TENSOR DECOMPOSITION FOR MULTILOOP MULTILEG HELICITY AMPLITUDES

Loops and Legs in Quantum Field Theory
Ettal – 26/04/2022

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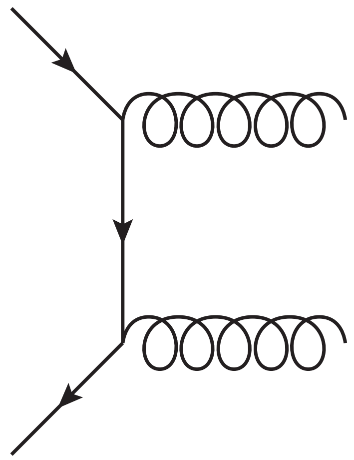
Collaboration with *Tiziano Peraro* arXiv:2012.00820, arXiv:1906.03298

HARD SCATTERING - FIXED ORDER CALCULATIONS

$$\sigma_{q\bar{q}\to gg} = \int [\mathrm{dPS}] \left|\mathcal{M}_{q\bar{q}\to gg}\right|^2$$

$$\left|\mathcal{M}_{q\bar{q}\to gg}\right|^2 = \boxed{\left|\mathcal{M}^{LO}_{q\bar{q}\to gg}\right|}^2 + \left(\frac{\alpha_s}{2\pi}\right)\left|\mathcal{M}^{NLO}_{q\bar{q}\to gg}\right|^2 + \left(\frac{\alpha_s}{2\pi}\right)^2\left|\mathcal{M}^{NNLO}_{q\bar{q}\to gg}\right|^2 + \ldots$$



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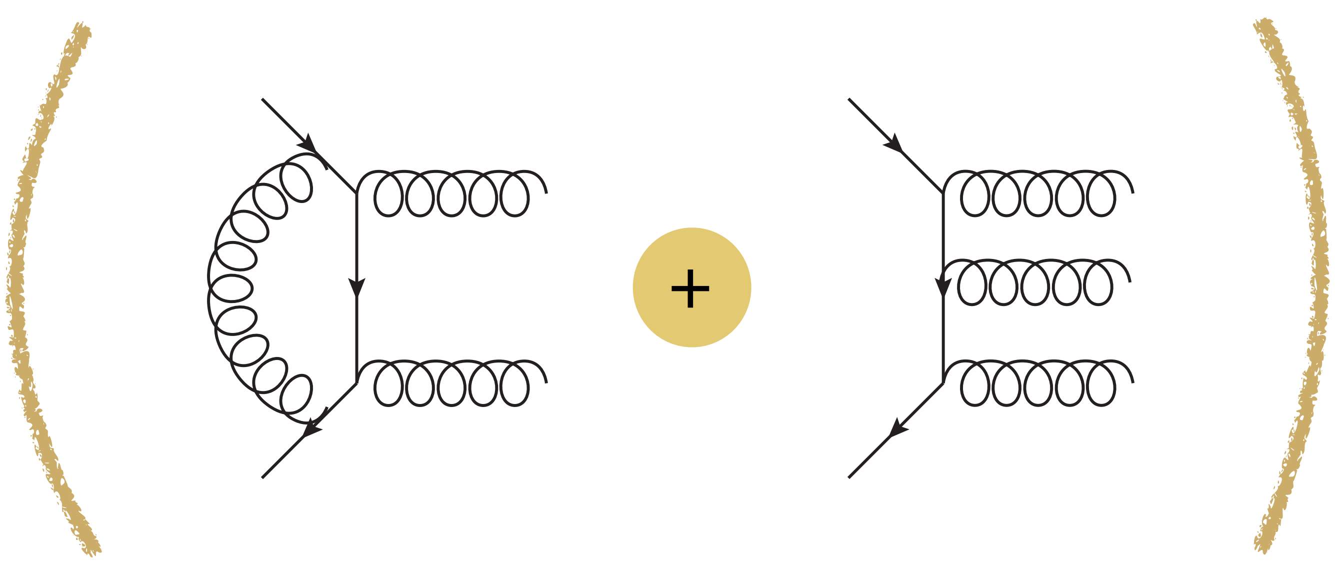

Virtual Real

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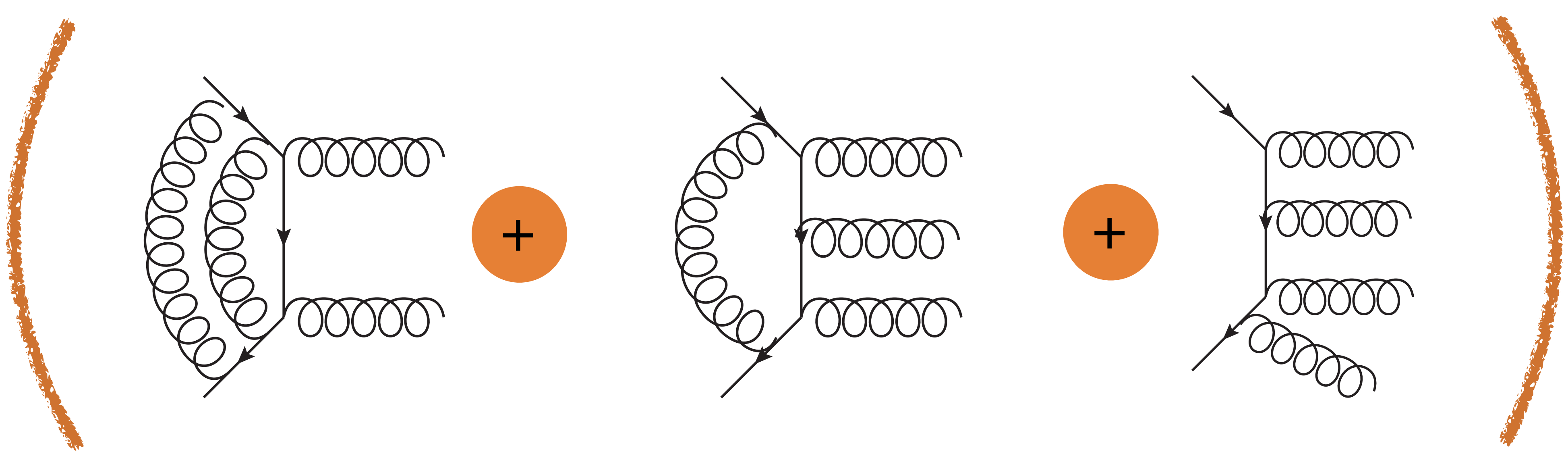

