Ansatz and the Amplitude: W+4-Parton at Two Loops

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based on [arXiv:2005.04195], [arXiv:2110.07541]

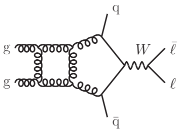
Motivation

- Experimental precision projected to reach 1% level.

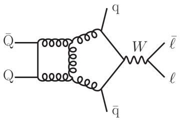
$$\sigma \sim \sigma_{LO} + \alpha_S \delta\sigma_{NLO} + \alpha_S^2 \delta\sigma_{NNLO} + \mathcal{O}(\alpha_S^3).$$

- Need two-loop amplitudes to match with NNLO precision.
- Must develop techniques for complex multi-scale amplitudes.

Focus of this talk: leading color 2-loop W-plus-4-partons.

$\mathcal{A}_g =$ 

$+ \cdots,$

$\mathcal{A}_Q =$ 

$+ \cdots.$

Recent Progress in Two-Loop Five-Point Calculations

Master integrals

- Massless: [Abreu, Chicherin, Dixon, Gehrmann, Henn, Herrmann, lo Presti, Mitev, BP, Papadopoulos, Sotnikov, Tommasini, Wasser, Wever, Zeng, Zoia].
- One-mass planar: [Abreu, Canko, Chicherin, Ita, Moriello, BP, Papadopoulos, Syrrakos, Sotnikov, Tschernow, Tommasini, Wever, Zeng, Zoia]. Vasily's talk.
- One-mass non-planar: [Abreu, Ita, Kardos, BP, Papadopoulos, Smirnov, Syrrakos, Tschernow, Wever] Costas' talk.

Amplitudes:

- Massless: [Abreu, Agarwal, Badger, Bronnum-Hansen, Buccioni, Chawdhry, Chicherin, Czakon, Dormans, Dixon, Febres Cordero, Gehrmann, Hartano, Henn, Heinrich, Herrmann, Ita, von Manteuffel, Marcoli, Mitov, Moodie, BP, Peraro, Poncelet, Sotnikov, Tancredi, Wasser, Zeng, Zhang, Zoia]
- One mass (planar): [Abreu, Badger, Febres Cordero, Guo, Hartanto, Ita, Klinkert, Krys, BP, Sotnikov, Wang, Yang, Zoia] This talk.

Five-Point NNLO Predictions (massless): [Badger, Chawdhry, Chen, Czakon, Gehrmann, Glover, Huss, Kallweit, Moodie, Marcoli, Mitov, Poncelet, Sotnikov, Wiesemann] Michał's talk

Amplitudes and Ansätze

- Amplitude built out of coefficients and master integrals.

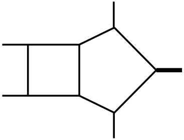
$$\mathcal{A}(p_1,\ldots,p_n) = \sum_j \underbrace{\mathcal{C}_j(\epsilon, p_1,\ldots,p_n)}_{\text{coefficients}} \underbrace{\mathcal{I}_j(p_1,\ldots,p_n)}_{\text{master integrals}}$$

- Both coefficients and integrals are governed by singularities.

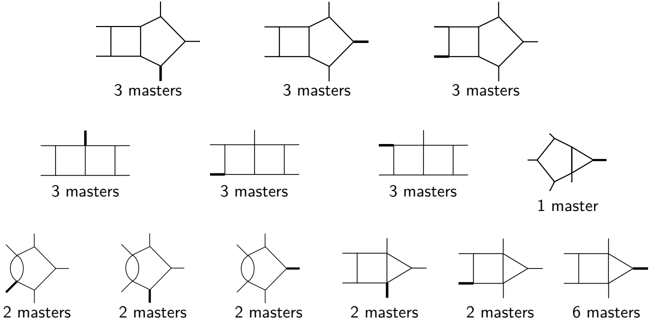
$$d\mathcal{I}_j = \epsilon \sum_\alpha M^\alpha_{jk} \mathrm{d}\log(W_\alpha)\mathcal{I}_k, \quad \leftrightarrow \quad \mathcal{C}_j = \frac{\mathcal{N}_j(\epsilon, p_1,\ldots,p_n)}{\prod_\alpha W_\alpha{}^{q_{j\alpha}}}$$

Our approach: Construct/fit Ansätze, built from knowledge of W_α.

