Status of double virtual NNLO QCD corrections for high multiplicity processes

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Outline

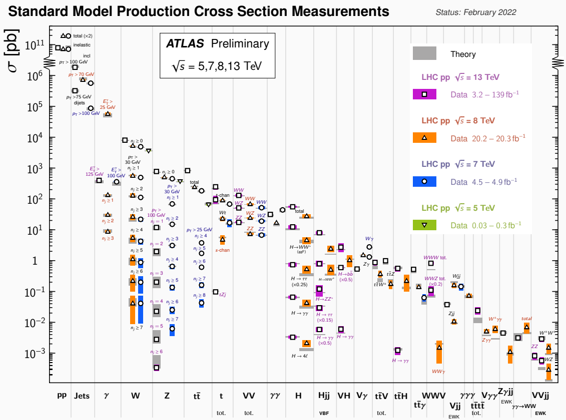
1. Introduction
2. Five-point two-loop amplitudes: challenges
3. Overview of recent results
4. One-mass pentagon functions

Introduction

Era of precision physics at the LHC

[ATL-PHYS-PUB-2022-009]

- LHC will still be running for years, future colliders
- Many observables probed at percent level precision
- Discovery via precision: search anomalous deviations from Standard Model



Theory must reach comparable precision target!

At least NNLO QCD and NLO EW corrections generally required

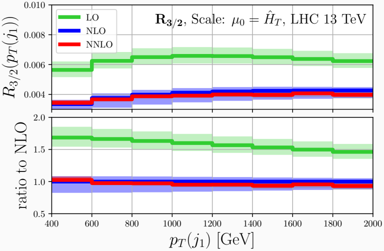
($\oplus$ parton shower, resummation, etc.)

Why fixed order?

$$\sigma \sim \overbrace{\underbrace{\sigma_{\text{LO}}\Big(1+\alpha_s\sigma^{(1)}}_{\text{NLO}}+\alpha_s^2\sigma^{(2)}}^{\text{NNLO}}+\mathcal{O}(\alpha_s^3)\Big)$$

naively $\sim 1\%$

- Stabilization of perturbation series



[Czakon, Mitov, Poncelet '21]

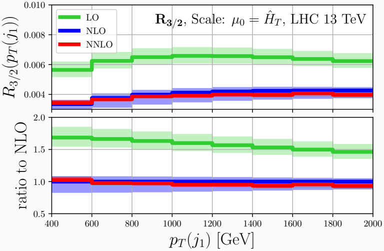
[Michal's talk]

Why fixed order?

$$\sigma \sim \overbrace{\underbrace{\sigma_{\text{LO}}(1+\alpha_s\sigma^{(1)}}_{\text{NLO}} + \alpha_s^2\sigma^{(2)}}^{\text{NNLO}} + \mathcal{O}(\alpha_s^3))$$

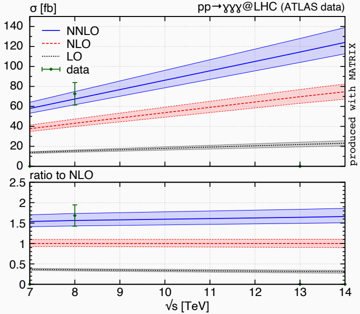
naively $\sim 1\%$

- Stabilization of perturbation series
- Giant K factors



[Czakon, Mitov, Poncelet '21]

[Michal's talk]



[Kallweit, VS, Wiesemann '20]

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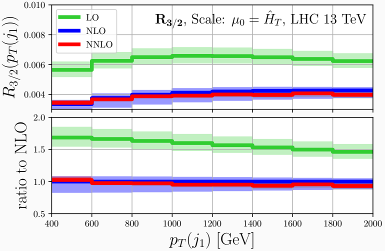

[Czakon, Mitov, Poncelet '21]

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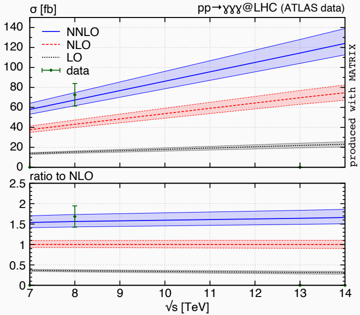

[Kallweit, VS, Wiesemann '20]

Clean theoretical prediction

NNLO QCD: status

State of the art

- Many $2 \to 2$ NNLO QCD known
- Current frontier $2 \to 3$, first cross sections

[Michal's talk] [Czakon, Mitov, Poncelet '21]
[Matteo's talk] [Chen, Gehrmann, Glover, Huss, Marcoli '21]

[Chawdry, Czakon, Mitov, Poncelet '21]
[Badger, Gehrmann, Marcoli, Moodie '21]

[Chawdry, Czakon, Mitov, Poncelet '19]
[Kallweit, VS, Wiesemann '20]
[now public: Stefan Kallweit's talk]

$$\sigma^{F+X}_{\mathrm{NNLO}} = \sigma^{F+X}_{\mathrm{NLO}} + \int_{\Phi_{F+2}} \mathrm{d}\sigma_{\mathrm{RR}} + \int_{\Phi_{F+1}} \mathrm{d}\sigma_{\mathrm{RV}} + \int_{\Phi_{F}} \mathrm{d}\sigma_{\mathrm{VV}}$$

IR divergences

Two-loop amplitudes

[Lorenzo Magnea's talk]
[Michal's talk] [Matteo's talk]