Double Virtual Real Virtual Double Real

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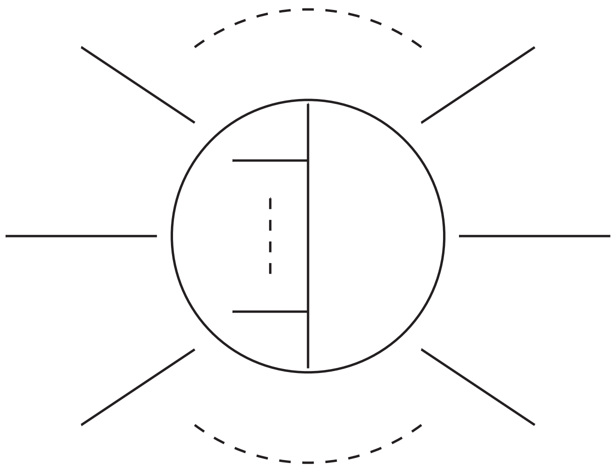
Calculations involve Feynman graphs with increasing numbers of loops and legs

MULTILOOP SCATTERING AMPLITUDES: THE STANDARD WAY

One way to go about it: **standard approach** ***(divide et impera)***



$$= \int \prod_{i=1}^{L} d^D k_i \; R_i(k_1, \ldots, k_L, p_1, \ldots, p_E, m_j)$$

Standard steps:

1) Obtain the **integrand** (From Feynman diagrams, Unitarity, …)

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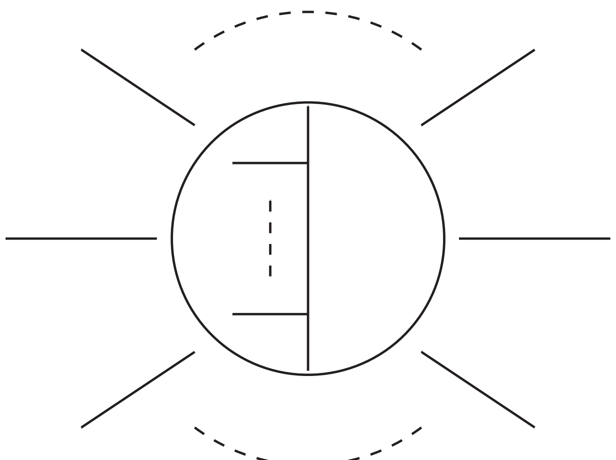

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$$= \sum_{i=1}^{N} \boxed{R_i(x_1, \ldots, x_r)}\, \mathcal{I}_i(x_1, \ldots, x_n)$$

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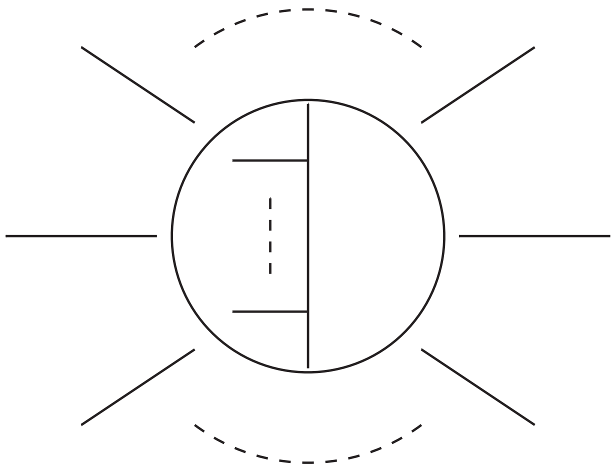

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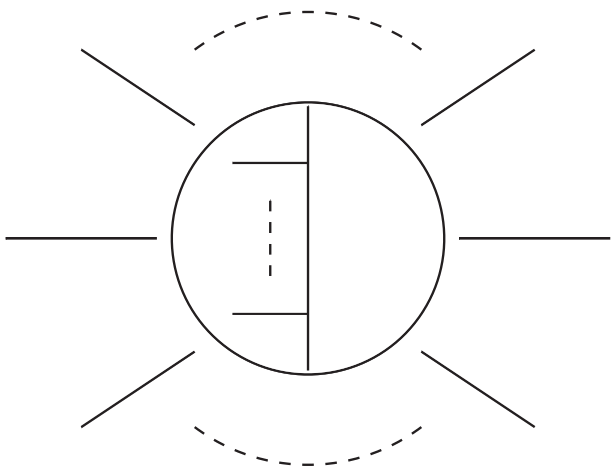
3) Compute the master integrals (Diff Equations, Canonical bases, polylogs etc…)

HOW TO GET THE INTEGRAND

First problem is ***"getting the integrand"***:



$$= \sum \text{Feynman Diagrams} \;\rightarrow\; ?$$

Problems:

- Number of diagrams ***grows factorially*** *(not a real problem though, at least for reasonable processes in QCD…)*

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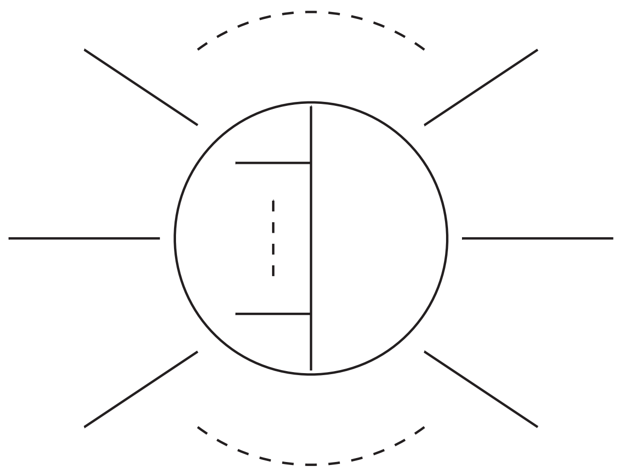
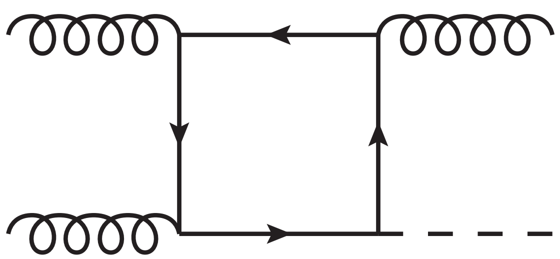


$$= \sum \text{Feynman Diagrams} \;\rightarrow\; ?$$

Problems:

- Number of diagrams ***grows factorially*** *(not a real problem though, at least for reasonable processes in QCD…)*
- **More serious problem(s):** *"tensor decomposition" -> each diagram produces thousands of terms!*

