Renormalisation of singlet operators to four loops

Loops and Legs in Quantum Field Theory

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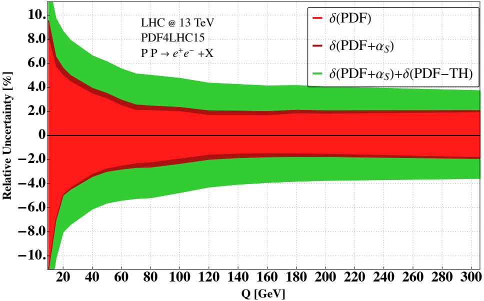
The N3LO frontier

Progress in N3LO calculations but N3LO PDFs are missing



Neutral Current DY at N3LO (Duhr, Mistlberger JHEP 03 (2022) 116).

$$\delta(\mathrm{PDF}-\mathrm{TH}) = \frac{1}{2}\left|\frac{\sigma^{\mathrm{NNLO,NNLO-PDFs}}(Q^2)-\sigma^{\mathrm{NNLO,NLO-PDFs}}(Q^2)}{\sigma^{\mathrm{NNLO,NNLO-PDFs}}(Q^2)}\right| \simeq O(2\%-3\%)$$

Percent-level accuracy at HL-LHC **requires** N3LO PDFs.

Scale Evolution of PDFs

N3LO PDFs → 4-loop evolution kernels

$$\mu^2 \frac{d}{d\mu^2} f_i\left(x, \mu^2\right) = \int_x^1 \frac{dy}{y} P_{ij}(\alpha_s, y)\, f_j\left(\frac{x}{y}, \mu^2\right), \qquad i = g, q, \bar{q}$$

$$P_{ij}(\alpha_s, x) = \underbrace{a\, P_{ij}^{(0)}}_{\text{LO}} + \underbrace{a^2\, P_{ij}^{(1)}}_{\text{NLO}} + \underbrace{a^3\, P_{ij}^{(2)}}_{\text{NNLO}} + \underbrace{a^4\, P_{ij}^{(3)}}_{\text{N3LO}}, \qquad a = \frac{\alpha_s}{4\pi}$$

Formulation in Mellin space

$$\gamma_{ij}^{(N)} = -\int_0^1 x^{N-1}\, P_{ij}(\alpha_s, x)\, dx$$

Computational strategies beyond NNLO

- Expansion of the DIS amplitude (Gorishnii, Larin, Tkachov 1983)
  - $N = 2 \ldots 8$ at N3LO (Moch, Ruijl, Ueda, Vermaseren, Vogt 2021)
- **Operator Matrix Elements** (OMEs)
  - N3LO and $N = 2, 3$ Mellin moments at N4LO for flavour Non Singlet (NS) splitting functions
    (Moch, Ruijl, Ueda, Vermaseren, Vogt 2017; + Herzog 2018)
  - NS and polarised singlet at NNLO
    (Blümlein, Marquard, Schneider, Schönwald 2021)[2]

Warning in the **Singlet** sector!

Potential flaws in the evolution of the **gluon** already at **2 loops**!

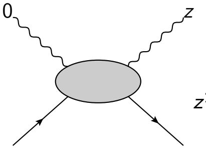
Discrepancies in 2-loop results (1977-1981) resolved in 1992 (Hamberg, van Neerven)

Can we extend the OME method to singlet sector beyond NLO?