Master Integrals and Singularities



Planar* 5-Point 1-Mass Masters

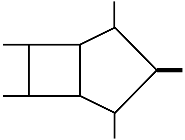


- DE + series solution: [Abreu, Ita, Moriello, BP, Tschernow, Zeng '20].
- Analytic solutions: [Canko, Papadopolous, Syrrakos, '20; Chicherin, Sotnikov, Zoia '21]. Vasily's talk, Costas' talk.

*Hexabox families also known. [Abreu, Ita, BP, Tschernow, '21; Kardos, Papadopolous, Smirnov, Syrrakos, Wever '22] Costas' talk

Scattering Kinematics and Notation

- Five-point one-mass kinematics.



$$\longrightarrow \quad p_v^2 \neq 0, \quad p_i^2 = 0, \quad [i = 1 \ldots, 4].$$

- Integrals depend on six independent Mandelstam variables.

$$\vec{s} = \{p_v^2, s_{v1}, s_{12}, s_{23}, s_{34}, s_{4v}\}.$$

- Also depend on square roots of pentagon and triangle grams.

$$\mathrm{tr}_5 = 4i\epsilon_{\alpha,\beta,\gamma,\delta} p_1^\alpha p_2^\beta p_3^\gamma p_4^\delta.$$

$$\sqrt{\Delta_3} = \sqrt{(p_v^2 - s_{12} - s_{34})^2 - 4s_{12}s_{34}},$$
$$\sqrt{\Delta_3^{\mathrm{nc}}} = \sqrt{(p_v^2 - s_{14} - s_{23})^2 - 4s_{14}s_{23}}.$$

Constructing Differential Equations from their Singularities

- Pure polylogarithmic integrals ⇒ simple differential equation.

$$d\mathcal{I}_j = \epsilon \sum_\alpha \underbrace{M^\alpha_{jk}}_{\text{rational numbers}} \, \mathrm{d}\log(\underbrace{W_\alpha}_{\text{algebraic functions}}) \mathcal{I}_k.$$

[Henn '13]

- Letters W_α are algebraic kinematic functions via square roots.

$$W_\alpha \in \mathbb{Q}\left(\vec{s}, \mathrm{tr}_5, \sqrt{\Delta_3}, \sqrt{\Delta_3^{\mathrm{nc}}}\right).$$

- Determine M^α_{jk} using letters as Ansatz and numerical IBPs.

[Abreu, BP, Zeng '18] [Schabinger, von Manteuffel '14; Peraro '16]

- Set of W_α are computed from cut differential equation.

[Abreu, Ita, Moriello, BP, Tschernow, Zeng '20]

Alphabet Structures

[Abreu, Ita, Moriello, BP, Tschernow, Zeng '21]

- Alphabet has 58 W_α, only 49 appear in integrals up to $\mathcal{O}(\epsilon)$.

$$\underbrace{\{W_1, \ldots, W_{49}\}}_{\text{relevant}} \cap \underbrace{\{W_{50}, \ldots, W_{58}\}}_{\text{irrelevant}}.$$

- Transforms non-trivially under Galois group $= \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

$$\mathrm{tr}_5 \to -\mathrm{tr}_5, \quad \sqrt{\Delta_3} \to -\sqrt{\Delta_3}, \quad \sqrt{\Delta_3^{\mathrm{nc}}} \to -\sqrt{\Delta_3^{\mathrm{nc}}}.$$

- Many W_α (odd and even) are compactly expressed with tr_+.

$$\mathrm{tr}_+(i_1 \cdots i_n) = \mathrm{tr}\left(\left[\frac{1+\gamma_5}{2}\right] \not{p}_{i_1} \cdots \not{p}_{i_n}\right).$$