TENSOR DECOMPOSITION

$\mathcal{M}_{gg\to Hg} \sim$ 

Strip it of Lorentz and Dirac structures

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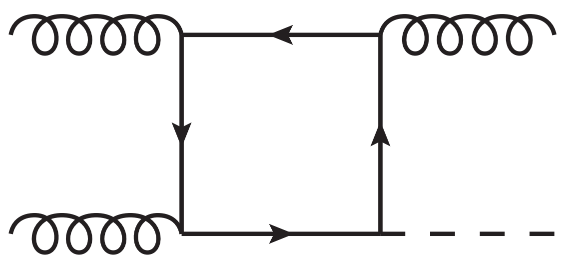
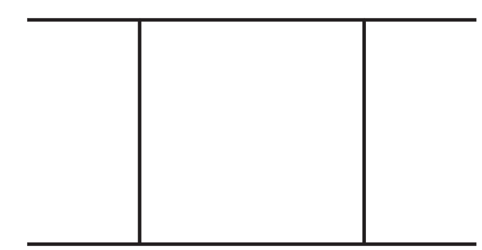


Strip it of Lorentz and Dirac structures

$$\sim \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(k-p_2)^2(k-p_2-p_3)^2(k-p_1-p_2-p_3)^2} =$$



Scalar Feynman Integrals are what we know how to compute

Can be achieved in different ways

It can become a real hassle at high loops and multiplicities

A widely used successful method is **projector / form factor method**

THE PROJECTOR–FORM FACTOR METHOD

The idea is very simple:

1. Use **Lorentz invariance**, **gauge invariance** (& any **symmetries**) to **parametrise** the scattering amplitude in terms of **tensor structures** and **scalar form factors**
2. Define **projector operators** to **extract** these form factors from the corresponding Feynman diagrams

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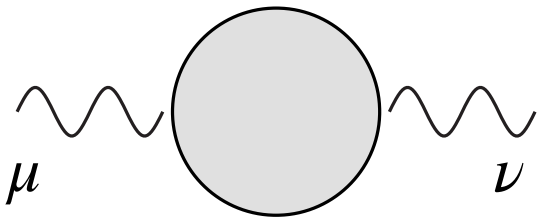

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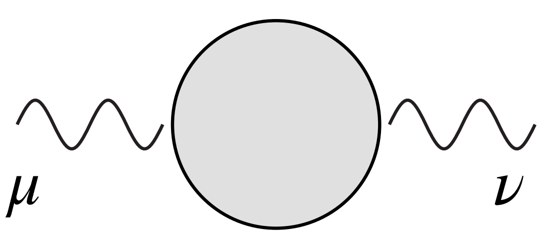

$$= \sum_{i=1}^{n} F_i \, T_i^{\mu\nu} = \left(F_1(p, m^2)\, p^\mu p^\nu + F_2(p, m^2)\, g^{\mu\nu} \right)$$

Lorentz Invariance

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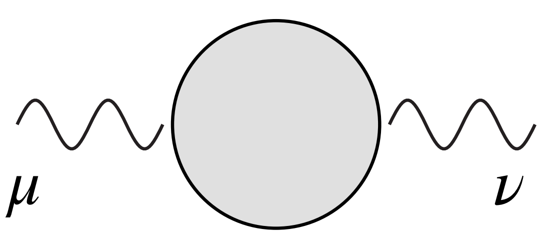

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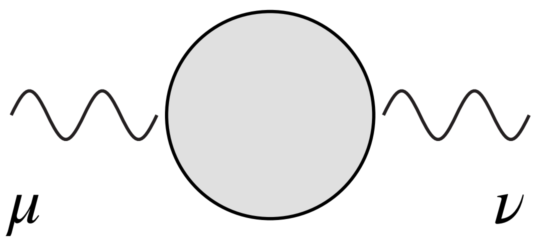
Lorentz Invariance

$$= \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) F(p, m^2)$$

Gauge Invariance

THE PROJECTOR-FORM FACTOR METHOD



$$= \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) F(p, m^2) = \Pi^{\mu\nu}$$

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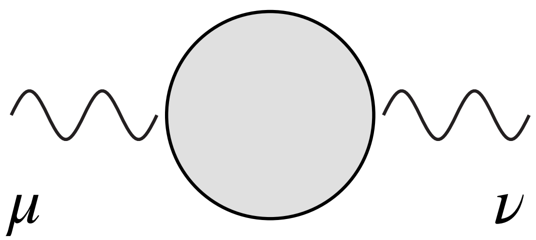

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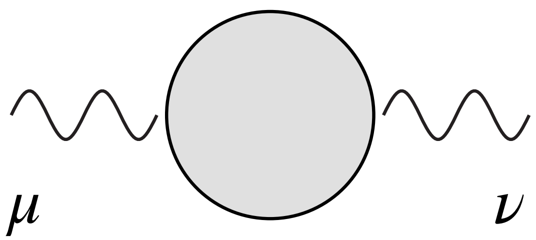
Non perturbative

To extract $F(p, m^2)$ I define a projector operator $P_{\mu\nu} = C(d, p, m^2)\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right)$

I can then determine the coefficient $C(d, p, m^2)$ by imposing $P_{\mu\nu}\Pi^{\mu\nu} = F(p, m^2)$

We find $$P_{\mu\nu} = \frac{1}{d-1}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right)$$

THE PROJECTOR-FORM FACTOR METHOD



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All **algebra** has to be performed in **d space-time dimensions** to be able to use the method in **CDR** (Conventional Dimensional Regularisation)

THE PROJECTOR-FORM FACTOR METHOD

Works in general, no restrictions of any kinds in principle:

1. Pick your favourite process
2. Use **Lorentz + gauge + any symmetry** (parity, Bose etc…) to find minimal set of tensor structures in d space-time dimensions: $\mathscr{A} = \sum_j F_j T_j$
3. Derive **projectors operators** to single out corresponding form factors: $\mathscr{P}_j \mathscr{A} = F_j$

$$M_{ij} = \sum_{pol} T_i^\dagger T_j \qquad \mathscr{P}_j = \sum_k \left(M^{-1}\right)_{jk} T_k^\dagger$$

4. Apply these projectors on your favourite representation for the scattering amplitude

PROBLEMS WITH PROJECTORS

Seems neat. Where are the **issues**?

Let's have a look at a more interesting example: **massless quark scattering** $q\bar{q} \to Q\bar{Q}$

Studied **up to 2 loops** first by N. Glover in hep-ph/0401119

$$0 \to q(p_1, \lambda_1) + \bar{q}(p_2, \lambda_2) + Q(p_3, \lambda_3) + \bar{Q}(p_4, \lambda_4)$$

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What is the **most general d-dimensional tensor structure**?

$$T \sim \bar{u}(p_1)\Gamma^{\mu_1,\ldots,\mu_n} u(p_2)\, \bar{u}(p_3)\Gamma_{\mu_1,\ldots,\mu_n} u(p_4)$$

When do I stop?

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What is the **most general d-dimensional tensor structure**?