Les Houches wishlist 2019 [2003.01700]

process	known	desired	
$pp \to 3\,\mathrm{jets}$	$\mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$	$\boxed{\mathrm{NNLO_{QCD}}}$	α_s running
$\vdots$	$\vdots$	$\vdots$	
$pp \to \gamma\gamma + j$	$\mathrm{NLO_{QCD}}$ $\mathrm{NLO_{EW}}$	$\boxed{\mathrm{NNLO_{QCD}}} + \mathrm{NLO_{EW}}$	$H(\to \gamma\gamma)$ p_T spectrum
$pp \to \gamma\gamma\gamma$	$\boxed{\mathrm{NNLO_{QCD}}}$		Anomalous gauge couplings
$\vdots$	$\vdots$	$\vdots$	
$pp \to V + 2j$	$\mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$ $\mathrm{NLO_{EW}}$	$\boxed{\mathrm{NNLO_{QCD}}}$	QCD precision physics
$pp \to V + b\bar{b}$	$\mathrm{NLO_{QCD}}$	$\boxed{\mathrm{NNLO_{QCD}}} + \mathrm{NLO_{EW}}$	$H \to b\bar{b}$ decay
$pp \to H + 2j$	$\mathrm{NLO_{HTL}} \otimes \mathrm{LO_{QCD}}$ $\mathrm{N^3LO^{(VBF^*)}_{QCD}}$ (incl.) $\mathrm{NNLO^{(VBF^*)}_{QCD}}$ $\mathrm{NLO^{(VBF)}_{EW}}$	$\boxed{\mathrm{NNLO_{HTL}} \otimes \mathrm{NLO_{QCD}}} + \mathrm{NLO_{EW}}$ $\mathrm{NNLO^{(VBF)}_{QCD}} + \mathrm{NLO^{(VBF)}_{EW}}$	VBF studies
$\vdots$	$\vdots$	$\vdots$	
$pp \to t\bar{t} + j$	$\mathrm{NLO_{QCD}}$ (w/ decays) $\mathrm{NLO_{EW}}$	$\boxed{\mathrm{NNLO_{QCD}}} + \mathrm{NLO_{EW}}$ (w/ decays)	Top mass
$pp \to t\bar{t} + Z$	$\mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$ (w/ decays)	$\boxed{\mathrm{NNLO_{QCD}}} + \mathrm{NLO_{EW}}$ (w/ decays)	Anomalous EW couplings
$pp \to t\bar{t} + W$	$\mathrm{NLO_{QCD}}$ $\mathrm{NLO_{EW}}$	$\boxed{\mathrm{NNLO_{QCD}}} + \mathrm{NLO_{EW}}$ (w/ decays)	
$pp \to t\bar{t} + H$	$\mathrm{NLO_{QCD}} + \mathrm{NLO_{EW}}$	$\boxed{\mathrm{NNLO_{QCD}}}$	Top Yukawa

Five-point two-loop amplitudes: challenges

Scattering amplitudes for phenomenology

Rational/algebraic
Feynman rules, particle content
Integral & tensor reduction

Transendental
Scattering kinematics
Feynman integrals

$$\mathcal{A} = \sum_{\mathbf{i}} \boxed{r_{\mathbf{i}}(\mathbf{s}, \epsilon)}\, \boxed{\mathbf{g}^{\mathbf{i}}(\mathbf{s}, \epsilon)}$$

Compact analytic form

Function basis
~~redundancy~~ ~~dim. reg. artefacts~~
analytic cancellation of IR divergences
numerical control

Goal: fast and stable evaluation over whole physical phase space

Transcendental part: the canonical way

Feynman integrals

Initial conditions
physics, limits, PSLQ

Canonical DE [Henn '13]

$$\mathrm{d}\vec{g} = \epsilon\, A\, \vec{g}$$

Letters of alphabet

Rational matrix

$$A = \sum_i \mathrm{d}\log W_i(\mathbf{s})\; A_i$$

Multiple Polylogarithms
(MPLs, GPLs)

Transcendental part: the canonical way

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Multiple Polylogarithms
(MPLs, GPLs)

Square roots ubiquitous $\dfrac{a+\sqrt{r}}{a-\sqrt{r}}$

- proliferation of spurious branch cuts
- analytic continuation
- explosion of number of MPLs

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Workarounds

Ansatz fitting, parametric integration
[Heller, von Manteuffel, Schabinger '19][Heller '21]
[Bonetti, Panzer, Smirnov, Tancredi '20]
[Kreer, Weinzierl '21][Duhr, Smirnov, Tancredi '21]

Simplified DE
[Papadopoulos, Tommasini, Wever '15] [Papadopoulos '14]
[Canko, Papadopoulos, Syrrakos '20]

⚠ Finding good representation in physical regions very difficult (if not impossible?!)
[Papadopoulos, Tommasini, Wever '15] [Gehrmann, Henn, Lo Presti '18]
[Canko, Papadopoulos, Syrrakos '20] [Duhr, Smirnov, Tancredi '21]

Transcendental part: the five-point massless story

Analytic work:
canonical DE, some MPL results

[Gehrmann, Henn, Lo Presti '15] [Papadopoulos, Tommasini, Wever '15] [Abreu, Dixon, Herrmann, Page, Zeng '18] [Abreu, Page, Zeng '18] [Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia '18]

First function basis (planar)
[Gehrmann, Henn, Lo Presti '18]

NNLO QCD corrections to three-photon production at the LHC
[Chawdry, Czakon, Mitov, Poncelet '19]

two-loop still difficult!