Galois Invariant Relevant Letters

[Abreu, Ita, Moriello, BP, Tschernow, Zeng '21]

$$\{W_1, \ldots, W_6\} \sim \{p_v^2,\ s_{23},\ s_{v1},\ s_{4v},\ s_{12},\ s_{34}\}\,.$$

$$\{W_7, \ldots, W_{13}\} \sim 2\,\{p_1\cdot p_4,\ p_v\cdot p_1, p_v\cdot p_2, p_1\cdot p_3\} + (1 \leftrightarrow 4, 2 \leftrightarrow 3).$$

$$\{W_{14}, \ldots, W_{21}\} \sim 2\,\{p_1\cdot p_{23}, p_1\cdot p_{34}, p_2\cdot p_{v1}, p_2\cdot p_{4v}\} + (1 \leftrightarrow 4, 2 \leftrightarrow 3).$$

$$\{W_{22}, \ldots, W_{30}\} \sim \{\mathrm{tr}_+(V1V4),\ \mathrm{tr}_+(V1V2),\ \mathrm{tr}_+(V1V3), \mathrm{tr}_+(V1V[3{+}4]), \mathrm{tr}_+([1{+}2]\,3\,[1{+}2]\,V)\} + (1 \leftrightarrow 4, 2 \leftrightarrow 3).$$

$$\{W_{31}, W_{32}\} \sim \{\mathrm{tr}_+(V\,1\,2\,3) - \mathrm{tr}_+(V\,1\,3\,4),\ \mathrm{tr}_+(V\,4\,3\,2) - \mathrm{tr}_+(V\,4\,2\,1)\}\,.$$

$$\{W_{48}, W_{49}\} = \{\sqrt{\Delta_3}, \mathrm{tr}_5\}.$$

Describes (little-group invariant) integral/amplitude singularities.

Galois Non-Trivial Relevant Letters

[Abreu, Ita, Moriello, BP, Tschernow, Zeng '21]

$$\{W_{33},\ldots,W_{36}\} = \left\{ \frac{s_{v1}+s_{v2}+\sqrt{\Delta_3}}{s_{v1}+s_{v2}-\sqrt{\Delta_3}}, \frac{s_{v1}+s_{4v}+\sqrt{\Delta_3^{\mathrm{nc}}}}{s_{v1}+s_{4v}-\sqrt{\Delta_3^{\mathrm{nc}}}}, \frac{s_{v3}+s_{v2}+\sqrt{\Delta_3^{\mathrm{nc}}}}{s_{v3}+s_{v2}-\sqrt{\Delta_3^{\mathrm{nc}}}} \right\} + (1 \leftrightarrow 4, 2 \leftrightarrow 3).$$

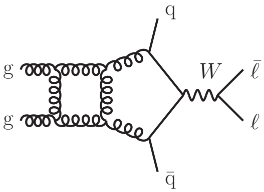
$$\{W_{37},W_{38},W_{39}\} = \left\{ \frac{s_{v1}-s_{v2}+\sqrt{\Delta_3}}{s_{v1}-s_{v2}-\sqrt{\Delta_3}}, \frac{s_{v1}-s_{4v}+\sqrt{\Delta_3^{\mathrm{nc}}}}{s_{v1}-s_{4v}-\sqrt{\Delta_3^{\mathrm{nc}}}} \right\} + (1 \leftrightarrow 4, 2 \leftrightarrow 3).$$

$$\{W_{40},\ldots,W_{46}\} = \left\{ \frac{\mathrm{tr}_+(1\,2\,3\,4)}{\mathrm{tr}_-(1\,2\,3\,4)}, \frac{\mathrm{tr}_+(V\,1\,2\,3)}{\mathrm{tr}_-(V\,1\,2\,3)}, \frac{\mathrm{tr}_+(3\,4\,V\,1)}{\mathrm{tr}_-(3\,4\,V\,1)}, \frac{\mathrm{tr}_+(V\,1\,3\,2)}{\mathrm{tr}_-(V\,1\,3\,2)} \right\} + (1 \leftrightarrow 4, 2 \leftrightarrow 3).$$