Problem: **γ-algebra is not closed in d-dimensions**

At arbitrary loops, arbitrary fermion lines with arbitrary numbers of matrices…

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Let's follow standard approach @ **2 loops**:

$$\mathscr{A}^{(2l)}_{qqQQ} = \sum_{j=1}^{6} A_j\, D_j$$

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$$\begin{aligned}
\mathcal{D}_1 &= \bar{u}(p_1)\gamma_{\mu_1} u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1} u(p_4), \\
\mathcal{D}_2 &= \bar{u}(p_1)\not{p}_3 u(p_2)\ \bar{u}(p_3)\not{p}_1 u(p_4), \\
\mathcal{D}_3 &= \bar{u}(p_1)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3} u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3} u(p_4), \\
\mathcal{D}_4 &= \bar{u}(p_1)\gamma_{\mu_1}\not{p}_3\gamma_{\mu_3} u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}\not{p}_1\gamma_{\mu_3} u(p_4), \\
\mathcal{D}_5 &= \bar{u}(p_1)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5} u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5} u(p_4), \\
\mathcal{D}_6 &= \bar{u}(p_1)\boxed{\gamma_{\mu_1}\gamma_{\mu_2}\not{p}_3\gamma_{\mu_4}\gamma_{\mu_5}} u(p_2)\ \bar{u}(p_3)\boxed{\gamma_{\mu_1}\gamma_{\mu_2}\not{p}_1\gamma_{\mu_4}\gamma_{\mu_5}} u(p_4).
\end{aligned}$$

Easy to see that at **2 loops** we cannot have more than **5 γ matrices** per fermion string

But at 3 loops I would need also strings with **7 γ**, etc etc…

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Define the matrix $M_{ij} = \sum_{pol} D_i^\dagger D_j$, its inverse provides us with the projectors

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$$
\begin{aligned}
\mathcal{P}(A_2) = & \frac{1}{32 s_{13}^2 s_{23}^2 s_{12}^2 (d-5)(d-7)(d-3)(d-4)} \times \Big(\\
& - s_{13}(35 s_{23}^2 d^3 - 55 s_{13} s_{23} d^3 + 1046 s_{13} s_{23} d^2 - 1872 s_{13}^2 d + 2432 s_{13}^2 - 454 s_{23}^2 d^2 \\
& -6040 s_{13} s_{23} d - 2688 s_{23}^2 + 368 s_{13}^2 d^2 + 1928 s_{23}^2 d - 20 s_{13}^2 d^3 + 11136 s_{13} s_{23}) \mathcal{D}_1^\dagger \\
& + 2 s_{13}(-2 s_{13}^2 d^2 - 9 s_{13} s_{23} d^2 + 142 s_{13} s_{23} d - 448 s_{13} s_{23} + 7 s_{23}^2 d^2 + 136 s_{23}^2 - 48 s_{13}^2 \\
& +28 s_{13}^2 d - 62 s_{23}^2 d) \mathcal{D}_3^\dagger \\
& + (-340 s_{13}^2 d^3 + 11008 s_{13}^2 - 740 s_{13} s_{23} d^3 + 44032 s_{13} s_{23} - 260 s_{23}^2 d^3 - 4144 s_{23}^2 d + 3712 s_{23}^2 \\
& +15 s_{13}^2 d^4 + 2852 s_{13}^2 d^2 - 28864 s_{13} s_{23} d + 1604 s_{23}^2 d^2 + 6944 s_{13} s_{23} d^2 - 9968 s_{13}^2 d \\
& +30 s_{13} s_{23} d^4 + 15 s_{23}^2 d^4) \mathcal{D}_2^\dagger \\
& - s_{13} s_{23}(12 s_{13} + s_{23} d - 4 s_{23} - s_{13} d) \mathcal{D}_5^\dagger \\
& + (-6 s_{23}^2 d + 24 s_{13}^2 + 2 s_{13} s_{23} d^2 - 40 s_{13} s_{23} d - 14 s_{13}^2 d + s_{13}^2 d^2 + 8 s_{23}^2 + s_{23}^2 d^2 + 192 s_{13} s_{23}) \mathcal{D}_6^\dagger \\
& - 2(5 s_{13}^2 d^3 + 5 s_{23}^2 d^3 + 10 s_{13} s_{23} d^3 - 240 s_{13} s_{23} d^2 - 100 s_{13}^2 d^2 - 56 s_{23}^2 d^2 + 580 s_{13}^2 d \\
& +1832 s_{13} s_{23} d + 196 s_{23}^2 d - 208 s_{23}^2 - 800 s_{13}^2 - 4224 s_{13} s_{23}) \mathcal{D}_4^\dagger \Big),
\end{aligned}
$$

with growth of number of tensors, the inversion can become extremely expensive!

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Artificial poles in $d \to 4$

They arise because the tensors we have chosen are actually NOT independent in d=4

Matrix not invertible in d=4

PROBLEMS WITH PROJECTORS

If we want to push this to **3 loops for $q\bar{q} \to Q\bar{Q}$**, it becomes more cumbersome…

- more structures
- more complicated projectors
- to be applied on much more complex Feynman diagrams

Can we do something about it?

See also alternative or complementary approaches by **[Chen '19], [Davies et al '20]; [Abreu et al '18]**

HELICITY AMPLITUDES AND PHYSICAL PROJECTORS

Ultimately, we are interested in **helicity amplitudes**, in d=4 in **'t Hooft-Veltman scheme**

Fix helicities, assuming that **external states** are in $d = 4$ **dimensions.**

$$\mathscr{A} = \sum_{i=1}^{n} F_j T_j \longrightarrow \mathscr{A}(\lambda_1, \ldots, \lambda_E) = \sum_{i=1}^{n} F_j T_j(\lambda_1, \ldots, \lambda_E) = \sum_{j=1}^{m<n} \bar{F}_j \, S_j(\lambda_1, \ldots, \lambda_E)$$

All "internal" indices are in d dimensions

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All "internal" indices are in d dimensions

Combinations of original form factors

Helicity amplitudes, spinor products, momentum twistors…

If we do things right, we'd expect that in 't Hooft-Veltman scheme ***there cannot be more independent form factors than independent helicity amplitudes***

For massless $q\bar{q} \to Q\bar{Q}$ there are 4 helicities, reduced to **2 by parity invariance…**

HELICITY AMPLITUDES AND PHYSICAL PROJECTORS

To get to minimal complexity, notice that:

$$
\begin{aligned}
\mathcal{D}_1 &= \bar{u}(p_1)\gamma_{\mu_1}u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}u(p_4),\\
\mathcal{D}_2 &= \bar{u}(p_1)\not{p}_3 u(p_2)\ \bar{u}(p_3)\not{p}_1 u(p_4),\\
\mathcal{D}_3 &= \bar{u}(p_1)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}u(p_4),\\
\mathcal{D}_4 &= \bar{u}(p_1)\gamma_{\mu_1}\not{p}_3\gamma_{\mu_3}u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}\not{p}_1\gamma_{\mu_3}u(p_4),\\
\mathcal{D}_5 &= \bar{u}(p_1)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}u(p_4),\\
\mathcal{D}_6 &= \bar{u}(p_1)\gamma_{\mu_1}\gamma_{\mu_2}\not{p}_3\gamma_{\mu_4}\gamma_{\mu_5}u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}\gamma_{\mu_2}\not{p}_1\gamma_{\mu_4}\gamma_{\mu_5}u(p_4).
\end{aligned}
$$