Lesson:
getting function basis right is important

Full function space (planar revised)
[Chicherin, VS '20]
`PentagonFunctions++`

Next-to-Next-to-Leading Order Study of Three-Jet Production at the LHC
Michal Czakon (Aachen, Tech. Hochsch.), Alexander Mitov (Cambridge U.), Rene Poncelet (Cambridge U.)
Jun 9, 2021

NNLO QCD corrections to diphoton production with an additional jet at the LHC
Herschel A. Chawdhry (Oxford U., Ctr. Quantum Comp. and Oxford U., Theor. Phys.), Michal Czakon (Aachen, Tech. Hochsch.), Alexander Mitov (Cambridge U.), Rene Poncelet (Cambridge U.)
May 14, 2021

Triphoton production at hadron colliders in NNLO QCD
Stefan Kallweit (Milan Bicocca U. and INFN, Milan Bicocca), Vasily Sotnikov (Munich, Max Planck Inst.), Marius Wiesemann (Munich, Max Planck Inst.)
Oct 9, 2020

Automation of antenna subtraction in colour space: gluonic processes
Xuan Chen (KIT, Karlsruhe and KIT, Karlsruhe, IKP), Thomas Gehrmann (Zurich U.), Nigel Glover (Durham U., IPPP), Alexander Huss (CERN), Matteo Marcoli (Zurich U.)
Mar 25, 2022

Next-to-leading order QCD corrections to diphoton-plus-jet production through gluon fusion at the LHC
Simon Badger (INFN, Turin and Turin U.), Thomas Gehrmann (Zurich U.), Matteo Marcoli (Zurich U.), Ryan Moodie (Durham U., IPPP)
Sep 24, 2021

Can we do the same beyond massless?

Rational coefficients: algebraic complexity

[Lorenzo Tancredi's talk]

Integrand $\displaystyle\sum_i \frac{m_i(\mathbf{s},\epsilon;\,\ell)}{\prod_j \rho_{i,j}}$

Tensor & IBP reduction

$$\sum_i c_i(\mathbf{s},\epsilon)\;\mathcal{I}_i$$

pure MIs

UV & IR renormalization

$$\sum_{\vec{\mathbf{i}}} r_{\vec{\mathbf{i}}}(\mathbf{s})\;\boldsymbol{g}^{\vec{\mathbf{i}}}(\mathrm{s})\;+\mathcal{O}(\epsilon)$$

pentagon functions

Key bottleneck
intermediate expression swell

Central lesson

- Coefficients $r_{\vec{\mathbf{i}}}$ simple
- Bypass complexity with exact numerics (finite fields)
- Reconstruct analytic form from numerical samples

[von Manteuffel, Schabinger '14] [Peraro '16]

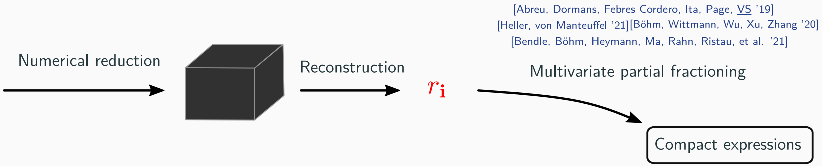
`FiniteFlow` [Peraro '19]
[Badger, Brønnum-Hansen, Hartanto, Peraro '18]
[Badger, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, Zoia '19]

Caravel
[Abreu, Dormans, Febres Cordero, Ita, Kraus, Page, Pascual, Ruf, VS '20]

Better IBPs: important insights from algebraic geometry

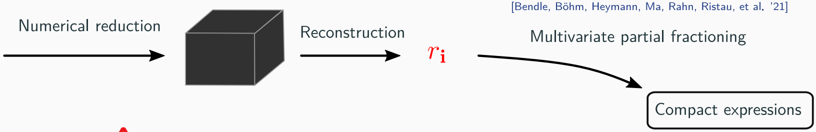
[Gluza, Kadja, Kosower '11] [Ita '15] [Larsen, Zhang '15]
[Georgoudis, Larsen, Zhang '16] [Böhm, Georgoudis, Larsen, Schulze, Zhang '17]
[Agarwal, Jones, von Manteuffel '20]

Analytic results from finite-field evaluations



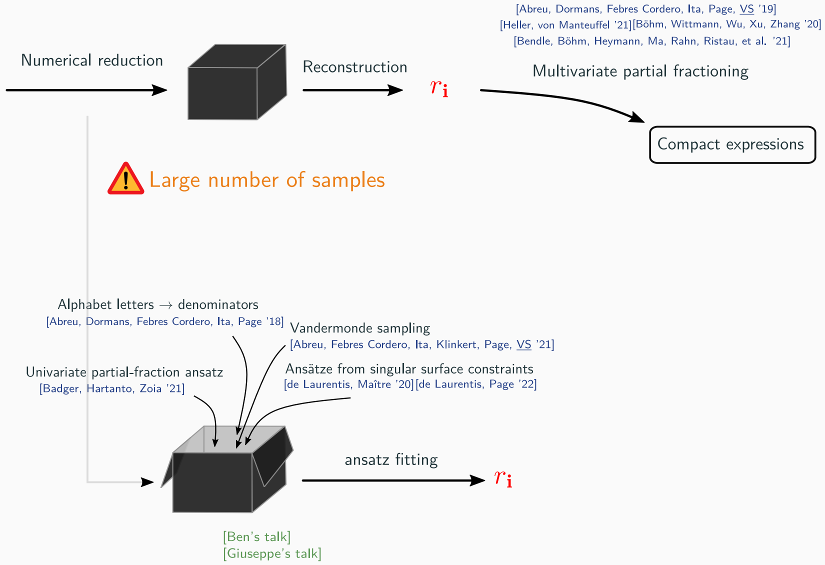
Analytic results from finite-field evaluations