$$W_{47} = \frac{\Omega^{--}\,\Omega^{++}}{\Omega^{+-}\,\Omega^{-+}}, \quad \text{where} \quad \Omega^{\pm\pm} = s_{v1}s_{4v} - s_{v1}s_{12} - s_{4v}s_{34} \pm s_{23}\sqrt{\Delta_3} \pm \mathrm{tr}_5.$$

Provide no new singular surfaces.

Ansätze and W + 4-Parton Amplitudes



Computational Setup

- Pole structure of scattering amplitudes known and universal.

$$\mathcal{A}_\kappa(\epsilon) = \mathcal{Z}_{UV}\mathcal{Z}_{IR}\mathcal{R}_\kappa(\epsilon).$$

[Catani '98; Becher, Neubert '09; Gardi, Magnea '09]

- Apply analytic reconstruction techniques to finite remainder.

$$\mathcal{R}_\kappa = \sum_{i \in B} r_i h_i, \quad \text{where} \quad r_i = r_i^+ + \mathrm{tr}_5 r_i^- .$$

- Used "pentagon functions" h_i from [Chicherin, Sotnikov, Zoia '21].

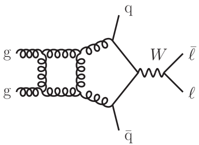
- Fit Ansatz with $\mathbb{F}_p$ evaluations of $r_i^\pm$ from Caravel.

  [Abreu, Febres Cordero, Ita, BP, Pascual, Ruf, Sotnikov '20]

Caravel

Back to Five-Point Kinematics [Abreu, Febres Cordero, Ita, Klinkert, BP, Sotnikov '21]

- Amplitudes depend on six points, but decay is simple.



$$+ \ldots = A^{\mu} J_{\mu}, \quad \text{where} \quad J_{\mu} = \overline{u}(p_6)\gamma_{\mu} v(p_5).$$

- Consider configuration where p_5 is collinear to p_i, i.e.

$$p_5^{(i)} = \frac{p_v^2}{2p_i \cdot p_v} p_i, \quad p_6^{(i)} = p_v - \frac{p_v^2}{2p_i \cdot p_v} p_i.$$

- Expand A^{μ} in terms of collinear "form factor" amplitudes:

$$A^{\mu} = \sum_{i=1}^{3} \mathcal{A}^{\{i\}} N_i^{\mu}, \quad \text{where} \quad \mathcal{A}^{\{i\}} = \mathcal{A}(p_5^{(i)}, p_6^{(i)}).$$

- We reconstruct its remainder $\mathcal{R}^{\{i\}}(s_{v1}, s_{12}, s_{23}, s_{34}, s_{4v}, p_v^2)$.

Ansätze for the Coefficients

- Singularities known, can consider common denominator form.

$$r_i^{\pm}(\vec{s}) = \frac{n_i^{\pm}(\vec{s})}{\prod_\alpha W_\alpha^{q_{i\alpha}^{\pm}}}$$

- Determine $q_{i\alpha}^{\pm}$ from univariate slice.

[Abreu, Dormans, Febres Cordero, Ita, BP '18]

see also [Heller, von Manteuffel '21]

- Given known W_α, $q_{i\alpha}^{\pm}$, can we build Ansatz with fewer terms?

- Perform univariate partial fraction during reconstruction.