HELICITY AMPLITUDES AND PHYSICAL PROJECTORS

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$$\begin{aligned}
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\mathcal{D}_6 &= \bar{u}(p_1)\gamma_{\mu_1}\gamma_{\mu_2}\not{p}_3\gamma_{\mu_4}\gamma_{\mu_5}u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}\gamma_{\mu_2}\not{p}_1\gamma_{\mu_4}\gamma_{\mu_5}u(p_4).
\end{aligned}$$

- The first two "tensors" are **confined in d=4**, even if I pick **d-dim γ^μ matrices**!
- They are **independent in d=4**, they span the whole physical space of the helicity ampl.
- Every **additional tensor**, has **extra components in d=-2ϵ**, cannot be physical (?)

HELICITY AMPLITUDES AND PHYSICAL PROJECTORS

Let then pick first 2: $T_j = D_j\,,\ \ j = 1,2$ $M^{2\times 2}_{ij} = T^{\dagger}_i T_j\,,$

$$\left(M^{2\times 2}\right)^{-1}_{ij} = \boxed{\frac{1}{d-3}} X_{ij}$$

with

$$X_{ij} = \frac{1}{4\,s_{12}^2}\begin{pmatrix} 1 & \dfrac{s_{12}+2s_{23}}{s_{23}(s_{12}+s_{23})} \\ \dfrac{s_{12}+2s_{23}}{s_{23}(s_{12}+s_{23})} & \dfrac{(d-2)s_{12}^2+4s_{23}(s_{12}+s_{23})}{s_{23}^2(s_{12}+s_{23})^2} \end{pmatrix}$$

the matrix is smooth in $d \to 4$

HELICITY AMPLITUDES AND PHYSICAL PROJECTORS

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Define the 2 projectors

$$\overline{P}_i = \sum_{j=1}^{2}\left(M_{ij}^{(2\times 2)}\right)^{-1}\,\overline{T}_j^\dagger$$

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Define the 2 projectors

$$\overline{P}_i = \sum_{j=1}^{2}\left(M^{(2\times 2)}_{ij}\right)^{-1}\,\overline{T}_j^\dagger$$

and the remaining tensors as

$$\overline{T}_i = T_i - \sum_{j=1}^{2}\left(\overline{P}_j T_i\right)\overline{T}_j\,, \quad \text{for} \quad i = 3, 4, 5, 6, \ldots$$

HELICITY AMPLITUDES AND PHYSICAL PROJECTORS

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We are effectively <u>**block-diagonalising the matrix**</u>!

HELICITY AMPLITUDES AND PHYSICAL PROJECTORS

The remaining tensors

$$\overline{T}_i = T_i - \sum_{j=1}^{2} \left(\overline{P}_j T_i\right) \overline{T}_j \,, \quad \text{for} \quad i = 3, 4, 5, 6, \ldots$$

are such that

$$\boxed{\bar{T}_j(\lambda_1, \ldots, \lambda_E) = 0 \,, \;\; j = 3,4,5,6,\ldots}$$

Exactly in d in tHV!

HELICITY AMPLITUDES AND PHYSICAL PROJECTORS

The remaining tensors

$$\overline{T}_i = T_i - \sum_{j=1}^{2} \left(\overline{P}_j T_i\right) \overline{T}_j \,, \quad \text{for} \quad i = 3, 4, 5, 6, \ldots$$

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Exactly in d in tHV!

New tensors are *smooth* linear combinations of the old ones:

$$\bar{T}_3 = \left(-3d - \frac{12 s_{23}}{s_{12}} - 4\right) \bar{T}_1 - \frac{24}{s_{12}} \bar{T}_2 + T_3$$

HELICITY AMPLITUDES AND PHYSICAL PROJECTORS

The remaining tensors

$$\overline{T}_i = T_i - \sum_{j=1}^{2} \left(\overline{P}_j T_i\right) \overline{T}_j \,, \quad \text{for} \quad i = 3, 4, 5, 6, \ldots$$

are such that

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Exactly in d in tHV!

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And the new 6x6 inverse matrix becomes **block-diagonal**

$$\left(\bar{M}_{ij}\right)^{-1} = \begin{pmatrix} \frac{X_{ij}}{d-3} & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & \mathbf{R}_{ij} & \\ 0 & & & \end{pmatrix}$$

R_{ij} contains the complexity that we saw before, but actually NEVER need to even compute it!

HELICITY AMPLITUDES AND PHYSICAL PROJECTORS

The provide us with natural basis to derive helicity amplitudes!

$$\bar{\mathcal{A}} = \sum_{i=1}^{2} \mathcal{F}_i \, T_i$$

$$T_1 = \bar{u}(p_2)\,\gamma_\alpha\,u(p_1) \times \bar{u}(p_4)\,\gamma^\alpha\,u(p_3)$$

$$T_2 = \bar{u}(p_2)\,\not{p}_3\,u(p_1) \times \bar{u}(p_4)\,\not{p}_2\,u(p_3)$$

By construction, helicity amplitudes only receive contributions from two form factors!

$$\bar{\mathcal{A}}^{q\bar{q}\to\bar{Q}Q}_{+-+-} = \mathcal{H}_1 \frac{\langle 13\rangle}{\langle 24\rangle}\,, \quad \bar{\mathcal{A}}^{q\bar{q}\to\bar{Q}Q}_{+--+} = \mathcal{H}_2 \frac{\langle 14\rangle}{\langle 23\rangle},$$

$$\mathcal{H}_1 = 2t\mathcal{F}_1 - tu\mathcal{F}_2, \quad \mathcal{H}_2 = 2u\mathcal{F}_1 + tu\mathcal{F}_2.$$

And the projectors that we need to apply on the Feynman diagrams are much simpler!

HELICITY AMPLITUDES AND PHYSICAL PROJECTORS

The remaining tensors and form factors can be thrown away, **except to get CDR results:**

$$\sum_{pol} (M^0)^* M^{Ll} = \sum_{i,j=1}^{2} c_{ij} F_i^* F_j + \epsilon \boxed{\sum_{i,j=3}^{N} c_{ij}(\epsilon) F_i^* F_j}$$

Evanescent tensors show up suppressed in ϵ *even in CDR*.

If interested in tHV, we can ignore them

IMPORTANT: they are instead *EXACTLY ZERO* in *'t Hooft-Veltman*!

PARTONIC CHANNEL $gg \to gg$: A SUBTLETY

$gg \to gg$: most complicated channel [Caola, Chakraborty, Gambuti, Manteuffel, Tancredi, '21]

8 helicity amplitudes $\sim$ 8 form factors for each color ordered amplitude

In fact: in CDR, **10 Form factors** (using Gauge Invariance), reduced to **8 in tHV**

$$\bar{\mathcal{A}} = \sum_{i=1}^{10} \mathcal{F}_i \, T_i \quad \longrightarrow \quad \bar{\mathcal{A}} = \sum_{i=1}^{8} \mathcal{F}_i \, T_i$$

$$\begin{aligned}
&T_1 = \epsilon_1 \cdot p_3 \, \epsilon_2 \cdot p_1 \, \epsilon_3 \cdot p_1 \, \epsilon_4 \cdot p_2 \,, \\
&T_2 = \epsilon_1 \cdot p_3 \, \epsilon_2 \cdot p_1 \, \epsilon_3 \cdot \epsilon_4 \,, \quad T_3 = \epsilon_1 \cdot p_3 \, \epsilon_3 \cdot p_1 \, \epsilon_2 \cdot \epsilon_4 \,, \quad T_4 = \epsilon_1 \cdot p_3 \, \epsilon_4 \cdot p_2 \, \epsilon_2 \cdot \epsilon_3 \,, \\
&T_5 = \epsilon_2 \cdot p_1 \, \epsilon_3 \cdot p_1 \, \epsilon_1 \cdot \epsilon_4 \,, \quad T_6 = \epsilon_2 \cdot p_1 \, \epsilon_4 \cdot p_2 \, \epsilon_1 \cdot \epsilon_3 \,, \quad T_7 = \epsilon_3 \cdot p_1 \, \epsilon_4 \cdot p_2 \, \epsilon_1 \cdot \epsilon_2 \,, \\
&T_8 = \epsilon_1 \cdot \epsilon_2 \, \epsilon_3 \cdot \epsilon_4 \,, \quad T_9 = \epsilon_1 \cdot \epsilon_4 \, \epsilon_2 \cdot \epsilon_3 \,, \quad T_{10} = \epsilon_1 \cdot \epsilon_3 \, \epsilon_2 \cdot \epsilon_4 \,.
\end{aligned}$$

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&T_5 = \epsilon_2 \cdot p_1\, \epsilon_3 \cdot p_1\, \epsilon_1 \cdot \epsilon_4\,, \quad T_6 = \epsilon_2 \cdot p_1\, \epsilon_4 \cdot p_2\, \epsilon_1 \cdot \epsilon_3\,, \quad T_7 = \epsilon_3 \cdot p_1\, \epsilon_4 \cdot p_2\, \epsilon_1 \cdot \epsilon_2\,, \\
&T_8 = \epsilon_1 \cdot \epsilon_2\, \epsilon_3 \cdot \epsilon_4\,, \quad T_9 = \epsilon_1 \cdot \epsilon_4\, \epsilon_2 \cdot \epsilon_3\,, \quad T_{10} = \epsilon_1 \cdot \epsilon_3\, \epsilon_2 \cdot \epsilon_4\,.
\end{aligned}
$$

Question: which one do I remove of the last three — $g^{\mu\nu}g^{\rho\sigma}, g^{\mu\rho}g^{\nu\sigma}, g^{\mu\sigma}g^{\nu\rho}$ — ?

PARTONIC CHANNEL $gg \to gg$: A SUBTLETY

To understand the answer, it is convenient to introduce the extra vector $v_\perp^\mu = \epsilon^{\mu\nu\rho\sigma} p_{1\nu} p_{2\rho} p_{3\sigma}$

such that p_1, p_2, p_3, v span the full four-dimensional space and $g^{\mu\nu}$ can be dropped!

$$
\begin{aligned}
&\tilde{T}_1 = \epsilon_1 \cdot p_3\, \epsilon_2 \cdot p_1\, \epsilon_3 \cdot p_1\, \epsilon_4 \cdot p_2\,, \\
&\tilde{T}_2 = \epsilon_1 \cdot p_3\, \epsilon_2 \cdot p_1\, \epsilon_3 \cdot v_\perp\, \epsilon_4 \cdot v_\perp\,, \quad \tilde{T}_3 = \epsilon_1 \cdot p_3\, \epsilon_3 \cdot p_1\, \epsilon_2 \cdot v_\perp\, \epsilon_4 \cdot v_\perp\,, \\
&\tilde{T}_4 = \epsilon_1 \cdot p_3\, \epsilon_4 \cdot p_2\, \epsilon_2 \cdot v_\perp\, \epsilon_3 \cdot v_\perp\,, \quad \tilde{T}_5 = \epsilon_2 \cdot p_1\, \epsilon_3 \cdot p_1\, \epsilon_1 \cdot v_\perp\, \epsilon_4 \cdot v_\perp\,, \\
&\tilde{T}_6 = \epsilon_2 \cdot p_1\, \epsilon_4 \cdot p_2\, \epsilon_1 \cdot v_\perp\, \epsilon_3 \cdot v_\perp\,, \quad \tilde{T}_7 = \epsilon_3 \cdot p_1\, \epsilon_4 \cdot p_2\, \epsilon_1 \cdot v_\perp\, \epsilon_2 \cdot v_\perp\,, \\
&\tilde{T}_8 = \epsilon_1 \cdot v_\perp\, \epsilon_2 \cdot v_\perp\, \epsilon_3 \cdot v_\perp\, \epsilon_4 \cdot v_\perp
\end{aligned}
$$

Natural basis, entirely confined in d=4 dimensions! To get rid of v notice now:

$$v_\perp^\mu v_\perp^\nu \sim g^{\mu\nu} \quad \longrightarrow \quad \boxed{v_\perp^\mu v_\perp^\nu v_\perp^\rho v_\perp^\sigma \sim g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}}$$

$$
\begin{aligned}
&T_1 = \epsilon_1 \cdot p_3\, \epsilon_2 \cdot p_1\, \epsilon_3 \cdot p_1\, \epsilon_4 \cdot p_2\,, \\
&T_2 = \epsilon_1 \cdot p_3\, \epsilon_2 \cdot p_1\, \epsilon_3 \cdot \epsilon_4\,, \quad T_3 = \epsilon_1 \cdot p_3\, \epsilon_3 \cdot p_1\, \epsilon_2 \cdot \epsilon_4\,, \quad T_4 = \epsilon_1 \cdot p_3\, \epsilon_4 \cdot p_2\, \epsilon_2 \cdot \epsilon_3\,, \\
&T_5 = \epsilon_2 \cdot p_1\, \epsilon_3 \cdot p_1\, \epsilon_1 \cdot \epsilon_4\,, \quad T_6 = \epsilon_2 \cdot p_1\, \epsilon_4 \cdot p_2\, \epsilon_1 \cdot \epsilon_3\,, \quad T_7 = \epsilon_3 \cdot p_1\, \epsilon_4 \cdot p_2\, \epsilon_1 \cdot \epsilon_2\,,
\end{aligned}
$$

$$T_8 = \epsilon_1 \cdot \epsilon_2\, \epsilon_3 \cdot \epsilon_4 + \epsilon_1 \cdot \epsilon_4\, \epsilon_2 \cdot \epsilon_3 + \epsilon_1 \cdot \epsilon_3\, \epsilon_2 \cdot \epsilon_4$$

SCATTERING WITH $N > 4$ EXTERNAL LEGS

For $N > 4$ the method becomes even simpler, because **momenta provide complete set of 4 vectors in d=4 dimensions!**

Take the prototypical case of **5-gluon scattering:**

$$g(p_1) + g(p_2) + g(p_3) + g(p_4) + g(p_5) \to 0$$

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Standard d-dimensional approach:

1. Rank-5 tensor out of $g^{\mu\nu}$, p_i^{μ}, $i = 1,\ldots,4$ contains **1724 tensor structures!**
2. Imposing **gauge invariance** reduced to **142 independent structures**
3. Projectors can *(painfully!)* be obtained inverting 142x142 matrix $\to \sim$ ***1GB of text file!***

SCATTERING WITH $N > 4$ EXTERNAL LEGS

$$g(p_1) + g(p_2) + g(p_3) + g(p_4) + g(p_5) \to 0$$

Typical tensors will be like:

$$T^{\mu_1\mu_2\mu_3\mu_4\mu_5} = p_i^{\mu_1} p_j^{\mu_2} p_k^{\mu_3} p_l^{\mu_4} p_s^{\mu_5}$$

$$T^{\mu_1\mu_2\mu_3\mu_4\mu_5} = p_i^{\mu_1} p_j^{\mu_2} p_k^{\mu_3} g^{\mu_4\mu_5}$$

$$T^{\mu_1\mu_2\mu_3\mu_4\mu_5} = p_i^{\mu_1} g^{\mu_2\mu_3} g^{\mu_4\mu_5}$$

SCATTERING WITH $N > 4$ EXTERNAL LEGS

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$$T^{\mu_1\mu_2\mu_3\mu_4\mu_5} = p_i^{\mu_1} p_j^{\mu_2} p_k^{\mu_3} p_l^{\mu_4} p_s^{\mu_5}$$

$$T^{\mu_1\mu_2\mu_3\mu_4\mu_5} = p_i^{\mu_1} p_j^{\mu_2} p_k^{\mu_3} g^{\mu_4\mu_5}$$

$$T^{\mu_1\mu_2\mu_3\mu_4\mu_5} = p_i^{\mu_1} g^{\mu_2\mu_3} g^{\mu_4\mu_5}$$

But since $p_1^\mu, \ldots, p_4^\mu$ are complete set in d=4, $g^{\mu\nu}$ **is not linear independent!**

I don't even need the construction that I have made for 4-point, I can drop all of them!

SCATTERING WITH $N > 4$ EXTERNAL LEGS

$$g(p_1) + g(p_2) + g(p_3) + g(p_4) + g(p_5) \to 0$$

Typical tensors will be like:

$$T^{\mu_1\mu_2\mu_3\mu_4\mu_5} = p_i^{\mu_1} p_j^{\mu_2} p_k^{\mu_3} p_l^{\mu_4} p_s^{\mu_5}$$

$$\cancel{T^{\mu_1\mu_2\mu_3\mu_4\mu_5} = p_i^{\mu_1} p_j^{\mu_2} p_k^{\mu_3} g^{\mu_4\mu_5}}$$

$$\cancel{T^{\mu_1\mu_2\mu_3\mu_4\mu_5} = p_i^{\mu_1} g^{\mu_2\mu_3} g^{\mu_4\mu_5}}$$

$p_1^\mu, \ldots, p_4^\mu \sim$ left with $2^5 = 32$ tensors $=$ # helicity amplitudes

Projectors $\sim$ 500 Kb versus $\sim$1Gb

Problem can be simplified further using symmetries/special combinations of these tensors!

Method used successfully for various **3 loop** $2 \to 2$ **calculations in QCD** and for **first full color calculation of a** $2 \to 3$ **scattering process** $q\bar{q} \to \gamma\gamma g$ **arXiv:2105.04585**

(See **G. Gambuti's** and **A. von Manteuffel's** talks)

CONCLUSIONS

- Tensor decomposition is a standard step towards calculation of multiloop scattering amplitudes
- Standard approach in CDR can be substantially simplified if one is interested in helicity amplitudes in tHV
- Decomposition proposed is 1-to-1 with # of helicity amplitudes, no spurious / evanescent structures are ever computed
- Particularly effective in the presence of multiple fermion lines or for $N > 4$ external legs
- Can be easily extended for different theories (masses, parity violating, etc)

THANK YOU FOR YOUR ATTENTION

BACK UP

PARTONIC CHANNEL $gg \to gg$

$gg \to gg$: most complicated channel [Caola, Chakraborty, Gambuti, Manteuffel, Tancredi, '21]

8 helicity amplitudes $\sim$ 8 form factors for each color ordered amplitude

$$\mathcal{A}^{a_1 a_2 a_3 a_4} = 4\pi\alpha_{s,b} \sum_{i=1}^{6} \mathcal{A}^{[i]} \mathcal{C}_i \longrightarrow \mathscr{A} = \sum_{j=1}^{8} \mathscr{F}_i T_i$$

$$\mathcal{C}_1 = \mathrm{Tr}[T^{a_1} T^{a_2} T^{a_3} T^{a_4}] + \mathrm{Tr}[T^{a_1} T^{a_4} T^{a_3} T^{a_2}] \quad \text{etc...}$$

Helicity amplitudes $\mathscr{A}_\lambda = s_\lambda H_\lambda$ where $\lambda = \{++++, -+++, +-++, etc\}$

$$s_{++++} = \frac{\langle 12\rangle\langle 34\rangle}{[12][34]}, \qquad s_{-+++} = \frac{\langle 12\rangle\langle 14\rangle[24]}{\langle 34\rangle\langle 23\rangle\langle 24\rangle}$$

$$s_{+-++} = \frac{\langle 21\rangle\langle 24\rangle[14]}{\langle 34\rangle\langle 13\rangle\langle 14\rangle}, \qquad s_{++-+} = \frac{\langle 32\rangle\langle 34\rangle[24]}{\langle 14\rangle\langle 21\rangle\langle 24\rangle}$$

$$s_{+++-} = \frac{\langle 42\rangle\langle 14\rangle[12]}{\langle 13\rangle\langle 23\rangle\langle 12\rangle}, \qquad s_{++--} = \frac{\langle 12\rangle[34]}{[12]\langle 34\rangle},$$

$$s_{+-+-} = \frac{\langle 13\rangle[24]}{[13]\langle 24\rangle}, \qquad s_{+--+} = \frac{\langle 14\rangle[23]}{[14]\langle 23\rangle}.$$

For example:

$$\longrightarrow H_{-+-+} = t^2 \left(\frac{\mathscr{F}_8}{su} - \frac{\mathscr{F}_3}{2s} + \frac{\mathscr{F}_6}{2u} - \frac{\mathscr{F}_1}{4} \right)$$

PARTONIC CHANNEL $q\bar{q} \to gg$

$q\bar{q} \to gg$: 4 helicity amplitudes ~ 4 form factors for each color ordered amplitude

[Caola, Chakraborty, Gambuti, Manteuffel, Tancredi, *To appear***]**

$$q(p_1) + \bar{q}(p_2) \to g(p_3) + g(p_4)\,, \quad \text{with} \quad p_i^2 = 0$$

To compute helicity amplitudes in standard way, start from generic tensor decomposition

$$\mathcal{A}(s,t) = \sum_{i=1}^{5} \mathcal{F}_i(s,t) T_i$$

$$T_i = \bar{u}(p_2)\,\Gamma_i^{\mu\nu}\,u(p_1)\epsilon_{3,\mu}\epsilon_{4,\nu}$$

$$\Gamma_1^{\mu\nu} = \gamma^\mu p_2^\nu\,, \quad \Gamma_2^{\mu\nu} = \gamma^\nu p_1^\mu\,,$$
$$\Gamma_3^{\mu\nu} = \not{p}_3\, p_1^\mu p_2^\nu\,, \quad \Gamma_4^{\mu\nu} = \not{p}_3\, g^{\mu\nu}$$
$$\Gamma_5^{\mu\nu} = \gamma^\mu \not{p}_3\, \gamma^\nu\,.$$

PARTONIC CHANNEL $q\bar{q} \to gg$

$q\bar{q} \to gg$: 4 helicity amplitudes $\sim$ 4 form factors for each color ordered amplitude

[Caola, Chakraborty, Gambuti, Manteuffel, Tancredi, *To appear*]

$$q(p_1) + \bar{q}(p_2) \to g(p_3) + g(p_4)\,, \quad \text{with} \quad p_i^2 = 0$$

To compute helicity amplitudes in standard way, start from generic tensor decomposition

$$\mathcal{A}(s,t) = \sum_{i=1}^{5} \mathcal{F}_i(s,t) T_i$$

$$T_i = \bar{u}(p_2)\,\Gamma_i^{\mu\nu}\,u(p_1)\epsilon_{3,\mu}\epsilon_{4,\nu}$$

$$\Gamma_1^{\mu\nu} = \gamma^\mu p_2^\nu\,, \quad \Gamma_2^{\mu\nu} = \gamma^\nu p_1^\mu\,,$$
$$\Gamma_3^{\mu\nu} = \not{p}_3\, p_1^\mu p_2^\nu\,, \quad \Gamma_4^{\mu\nu} = \not{p}_3\, g^{\mu\nu}$$
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Not linearly independent in d=4

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Not linearly independent in d=4

$$\lim_{d\to 4}\left(T_5 - \frac{u}{s}T_1 + \frac{u}{s}T_2 - \frac{2}{s}T_3 + T_4\right) = 0$$

PARTONIC CHANNEL $q\bar{q} \to gg$

Helicity amplitudes in tHV can be computed by fixing helicities on the tensors in d=4

$$\mathscr{A}_{\lambda_q\lambda_3\lambda_4}(s,t) = \sum_{i=1}^{5} \mathscr{F}_i(s,t)\left[T_i\right]_{\lambda_q\lambda_3\lambda_4, d=4}$$

One of five tensors is not independent in d=4

$$\lim_{d\to 4}\left(T_5 - \frac{u}{s}T_1 + \frac{u}{s}T_2 - \frac{2}{s}T_3 + T_4\right) = 0$$

We can identify one **evanescent tensor structure**

$$\overline{T}_i = T_i\,, \quad i = 1, \ldots, 4\,,$$

$$\overline{T}_5 = T_5 - \frac{u}{s}T_1 + \frac{u}{s}T_2 - \frac{2}{s}T_3 + T_4$$

PARTONIC CHANNEL $q\bar{q} \to gg$

In new basis of tensors, **by definition** only first four contribute to hel amplitudes

$$\mathscr{A}_{\lambda_q\lambda_3\lambda_4}(s,t) = \sum_{i=1}^{5} \overline{\mathscr{F}}_i(s,t)\left[\overline{T}_i\right]_{\lambda_q\lambda_3\lambda_4,d=4} = \sum_{i=1}^{4} \overline{\mathscr{F}}_i(s,t)\left[\overline{T}_i\right]_{\lambda_q\lambda_3\lambda_4,d=4}$$

Fifth tensor required to recover **CDR result** starting at $\mathscr{O}(\epsilon)$

$$\begin{aligned}
\sum \mathcal{A}^{(n)}\mathcal{A}^{(m)*} = {} & \frac{2(s-t)t}{u}\,\overline{\mathcal{F}}_4^{(n)}\left[-s\overline{\mathcal{F}}_1^{(m)*} + s\overline{\mathcal{F}}_2^{(m)*} - st\overline{\mathcal{F}}_3^{(m)*} - \frac{2\overline{\mathcal{F}}_4^{(m)*}\left(-s^2-t^2+u^2\epsilon\right)}{(s-t)}\right] \\
& + \frac{2st}{u}\,\overline{\mathcal{F}}_1^{(n)}\left[-2s(\epsilon-1)\overline{\mathcal{F}}_1^{(m)*} - s\overline{\mathcal{F}}_2^{(m)*} + st\overline{\mathcal{F}}_3^{(m)*} - \overline{\mathcal{F}}_4^{(m)*}(s-t)\right] \\
& + \frac{2st}{u}\,\overline{\mathcal{F}}_2^{(n)}\left[-s\overline{\mathcal{F}}_1^{(m)*} - 2s(\epsilon-1)\overline{\mathcal{F}}_2^{(m)*} - st\overline{\mathcal{F}}_3^{(m)*} + \overline{\mathcal{F}}_4^{(m)*}(s-t)\right] \\
& + \frac{2st^2}{u}\,\overline{\mathcal{F}}_3^{(n)}\left[s\overline{\mathcal{F}}_1^{(m)*} - s\overline{\mathcal{F}}_2^{(m)*} + st\overline{\mathcal{F}}_3^{(m)*} - \overline{\mathcal{F}}_4^{(m)*}(s-t)\right] \\
& + 4tu\,\epsilon\,(2\epsilon-1)\overline{\mathcal{F}}_5^{(m)*}\overline{\mathcal{F}}_5^{(n)}\,,
\end{aligned}$$