[Abreu, Dormans, Febres Cordero, Ita, Page, VS '19]
[Heller, von Manteuffel '21][Böhm, Wittmann, Wu, Xu, Zhang '20]
[Bendle, Böhm, Heymann, Ma, Rahn, Ristau, et al. '21]

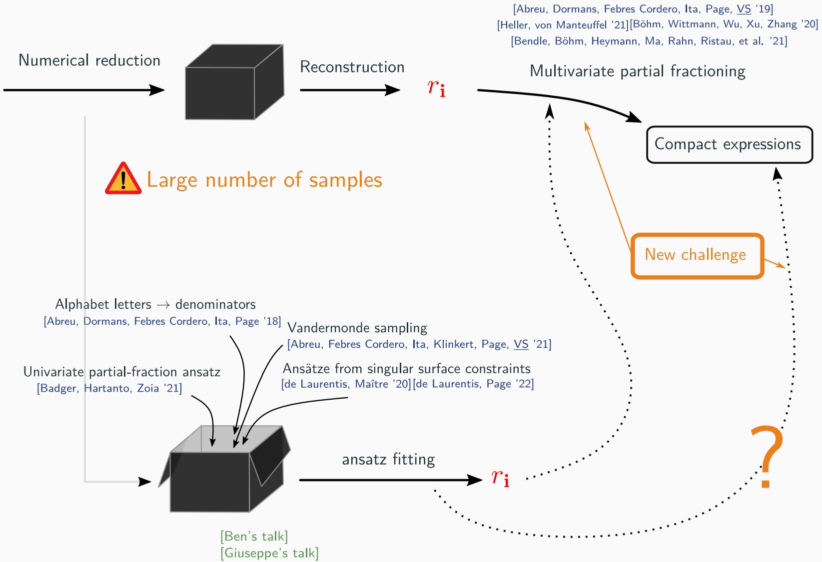


⚠ Large number of samples

Analytic results from finite-field evaluations



Analytic results from finite-field evaluations



Overview of recent results

Summary two-loop five-point massless amplitudes

	Comment	Complete analytic results	Public numerical code	Cross sections
$pp \to jjj$	l.c.	[1]	[1]	[8, 9]
$pp \to \gamma\gamma j$	l.c.*	[2, 3]	[2]	[10]
$pp \to \gamma\gamma\gamma$	l.c.*	[4, 5]	[4]	[11, 12]
$pp \to \gamma\gamma j$		[6]		
$gg \to \gamma\gamma g$	NLO loop induced	[7]	[7]	[13]

[1] [Abreu, Febres Cordero, Ita, Page, VS '21]

[2] [Agarwal, Buccioni, von Manteuffel, Tancredi '21]

[3] [Chawdry, Czakon, Mitov, Poncelet '21]

[4] [Abreu, Page, Pascual, VS '20]

[5] [Chawdry, Czakon, Mitov, Poncelet '20]

[6] [Agarwal, Buccioni, von Manteuffel, Tancredi '21]

[7] [Badger, Brønnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro, Zoia '21]

[8] [Czakon, Mitov, Poncelet '21]

[9] [Chen, Gehrmann, Glover, Huss, Marcoli '21]

[10] [Chawdry, Czakon, Mitov, Poncelet '21]

[11] [Chawdry, Czakon, Mitov, Poncelet '19]

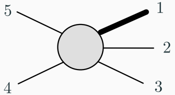
[12] [Kallweit, VS, Wiesemann '20]

[13] [Badger, Gehrmann, Marcoli, Moodie '21]

One-mass Feynman integrals and functions

One-mass kinematics

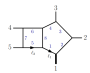
e.g. $pp \to Vjj$



$p_1^2, \; s_{12}, \; s_{23}, \; s_{34}, \; s_{45}, \; s_{15}$

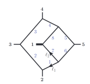
One-mass Feynman integrals and functions

Canonical DE

Planar 

[Abreu, Ita, Moriello, Page, Tschernow, Zeng '20]

[Ben's talk]

Hexa-box 

[Abreu, Ita, Page, Tschernow '21]

GPL results

[Costas's talk]

[Papadopoulos, Tommasini, Wever '15]
[Canko, Papadopoulos, Syrrakos '20] [Syrrakos '20]
[Kardos, Papadopoulos, Smirnov, Syrrakos, Wever '22]

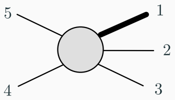
Function basis (planar)

[Badger, Hartanto, Zoia '21] color-ordered, numerical evaluation

[Chicherin, VS, Zoia '21] ← `PentagonFunctions++`

One-mass kinematics

e.g. $pp \to Vjj$

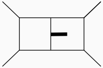


$p_1^2, \ s_{12}, \ s_{23}, \ s_{34}, \ s_{45}, \ s_{15}$

Semi-numerical DE solution

`DiffExp` [Moriello '19] [Hidding '20]
`AMFLow` [Liu, Ma, Wang '17] [Liu, Ma '21]

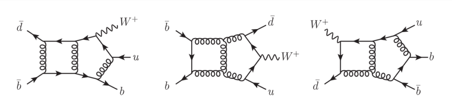
Initial values, validation, small scale sampling

 Missing: W.I.P. 

One-mass two-loop five-point amplitudes

$pp \to Wb\bar{b}$

[Badger, Hartanto, Zoia '21]



Leading color ($N_c \to \infty$), unpolarized, on-shell W, 5FNS (b massless)

Setup: Feynman diagrams, `FiniteFlow` (Laporta IBPs), custom function basis (iterated integrals, similar to [Chicherin, VS '20]) numerical evaluation with `DiffExp`

Reconstruction: univariate partial-fraction ansatz (size 45k)

$pp \to Hb\bar{b}$

[Badger, Hartanto, Kryś, Zoia '21]

Leading color($N_c \to \infty$, $N_f \sim N_c$), 5FNS (b massless), bottom Yukawa finite

Setup: as in $pp \to Wb\bar{b}$

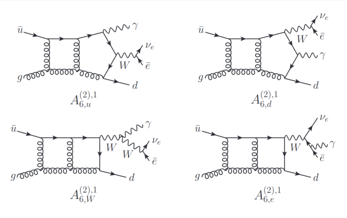
Reconstruction: univariate partial-fraction ansatz (size 30k)

One-mass two-loop five-point amplitudes

$pp \to W(l\nu)\gamma j$

[Badger, Hartanto, Kryś, Zoia '21]

Leading color $(N_c \to \infty)^\star$



Setup: Feynman diagrams, `FiniteFlow` (Laporta IBPs), one-mass pentagon functions

Reconstruction: functional reconstruction [Peraro '16]

sample size 1000k

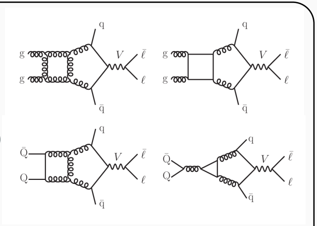
$pp \to W(l\nu)jj$

[Ben's talk]

[Abreu, Febres Cordero, Ita, Klinkert, Page, VS '21]

Leading color $N_c \to \infty$, $N_f \sim N_c$

$(pp \to Z(l\bar{l})jj$ with $N_c \to \infty)$



Numerical benchmark
[Hartanto, Badger, Brønnum-Hansen, Peraro '19]

Setup: numerical unitarity
one-mass pentagon functions
Caravel

Reconstruction: univariate partial-fraction ansatz, Vandermonde sampling
ansatz size 500k

One-mass pentagon functions

Basis construction

[Chicherin, VS, Zoia '21] (see also [Chicherin, VS '20] [Badger, Hartanto, Zoia '21])

Feynman integrals

Initial conditions
physics, limits, PSLQ

Canonical DE

$$\mathrm{d}\vec{g} = \epsilon\, A\, \vec{g}$$

$$A = \sum_i \mathrm{d}\log W_i(\mathbf{s})\ A_i$$

Multiple Polylogarithms
(MPLs, GPLs)

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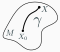
weight = length = ϵ order

Chen iterated integrals [Chen '77]

$$[W_1, \ldots, W_n]_\gamma =$$



$$\int_0^1 \mathrm{d}\log W_n(t_n)\ \ldots\ \int_0^{t_2} \mathrm{d}\log W_n(t_1)$$

Vector subspace, weight-graded

$$\mathbf{G} = \bigoplus_w \mathbf{G}^{(w)}$$

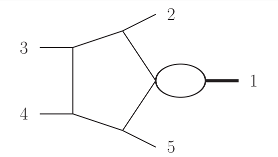
+ shuffle product

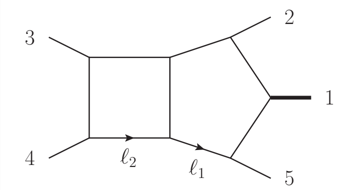
$$\mathbf{G}^{w_1} \times \mathbf{G}^{(w_2)} \mapsto \mathbf{G}^{(w_1+w_2)}$$

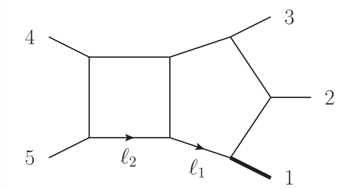
$$[W_1, \ldots, W_r]_\gamma [W_{r+1}, \ldots, W_n]_\gamma = \sum_{\mathbf{i} \in \text{shuffles}} [W_{i_1}, \ldots, W_{i_n}]_\gamma$$

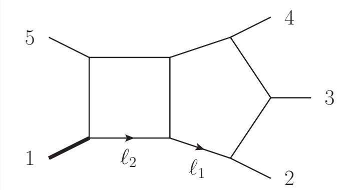
Basis in $\mathbf{G}^{(w)}$ mod products

- ✓ complete
- ✓ non-redundant
- ✓ amplitudeology friendly









Loop order	Master integrals, σ_{id}	Master integrals, S_4
1	13	56
2	167	1361

The alphabet

[Abreu, Ita, Moriello, Page, Tschernow, Zeng '20]

$$A = \sum_i \mathrm{d}\log W_i(\mathbf{s})\ A_i$$

156 letters, 108 contribute up to weight 4

Square roots:

$$\begin{aligned}
\Delta_5 &= 16\, G(p_1, p_2, p_3, p_4) \\
\Delta_3^{(1)} &= -4\, G\,(p_1, p_2 + p_3) \\
\Delta_3^{(2)} &= -4\, G\,(p_1, p_2 + p_4) \\
\Delta_3^{(3)} &= -4\, G\,(p_1, p_3 + p_4)
\end{aligned}$$

Rational

- 27 linear, $s_{12} - p_1^2$
- 54 quadratic, $s_{12}s_{23} + p_1^2 s_{45} - s_{12}s_{45}$

One square root

- 3+12 letters with $\sqrt{\Delta_3^{(i)}}$, $\frac{p_1^2 - s_{23} + s_{45} + \sqrt{\Delta_3^{(1)}}}{p_1^2 - s_{23} + s_{45} - \sqrt{\Delta_3^{(1)}}}$
- 1+8 letters with $\sqrt{\Delta_5}$

Two square roots

3 letters with $\sqrt{\Delta_5}$ and one of $\sqrt{\Delta_3^{(i)}}$, $\frac{\Omega^{--}\Omega^{++}}{\Omega^{-+}\Omega^{+-}}$
$\Omega^{\pm\pm} = s_{12}s_{15} - s_{12}s_{23} - s_{15}s_{45} \pm s_{34}\sqrt{\Delta_3^{(1)}} \pm \sqrt{\Delta_5}$

Basis features

Weight	Lin. indep.	Irreducible	Cyclic
1	11	11	6
2	86	25	8
3	483	145	31
4	1187	675	113

cf. 1417 master integrals

- constants except $i\pi$ and ζ_3 absorbed into basis functions

New

- permutation orbits $\sigma\left(f_i^{(w)}\right) \to \sum_j c_{ij} f_j^{(w)} + \ldots$
$\implies$ crossing between channels simple

$$\left\{f_i^{(3)}\right\}_{i=1}^{145} : \quad \overbrace{1, \ldots, 29, \underbrace{30, 31}_{\mathcal{S}}}^{\text{cyclic}}, \overbrace{32, \ldots, 114, \underbrace{115, \ldots, 128}_{\mathcal{S}}, 129, \ldots, 145}^{\text{noncyclic}} .$$

$$\left\{f_i^{(4)}\right\}_{i=1}^{675} : \quad \overbrace{1, \ldots, 67, \underbrace{68, \ldots, 106}_{\mathcal{S}}, \underbrace{107, \ldots, 112}_{}, \underbrace{113}_{\mathcal{S}, \sqrt{\Delta_5}}}^{\text{cyclic}},$$

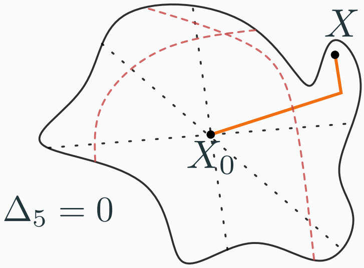
$$\overbrace{114, \ldots, 441, \underbrace{442, \ldots, 664}_{\mathcal{S}}, \underbrace{665, \ldots, 672}_{\sqrt{\Delta_5}}, \underbrace{673, \ldots, 675}_{\mathcal{S}, \sqrt{\Delta_5}}}^{\text{non-cyclic}}$$

drop from amplitudes
[Badger, Hartanto, Zoia '21]

drop from finite remainders
[Chicherin, Henn, Papathanasiou '20]

$\mathcal{S}$ — spurious singularities

Numerical evaluation



Weights 1 and 2

Well-defined combinations of $\log$, Li_2 functions

Weights 3 and 4

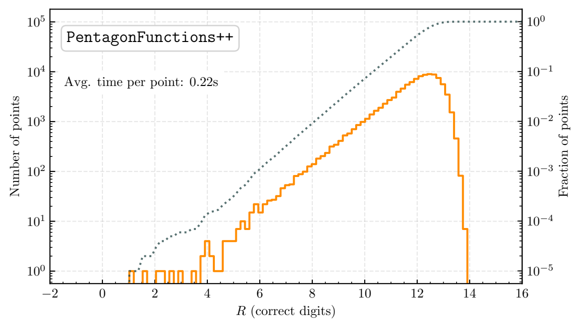
- Numerical one-fold integration [Caron-Huot, Henn '14] of analytic integrands
  $\implies$ exponential convergence [Takahasi, Mori '73]
- No crossing of physical thresholds $\implies$ no analytic continuation needed
- Dedicated series expansions around spurious singularities

Numerical performance

Evaluation of all functions: any one mass planar five-point amplitude in all "crossings"

Sample over physical phase space

Precision	Digits	Timing (s)
double	12	0.19
quadruple	28	159
octuple	60	1695



(vs. quad precision targets)

[Chicherin, VS, Zoia '21]

Available as a C++ library `PentagonFunctions++`

https://gitlab.com/pentagon-functions/PentagonFunctions-cpp

(also `Mathematica` interface)

Beyond planar

Hexa-box alphabet [Abreu, Ita, Page, Tschernow '21]

204 letters, all appear at weight 4 ($+96$ from planar)

6 new square roots:

$$\Sigma_5^{(1)} = (s_{12}s_{15} - s_{12}s_{23} - s_{15}s_{45} + s_{34}s_{45} + s_{23}s_{34})^2 - 4s_{23}s_{34}s_{45}(s_{34} - s_{12} - s_{15})$$
$$+ \ 5 \text{ permutations}$$

$6 + 24$ letters with $\sqrt{\Sigma_5^{(i)}}$,

$$\frac{s_{12}s_{15} - s_{45}s_{15} - s_{12}s_{23} - s_{23}s_{34} + s_{34}s_{45} - \sqrt{\Sigma_5^{(1)}}}{s_{12}s_{15} - s_{45}s_{15} - s_{12}s_{23} - s_{23}s_{34} + s_{34}s_{45} + \sqrt{\Sigma_5^{(1)}}}$$

6 letters with $\sqrt{\Delta_5}$ and one of $\sqrt{\Sigma_5^{(i)}}$,

$$\frac{\tilde{\Omega}^{--}\tilde{\Omega}^{++}}{\tilde{\Omega}^{-+}\tilde{\Omega}^{+-}},$$

$$\tilde{\Omega}^{\pm\pm} = p_1^2 s_{34} \pm \sqrt{\Delta_5} \pm \sqrt{\Sigma_5^{(1)}}$$

Beyond planar

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$$\tilde{\Omega}^{\pm\pm} = p_1^2 s_{34} \pm \sqrt{\Delta_5} \pm \sqrt{\Sigma_5^{(1)}}$$

- Nightmare for rationalization-based methods
- A minor inconvenience for our approach:
  Σ_5 can vanish inside physical phase space $\implies$ new spurious divergences

Conclusions & Outlook

Conclusions & Outlook

Two-loop amplitudes remain major bottleneck in $2 \to 3$ NNLO QCD computations

- Leading color for jet and photon production known, full color still in progress
- First results beyond massless scattering
  - ramp up in complexity, but techniques not stretched to limits
  - planar function basis available, reliable evaluation
  - analytic two-loop amplitudes for Wjj, $Wj\gamma$, $Hb\bar{b}$

Outlook

Need NNLO corrections for Vjj, $Vb\bar{b}$, Hjj, $t\bar{t}j$, $t\bar{t}V$, $t\bar{t}H$, etc.

- Full two-loop five-point one-mass function space
- Top quarks in loops: function bases with non-logarithmic kernels?
- N^{3}LO applications

Looking forward to more exciting NNLO phenomenology!

Acknowledgments

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European Research Council
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Swiss National Science Foundation

Backup

Initial values

We choose an initial point $X_0 \in \mathcal{P}_{45}^{+}$,

$$X_0 := \left(p_1^2 = 1\,, s_{12} = 3\,, s_{23} = 2\,, s_{34} = -2\,, s_{45} = 7\,, s_{15} = -2\right)\,,$$

which satisfies the following requirements:

1. X_0 introduces the minimal number of distinct prime factors.
2. X_0 is invariant under the exchanges of momenta $2 \leftrightarrow 3$ and $4 \leftrightarrow 5$ (automorphisms of $\mathcal{P}_{45}$).
3. The four linear letters which have indefinite sign vanish at X_0.

Algebraic relations between initial values required.

- Numerical evaluation of available GPL expressions [Canko, Papadopoulos, Syrrakos '20], [Syrrakos '20] to 3000 digits
- Relations from PSLQ $\Longrightarrow$ generating set

	Linear span ($\oplus$ products)		Irreducible	
Weight	Re	Im	Re	Im
1	4*	1	4*	1
2	12	4	5	0
3	67	23	23	7
4	305	135	90	40

Step 1: canonical DE

We start with canonical DEs

$$\mathrm{d}\vec{g}_{\tau,\sigma}(X) = \epsilon\,\mathrm{d}\tilde{A}_{\tau,\sigma}(X)\,\vec{g}_{\tau,\sigma}(X)\,,$$
$$\mathrm{d}\tilde{A}_{\tau,\sigma}(X) = \sum_{i\in\mathbb{A}} a^{(i)}_{\tau,\sigma}\,\mathrm{d}\log W_i(X)\,.$$

for all relevant integral families τ and all permutations σ.

E.g. for planar five-point one-mass $\tau = \{\mathrm{1L}, \mathrm{zmz}, \mathrm{zzm}, \mathrm{zzz}, \mathrm{1L}^2\}$, $\sigma \in S_4$.

Step 2: solution through iterated integrals

1. Choose a "good" base point X_0 for all τ and σ inside a physical phase space $\mathcal{P}_{ij}$, $ij \to kl \ldots$

2. Determine initial values $\vec{g}^{(w)}_{\tau,\sigma}(X_0)$ and algebraic relations between them

3. Write solutions DEs along a generic path $\gamma(t)$ ($\gamma(0) = X_0$) entirely within $\mathcal{P}_{ij}$ through iterated integrals

$$\vec{g}^{(w)}_{\tau,\sigma} = \sum_{w'=0}^{w} \sum_{i_1,\ldots,i_{w'} \in \mathbb{A}} a^{(i_1)}_{\tau,\sigma} \cdot a^{(i_2)}_{\tau,\sigma} \cdots a^{(i_{w'})}_{\tau,\sigma} \cdot \vec{g}^{(w-w')}_{\tau,\sigma}(X_0) \; \Big[W_{i_1}, \ldots, W_{i_{w'}}\Big]_{X_0}$$

Step 2: solution through iterated integrals

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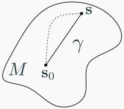
- $\bigcup_{\tau,\sigma} \vec{g}^{(w)}_{\tau,\sigma}$ spans a vector space at each weight ω
- No analytic continuation required by construction

Iterated integrals

[Chen '77] (see also *"Iterated integrals in QFT"* [Brown '11])

Let $\omega_1, \ldots, \omega_n$ be differential 1-forms on M (phase space), and path $\gamma : [0,1] \to M$. Pull the forms back on the path $\omega_i(\mathbf{s}) \xrightarrow{\gamma^\star} w_i(t)\,\mathrm{d}t$. Iterated integrals iints are



$$I_\gamma[\omega_1, \ldots, \omega_n] = \int_0^1 w_n(t_n)\,\mathrm{d}t_n \ldots \int_0^{t_2} w_1(t_1)\,\mathrm{d}t_1 \tag{ii}$$

We need only logarithmic forms $\omega_i = \mathrm{d}\log(W_i)$, use notation

$$[W_1, \ldots, W_n]_\gamma \coloneqq I_\gamma[\omega_1, \ldots, \omega_n]$$

Shuffle product

$$I_\gamma[\omega_1, \ldots, \omega_r]\; I_\gamma[\omega_{r+1}, \ldots, \omega_n] = \sum_{\boldsymbol{i} \in \{1,\ldots,r\} \sqcup\!\sqcup \{r+1,n\}} I_\gamma[\omega_{i_1}, \ldots, \omega_{i_n}] \tag{$\sqcup\!\sqcup$}$$

- All functional relations manifest
- Similar to *symbol*, but complete information preserved

Step 3: construct basis

Proceed recursively order-by-order in ϵ (or transcendental weight w).

Denote weight w basis as $\{f_i^{(w)}\}$.

Start from $w = 0$: integrals are rational numbers $\iff$ $\{f_i^{(0)}\} := \{1\}$.

- Consider subspace $\mathbf{G}^{(w)} := \bigcup_{\tau,\sigma} \vec{g}_{\tau,\sigma}^{(w)}$ in vector space spanned by all iints in $\mathbf{G}^{(w)}$
- Add all products of lower-weight functions $\{f_i^{(w'<w)}\}$ to $\mathbf{G}^{(w)}$,

$$\mathbf{F}^{(w)} := \mathbf{G}^{(w)} \bigcup_{\vec{\boldsymbol{w}},\vec{\boldsymbol{i}}} \boldsymbol{f}_{\vec{\boldsymbol{i}}}^{(\vec{\boldsymbol{w}})}, \qquad \boldsymbol{f}_{\vec{\boldsymbol{i}}}^{(\vec{\boldsymbol{w}})} = f_{i_1}^{(w_1)} \cdots f_{i_2}^{(w_n)}, \qquad \sum_k w_k = w$$

  Use shuffle algebra of iints to linearize identities in $\mathbf{F}^{(w)}$.

- Use linear algebra to choose a basis in $\mathbf{F}^{(w)}$, preferring products of lower-weight functions. This basis is functions $\{f_i^{(w)}\}$.

Explicit representation: weights one and two

Explicit real-analytic $\log, \mathrm{Li}_2, \mathrm{Cl}_2$ functions, e.g.

$$g_{1,4}^{(1)} \equiv [W_4] = \log(s_{45})$$

$$g_{2,10}^{(1)} = [W_{31}] + \frac{\mathrm{i}\,\pi}{2} + \frac{\log(3)}{2} = \log\Big(\sqrt{\Delta}\Big)$$

$$\begin{aligned} g_{1,3}^{(2)} &\equiv -[W_2, W_2] + [W_2, W_{14}] + [W_4, W_2] - [W_4, W_{14}] + ([W_2] - [W_4])\log(2) + \frac{\pi^2}{12} \\ &= -\mathrm{Li}_2\left(\frac{s_{45}}{s_{23}}\right) - \log\left(-\frac{s_{45}}{s_{23}}\right)\log\left(1 - \frac{s_{45}}{s_{23}}\right) \end{aligned}$$

Positivity properties of the alphabet are important!

Weight 3: one-fold integral representation

Weight 3 functions are one-fold integrals of weight 2 functions by definition

$$f_i^{(3)}(X) = \sum_{j,k} c_{i,j,k} \int_0^1 \mathrm{d}\log W_j(t)\, h_k^{(2)}(t) + \tau_i^{(3)}$$

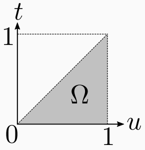
Integrands analytic on the integration domain $\implies$ integration well-defined.

- Efficient numerical integration possible
- Some care to avoid numerical cancellations if $\mathrm{d}\log(W_j)$ can vanish along the path

Weight 4: one-fold integral representation

Change order of integration [Caron-Huot, Henn '14]

$$I_\gamma[\omega_1,\ldots,\omega_n] =$$
$$\int_0^1 (\gamma^\star \circ \omega_n)(t) \int_0^t (\gamma^\star \circ \omega_{n-1})(u)\; I_{\gamma(u)}[\omega_1,\ldots,\omega_{n-2}]$$
$$= \int_0^1 (\gamma^\star \circ \omega_{n-1})(u) \left(\int_u^1 (\gamma^\star \circ \omega_n)(t)\right)\; I_{\gamma(u)}[\omega_1,\ldots,\omega_{n-2}]$$



For logarithmic forms the last integration is trivial

$$\int_u^1 (\gamma^\star \circ \omega_n)(t) = \int_u^1 \mathrm{d}\log\left(W_n(t)\right) = \log(W_n(1)) - \log(W_n(u))$$

Integrands are analytic, except possibly at loci of vanishing spurious letters, where integrable logarithmic divergences can occur.

- Linear letters: only at X_0
- Quadratic: anywhere along the path $\Longrightarrow$ efficient numerical integration over multiple line segments