[Badger, Hartanto, Zoia '21]

see Giuseppe's talk for alternative approach

Univariate Partial Fractions Ansatz

- Partial fraction w.r.t s_{34}: $\vec{s} = \underbrace{\{s_{v1}, s_{12}, s_{23}, s_{4v}, p_v^2\}}_{\vec{s}_{\text{rem}}} \cap \{s_{34}\}$.
- Rational function depends on simpler polynomials $P_{ij}^{\pm}$ as

$$r_i^{\pm}(\vec{s}) = \sum_j \frac{P_{ij}^{\pm}(\vec{s}_{\text{rem}})}{\underbrace{W_{k_{ij}}(s_{34}, \vec{s}_{\text{rem}})}_{\text{poles in } s_{34}} \prod_l \underbrace{W_l(\vec{s}_{\text{rem}})^{\gamma_{ijl}^{\pm}}}_{\text{spurious \& physical}}}.$$

- Choose $\vec{s}_{\text{rem}}^{(k)}$ so that $P_{ij}^{\pm}$ Ansatz matrix in Vandermonde form.

  [Klappert, Klein, Lange '20]

  [Abreu, Febres Cordero, Ita, Klinkert, BP, Sotnikov '21]
- Can be solved in $\mathcal{O}(N^2)$ time, can adapt to zero coefficients.

Summary of Results [Abreu, Febres Cordero, Ita, Klinkert, BP, Sotnikov '21]

$\mathcal{R}_Q^{N_f^0}$	$p_5 \parallel p_i$	# Terms $n_i^{\pm}$	$\sum_j$ # Terms $P_{ij}^{\pm}$	# Result Terms
$+-$	2	1700 k	210 k	56 k
$-+$	3	5500 k	220 k	48 k

$\mathcal{R}_g^{N_f^0}$	$p_5 \parallel p_i$	# Terms $n_i^{\pm}$	$\sum_j$ # Terms $P_{ij}^{\pm}$	# Result Terms
$++$	2	1100 k	82 k	14 k
$+-$	2	7000 k	480 k	110 k
$-+$	3	24 000 k	430 k	99 k

- Greatly reduced term/sample count in partial fraction Ansatz.
- Reconstructed expression is yet more sparse.
- Analytic results are publicly available in ancillary files.

Analytic Structure of The Remainder

Some Galois invariant letters cancel in amplitude/remainder:

(Also observed in amplitudes of [Badger, Hartanto, Krys, Zoia])

$$\{W_1, \ldots, W_6\} \sim \{p_v^2,\, s_{23},\, s_{v1},\, s_{4v},\, s_{12},\, s_{34}\}\,.$$

$$\{W_7, \ldots, W_{13}\} \sim 2\,\{p_1 \cdot p_4,\; p_v \cdot p_1, p_v \cdot p_2, p_1 \cdot p_3\} + (1 \leftrightarrow 4, 2 \leftrightarrow 3).$$

$$\{W_{14}, \ldots, W_{21}\} \sim 2\,\{p_1 \cdot p_{23}, \cancel{p_1 \cdot p_{34}}, p_2 \cdot p_{v1}, p_2 \cdot p_{4v}\} + (1 \leftrightarrow 4, 2 \leftrightarrow 3).$$

$$\{W_{22}, \ldots, W_{30}\} \sim \{\mathrm{tr}_+(V1V4),\, \mathrm{tr}_+(V1V2),\, \mathrm{tr}_+(V1V3), \cancel{\mathrm{tr}_+(V1V[3+4])}, \cancel{\mathrm{tr}_+([1+2]\,3\,[1+2]\,V)}\} + (1 \leftrightarrow 4, 2 \leftrightarrow 3).$$

$$\{W_{31}, W_{32}\} \sim \{\mathrm{tr}_+(V\,1\,2\,3) - \mathrm{tr}_+(V\,1\,3\,4),\, \mathrm{tr}_+(V\,4\,3\,2) - \mathrm{tr}_+(V\,4\,2\,1)\}\,.$$

$$\{W_{48}, W_{49}\} = \{\sqrt{\Delta_3}, \cancel{\mathrm{tr}_5}\}.$$

Summary and Conclusions

- Understanding of singularities helping to construct amplitudes.
- Leading-colour W-plus-4-parton amplitudes are now known.
- Unexpected analytic cancellations $\Rightarrow$ Better ways to compute?