Operator Matrix Elements (OMEs)

- Directly from the lightcone expansion for DIS

(Gross, Wilczek 1973; Georgi, Politzer 1973)



$$\underset{z^2\to 0}{\simeq} \sum_i C_i(z^2) \times z^{\mu_1} \ldots z^{\mu_N} \underbrace{\langle P|\mathcal{O}^{(N),i}_{\mu_1\ldots\mu_N}(0)|P\rangle}_{\substack{\text{Operator of min.} \\ \text{Twist = dim. - spin = 2}}}$$

- Separate short-distance (C_i) from long-distance physics (OMEs).
- RGE of the operators → PDF evolution

$$\mu^2 \frac{d}{d\mu^2} \mathcal{O}^{(N),i}_{\mu_1\ldots\mu_N} = -\gamma^{(\mathbf{N})}_{\mathbf{ij}} \, \mathcal{O}^{(N),j}_{\mu_1\ldots\mu_N}$$

Twist-2 operators

Twist-2 operators of QCD fields

$$\mathcal{O}^{(N)}_{g;\mu_1\ldots\mu_N} = \frac{1}{2}\,\mathcal{S}_T\left\{F^{a_1}_{\rho\mu_1}\, D^{a_1a_2}_{\mu_2}\,\ldots\, D^{a_{N-2}a_{N-1}}_{\mu_{N-1}}\, F^{a_N;\rho}{}_{\mu_N}\right\},$$

$$\mathcal{O}^{(N)}_{q;\mu_1\ldots\mu_N} = \mathcal{S}_T\left\{\bar{\psi}_{i_1}\,\gamma_{\mu_1}\, D^{i_1i_2}_{\mu_2}\,\ldots\, D^{i_{N-1}i_N}_{\mu_N}\,\psi_{i_N}\right\},$$

$$\mathcal{O}^{(N),\rho}_{\mathrm{ns};\mu_1\ldots\mu_N} = \mathcal{S}_T\left\{\bar{\psi}_{i_1}\,(\lambda^\rho)\,\gamma_{\mu_1}\, D^{i_1i_2}_{\mu_2}\,\ldots\, D^{i_{N-1}i_N}_{\mu_N}\,\psi_{i_N}\right\},$$

$\lambda^\rho \to$ generator of $\mathrm{SU}(n_f)$.

$\mathcal{S}_T \to$ symmetrise over $\mu_1\ldots\mu_N$ and remove trace terms.

- Construct projectors with a **lightlike** vector Δ^μ

$$F^a_\nu = F^a_{\nu\mu}\Delta^\mu, \quad D = D_\mu\,\Delta^\mu, \quad \partial = \partial_\mu\,\Delta^\mu, \quad A^a = A^a_\mu\,\Delta^\mu,$$

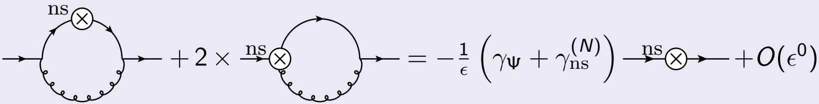
- In this talk $\to$ focus on **gluonic** operator

$$\mathcal{O}^{(N)}_1 = \mathcal{O}^{(N)}_{g;\mu_1\ldots\mu_N}\,\Delta^{\mu_1}\ldots\Delta^{\mu_N} = \frac{1}{2}\mathrm{Tr}\left[F_\rho\, D^{N-1}\, F^\rho\right]$$

Renormalisation: Non-singlet vs Singlet operators

Non-singlet operators • Diagonal RGE

$$\mathcal{O}_{\mathrm{ns}}^{(N),R}(\mu^2) = Z_{\mathrm{ns}}^{(N)}(\mu^2)\, \mathcal{O}_{\mathrm{ns}}^{(N),\mathrm{bare}}$$



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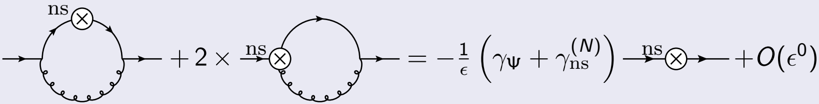
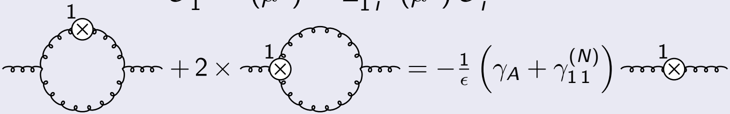


Singlet operators • