1s} - i\delta] \to G[1, x] + G\left[\frac{\mathrm{S45}}{\mathrm{S12}}, x\right] + \mathrm{Log}[-\mathrm{S45}], \\
&\mathrm{Log}[-\mathrm{s34} - i\delta] \to \mathrm{Log}[-\mathrm{S51}] + \mathrm{Log}[x], \\
&\mathrm{Log}[-\mathrm{s12} - i\delta] \to G\left[\frac{\mathrm{S12} - \mathrm{S34}}{\mathrm{S12}}, x\right] + \mathrm{Log}[\mathrm{S12} - \mathrm{S34}] + \mathrm{Log}[x], \\
&\mathrm{Log}[-\mathrm{s45} - i\delta] \to \mathrm{Log}[-\mathrm{S12}] + 2\,\mathrm{Log}[x], \\
&\mathrm{Log}[-\mathrm{s15} - i\delta] \to G\left[\frac{\mathrm{S45}}{-\mathrm{S23} + \mathrm{S45}}, x\right] + \mathrm{Log}[-\mathrm{S45}]
\end{aligned}
$$

Pentabox - one leg off-shell: Kinematical Regions

- In general many letters will be now in $[0,x]$. This has two consequences:
  1. Need to fix infinitesimal imaginary part of $\frac{l_i}{x}$
  2. Increasing CPU time in `GiNaC`.
- Since the $\mathcal{F}$ polynomial maintains the sign of the $i0$ prescription of Feynman propagators with all original invariants assuming $s_{ij}(p_{1s}) \to s_{ij}(p_{1s}) + i\delta$, we determine the corresponding infinitesimal imaginary part of $\frac{l_i}{x}$ from

$$p_{1s} + i\delta = (1-x)(S_{45} - S_{12}x),\; s_{12} + i\delta = (S_{34} - S_{12}(1-x))\,x,$$
$$s_{23} + i\delta = S_{45},\; s_{34} + i\delta = S_{51}x,$$
$$s_{45} + i\delta = S_{12}x^2,\; s_{15} + i\delta = S_{45} + (S_{23} - S_{45})x$$

  with $S_{ij} \to S_{ij} + i\delta\eta_{ij}$, $x \to x + i\delta\eta_x$,
- Building a Fibration Basis using for instance `PolyLogTools`.

Pentabox - one leg off-shell: Validation

- All regions of AIMPTZ checked @precision

- One-loop pentagon at order $\mathcal{O}(\varepsilon^4)$ [any order, analytic]

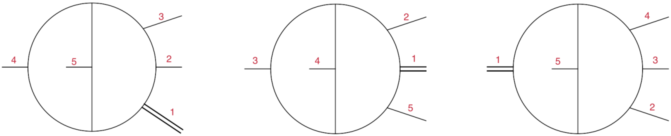
  N. Syrrakos, "Pentagon integrals to arbitrary order in the dimensional regulator," arXiv:2012.10635 [hep-ph].

- Taken the limit $x = 1$ in all families to obtain the result for on-shell planar 5box

> SDE is not only capable to produce analytic results for off-shell MI but it can also give, almost for free, the on-shell MI.

- Evaluating phase-space points for $pp \to W^+ j_1 j_2$ generated by `HELAC-PHEGAS`, i.e. arbitrary floating points.

HEXABOX - ONE LEG OFF-SHELL



$$r_1 = \sqrt{\lambda(p_{1s},\ s_{23},\ s_{45})}$$
$$r_2 = \sqrt{\lambda(p_{1s},\ s_{24},\ s_{35})}$$
$$r_3 = \sqrt{\lambda(p_{1s},\ s_{25},\ s_{34})}$$
$$r_4 = \sqrt{\det \mathbb{G}(q_1,\ q_2,\ q_3,\ q_4)}$$
$$r_5 = \sqrt{\Sigma_5^{(1)}}$$
$$r_6 = \sqrt{\Sigma_5^{(2)}}$$

- For topology N_1, the square roots r_1 and r_4 appear in its alphabet and are rationalized.

$$\partial_x \mathbf{g} = \epsilon \left(\sum_{i=1}^{l_{max}} \frac{\mathbf{M}_i}{x - l_i} \right) \mathbf{g}$$

  $l_{max} = 21$ from 39 letters in the original alphabet

- For topologies N_2 and N_3, the square roots appearing are $\{r_1,\ r_2,\ r_4,\ r_5\}$ and $\{r_1,\ r_3,\ r_4,\ r_6\}$ not *simultaneous* rationalisation possible !
  The more general form of the SDE takes the form:

$$\partial_x \mathbf{g} = \epsilon \left(\sum_{a=1}^{l_{max}} \frac{dL_a}{dx} \mathbf{M}_a \right) \mathbf{g}$$

  where most of the L_a are simple rational functions of x, as in (1), whereas the rest are algebraic functions of x involving the non-rationalisable square roots.

For instance element 11 of N_2 is given as

$$g_{11}^{(2)} = 8\Bigg(2\mathcal{G}(0,-y)\left(\mathcal{G}\left(1,y\right)-\mathcal{G}\left(\frac{\tilde{S}_{45}}{\tilde{S}_{12}},y\right)\right)+2\mathcal{G}\left(0,\frac{\tilde{S}_{45}}{\tilde{S}_{12}},y\right)-\mathcal{G}\left(1,y\right)\log\left(\frac{\tilde{S}_{45}}{\tilde{S}_{12}}\right)$$
$$+\log\left(\frac{\tilde{S}_{45}}{\tilde{S}_{12}}\right)\mathcal{G}\left(\frac{\tilde{S}_{45}}{\tilde{S}_{12}},y\right)-2\mathcal{G}\left(0,1,y\right)\Bigg)$$

where the new parametrization of the external momenta is given by

$$q_1 \to \tilde{p}_{123} - y\tilde{p}_{12},\ q_2 \to y\tilde{p}_2,\ q_3 \to -\tilde{p}_{1234},\ q_4 \to y\tilde{p}_1$$

with the new momenta $\tilde{p}_i,\ i = 1\ldots5$ satisfying as usual, $\sum_1^5 \tilde{p}_i = 0,\ \tilde{p}_i^2 = 0,\ i = 1\ldots5$, with $\tilde{p}_{i\ldots j} := \tilde{p}_i + \ldots + \tilde{p}_j$. The set of independent invariants is given by $\{\tilde{S}_{12}, \tilde{S}_{23}, \tilde{S}_{34}, \tilde{S}_{45}, \tilde{S}_{51}, y\}$, with $\tilde{S}_{ij} := \left(\tilde{p}_i + \tilde{p}_j\right)^2$. The explicit mapping between the two sets of invariants is given by

$$q_1^2 = (1-y)(\tilde{S}_{45} - \tilde{S}_{12}y),\ s_{12} = \tilde{S}_{45}(1-y) + \tilde{S}_{23}y,\ s_{23} = -y\left(\tilde{S}_{12} - \tilde{S}_{34} + \tilde{S}_{51}\right),$$
$$s_{34} = \tilde{S}_{51}y, s_{45} = y\left(\tilde{S}_{23} - \tilde{S}_{45} - \tilde{S}_{51}\right),\ s_{15} = y\left(\tilde{S}_{34} - \tilde{S}_{12}(1-y)\right).$$

- By identifying $f_- = y$ and $f_+ = y\frac{\tilde{S}_{12}}{\tilde{S}_{45}}$, which in terms of (38) are given as

$$f_\pm = \frac{S_{45} + x\left(-S_{23} - S_{34} + 2S_{51} + S_{12}x\right) \pm r_2}{2\left(S_{12} - S_{34} + S_{51}\right)x}$$

we can write the DE for this element in the simple and compact form

$$\frac{d}{dx}g_{11}^{(2)} = -8\left(\mathrm{dlog}\left(\frac{f_+ - 1}{f_- - 1}\right)\log\left(f_- f_+\right) - \mathrm{dlog}\left(\frac{f_+}{f_-}\right)\log\left(\left(f_- - 1\right)\left(f_+ - 1\right)\right)\right).$$

The form of the DE makes the determination of the ansatz rather straightforward, with the result

$$g_{11}^{(2)} = -8\Big(-\log(f_- f_+)\Big(\mathcal{G}(1, f_-) - \mathcal{G}(1, f_+)\Big) + 2\mathcal{G}(0,1,f_-) - 2\mathcal{G}(0,1,f_+)\Big).$$

- Concerning the other non-rationalisable square root in the family N_2, r_5, it also appears for the first time at weight 2 in the basis element 73 only (see the ancillary file), which is one of the new integrals to be calculated.

$$g_{73}^{(2)} = 16\log\left(f_- f_+\right)\Big(\mathcal{G}(1, f_-) - \mathcal{G}(1, f_+)\Big) - 32\Big(\mathcal{G}(0,1,f_-) - \mathcal{G}(0,1,f_+)\Big)$$

with

$$f_\pm = \frac{S_{45}\left(2S_{12}x - S_{34}x + S_{51}\right) + x\left(S_{23}S_{34} - S_{12}S_{23} + xS_{12}S_{51}\right) \pm r_5}{2S_{45}\left(S_{12} - S_{34} + S_{51}\right)}$$

- The pure basis elements can be written in general as follows:

$$g = Ce^{2\gamma_E \epsilon} \int \frac{d^d k_1}{i\pi^{d/2}} \frac{d^d k_2}{i\pi^{d/2}} \frac{P\left(\{D_i\}, \{S_{ij}, x\}\right)}{\prod\limits_{i \in \tilde{S}} D_i^{a_i}} \tag{1}$$

where D_i, $i = 1...11$, represent the inverse scalar propagators, $\tilde{S}$ the set of indices corresponding to a given sector, S_{ij}, x the kinematic invariants, P is a polynomial, a_i are positive integers and C a factor depending on S_{ij}, x.

- This form is usually decomposed in terms of FI, F_i,

$$g = C \sum c_i \left(\{S_{ij}, x\}\right) F_i$$

with c_i being polynomials in S_{ij}, x.

- An alternative approach, would be to build-up the Feynman parameter representation for the whole basis element, by considering the integral as a tensor integral in its Feynman parameter representation.

  J. Gluza, K. Kajda, T. Riemann and V. Yundin, Eur. Phys. J. C **71** (2011), 1516 [arXiv:1010.1667 [hep-ph]].
  S. C. Borowka, [arXiv:1410.7939 [hep-ph]].

  Then, by using the expansion by regions approach

  B. Jantzen, A. V. Smirnov and V. A. Smirnov, Eur. Phys. J. C **72** (2012), 2139 [arXiv:1206.0546 [hep-ph]].
  A. V. Smirnov, Comput. Phys. Commun. **204** (2016), 189-199 [arXiv:1511.03614 [hep-ph]].

$$b = \sum_I N_I \int \prod_{i \in S_I} dx_i \, U_I^{a_I} F_I^{b_i} \Pi_I$$

  where I runs over the set of contributing regions, U_I and F_I are the limits of the usual Symanzik polynomials, Π_I is a polynomial in the Feynman parameters, x_i, and the kinematic invariants S_{ij}, and S_I the subset of surviving Feynman parameters in the limit.
- In this way a significant reduction of the number of regions to be calculated is achieved, namely from 208 to 9. Notice that in contrast to the approach described in the previous paragraphs, only the regions $x^{-2\epsilon}$ and $x^{-4\epsilon}$ contribute to the final result, making thus the evaluation of the region-integrals simpler.
- Moreover, this approach overpasses the need for an IBP reduction of the basis elements in terms of MI.

Weight 3:
The differential equation (1) can be written in the form:

$$\partial_x g_I^{(3)} = \sum_a \left(\partial_x \log L_a\right) \sum_J c_{IJ}^a g_J^{(2)}$$

Since the lower limit of integration corresponds to $x = 0$, we need to subtract the appropriate term so that the integral is explicitly finite. This is achieved as follows:

$$\partial_x g_I^{(3)} = \sum_a \frac{l_a}{x} \sum_J c_{IJ}^a g_{J,0}^{(2)} + \left(\sum_a \left(\partial_x \log L_a\right) \sum_J c_{IJ}^a g_J^{(2)} - \sum_a \frac{l_a}{x} \sum_J c_{IJ}^a g_{J,0}^{(2)} \right)$$

where $g_{I,0}^{(2)}$ are obtained by expanding $g_I^{(2)}$ around $x = 0$ and keeping terms up to order $\mathcal{O}\left(\log(x)^2\right)$, and $l_a \in \mathbb{Q}$ are defined through

$$\partial_x \log L_a = \frac{l_a}{x} + \mathcal{O}\left(x^0\right).$$

The DE can now be integrated from $x = 0$ to $x = \bar{x}$, and the result is given by

$$g_I^{(3)} = g_{I,\mathcal{G}}^{(3)} + b_I^{(3)} + \int_0^{\bar{x}} \mathrm{d}x \left(\sum_a \left(\partial_x \log L_a\right) \sum_J c_{IJ}^a g_J^{(2)} - \sum_a \frac{l_a}{x} \sum_J c_{IJ}^a g_{J,0}^{(2)} \right)$$

with $b_I^{(3)}$ being the boundary terms at $\mathcal{O}(\epsilon^3)$ and

$$g_{I,\mathcal{G}}^{(3)} = \int_0^{\bar{x}} \mathrm{d}x \sum_a \frac{l_a}{x} \sum_J c_{LJ}^a g_{J,0}^{(2)} \bigg|_{\mathcal{G}}$$

with the subscript $\mathcal{G}$, indicating that the integral is represented in terms of GPLs (see ancillary file), following the convention

$$\int\limits_0^{\bar{x}} dx \frac{1}{x} \mathcal{G}\left(\underbrace{0, \ldots 0}_{n}; x\right) = \mathcal{G}\left(\underbrace{0, \ldots 0}_{n+1}; \bar{x}\right).$$

Weight 4:
At weight 4, the differential equation (1) can be written in the form:

$$\partial_x g_I^{(4)} = \sum_a \left(\partial_x \log L_a\right) \sum_J c_{IJ}^a g_J^{(3)}$$

which after doubly-subtracting, in order to obtain integrals that are explicitly finite as in (2), is written as

$$\partial_x g_I^{(4)} = \sum_a \partial_x (\log L_a - LL_a) \sum_J c_{IJ}^a g_J^{(3)} + \sum_a \partial_x (LL_a) \sum_J c_{IJ}^a (g_J^{(3)} - g_{J,0}^{(3)}) + \sum_a \frac{l_a}{x} \sum_J c_{IJ}^a g_{J,0}^{(3)}$$

where LL_a are obtained by expanding $\log(L_a)$ around $x = 0$ and keeping terms up to order $\mathcal{O}\big(\log(x)\big)$, and

$$g_{I,0}^{(3)} = g_{I,\mathcal{G}}^{(3)} + b_I^{(3)}.$$

Now, by integrating by parts and using (2) we can write the final result as follows:

$$
\begin{aligned}
g_I^{(4)} =& g_{I,\mathcal{G}}^{(4)} + b_I^{(4)} + \left(\sum_a \log L_a \sum_J c_{IJ}^a g_J^{(3)}\right) - \left(\sum_a LL_a \sum_J c_{IJ}^a g_{J,0}^{(3)}\right) \\
& - \int_0^{\bar{x}} \mathrm{d}x \sum_a (\log L_a - LL_a) \sum_J c_{IJ}^a \sum_b \frac{l_b}{x} \sum_K c_{JK}^b g_{K,0}^{(2)} \\
& - \int_0^{\bar{x}} \mathrm{d}x \sum_a \log L_a \sum_J c_{IJ}^a \left(\sum_b (\partial_x \log L_b) \sum_K c_{JK}^b g_K^{(2)} - \sum_b \frac{l_b}{x} \sum_K c_{JK}^b g_{K,0}^{(2)}\right)
\end{aligned}
$$

with a, b running over the set of contributing letters, I, J, K running over the set of basis elements, $b_I^{(4)}$ being the boundary terms at $\mathcal{O}(\epsilon^4)$ and

$$
g_{I,\mathcal{G}}^{(4)} = \int_0^{\bar{x}} \mathrm{d}x \left(\sum_a \frac{l_a}{x} \sum_J c_{IJ}^a g_{J,0}^{(3)}\right)\Bigg|_{\mathcal{G}}
$$

where the subscript $\mathcal{G}$ indicates that the integral is represented in terms of GPLs (see ancillary file).

As a proof of concept, we have implemented the final formulae in `Mathematica`. We use `NIntegrate` to perform the one-dimensional integrals, after expressing all weight-2 functions in terms of classical polylogarithms following reference

H. Frellesvig, D. Tommasini and C. Wever, JHEP **03** (2016), 189 [arXiv:1601.02649 [hep-ph]].

- The user can easily assess the performance of this straightforward implementation by running the provided codes and look at the minimum number of digits in agreement with the high-precision results from Abreu et. al, as well as at the number of integrand evaluations performed by `NIntegrate`.
- Notice that the integrand expressions involve logarithms and classical polylogarithms Li_2 that are evaluated using very little CPU time.
- The parts of the formulae that can be represented in terms of GPLs up to weight four, as well as the results for the N_1 family, for which we have all basis elements in terms of GPLs up to weight four, are evaluated with `GiNaC`, as implemented in `PolyLogTools`.

J. Vollinga, Nucl. Instrum. Meth. A **559** (2006), 282-284 [arXiv:hep-ph/0510057 [hep-ph]].

C. Duhr and F. Dulat, JHEP **08** (2019), 135 [arXiv:1904.07279 [hep-th]].

HEXABOX - ONE LEG OFF-SHELL: PERFORMANCE

- In the current implementation we use the default parameters for `GiNaC` and the default parameters for `NIntegrate` with the exception of `WorkingPrecision` and `PrecisionGoal`, in order to obtain reasonable results within reasonable time, taking into account that the provided implementation serves merely as a demonstration of the correctness of our representations.
- For the Euclidean point the precision is typically of the order of 32 digits, which is compatible with `GiNaC` setup.
- For the physical point, the typical precision is of the order of 25 digits, which is compatible with the expected one taking into account the numerical value of the infinitesimal imaginary part assigned to the kinematical invariants.

5-point 2-loop - one leg off-shell: Outlook

- Non-planar families
  - We have completed the hexa-box families, N_1, N_2, N_3.
  - Preliminary checks against AIMPTZ group results successful.
  - Next task: double-pentagon families, N_4, N_5.
- SDE approach: all MI up to 4-point with up to 2 off shell legs and 5-point with up to one off-shell leg.
- Speed-up numerical evaluation
  - Improving GPLs analytic continuation.
  - Study letters ordering in physical regions, use different mappings and/or fibrations.
  - Combine analytics with numerics.
- Massive internal particles.
- HELAC2LOOP: generic approach to amplitude reduction and evaluation

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  - SDE@1-loop N. Syrrakos, "One-loop Feynman integrals for $2 \to 3$ scattering involving many scales including internal masses," JHEP **10** (2021), 041 [arXiv:2107.02106 [hep-ph]].
  - SDE@3-loop D. D. Canko and N. Syrrakos, "Planar three-loop master integrals for $2 \to 2$ processes with one external massive particle," [arXiv:2112.14275 [hep-ph]].
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Acknowledging

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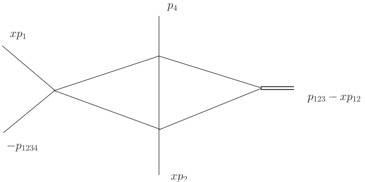
Thank you for your attention !

HOCTools-II: post-doc + phd

Backup slides



The two-loop diagram representing the decoupling basis element.

- Basis element 46 for P2 (53 for P3) known from double box P23 family; starts at $\mathcal{O}(\epsilon^4)$. [decoupling]

$$q_1 \to P_{123} - yP_{12},\; q_2 \to yP_1,\; q_3 \to P_4,\; q_4 \to -P_{1234},\; q_5 \to yP_2$$

$$q_1 \to p_{123} - xp_{12},\; q_2 \to p_4,\; q_3 \to -p_{1234},\; q_4 \to xp_1,\; q_5 \to xp_2$$

$$q_1^2 = (1-y)(S'_{45} - S'_{12}y),\; s_{12} = S'_{45} - (S_{12} + S'_{23})y,\; s_{23} = \left(S'_{34} - S_{12}(1-y)\right) y,$$
$$s_{34} = S'_{45},\; s_{45} = -(S'_{12} - S'_{34} + S'_{51})y,\; s_{15} = S'_{45} + S'_{23}y$$

$$q_1^2 = (1-x)(S_{45} - S_{12}x),\; s_{12} = (S_{34} - S_{12}(1-x))\,x,\; s_{23} = S_{45},\; s_{34} = S_{51}x,$$
$$s_{45} = S_{12}x^2,\; s_{15} = S_{45} + (S_{23} - S_{45})x$$

Pentabox - one leg off-shell: P2-3

$$d\vec{g} = \left[\sum_b d\log\left(x-\ell_b\right)M_b + \sum_c d\log\left(y-\ell_c\right)\bar{M}_c + d\log\left(W_{58}\left(x,y\right)\right)\tilde{M}_{58}\right]\vec{g}$$

- all letters W_a, except W_{58}, are linear functions only of x or y.
- M matrices have zeroes in the row and the column corresponding to the basis element 46 for P2 (53 for P3).
- $\bar{M}$ matrices have non-zero matrix elements only in the row and the column corresponding to the basis element 46 for P2 (53 for P3).
- $\tilde{M}$ matrix have non-zero matrix elements only in the column corresponding to the basis element 46 for P2 (53 for P3).

$$\frac{d\vec{g}'}{dx} = \sum_a \frac{1}{x-\ell_a} M_a \vec{g}'$$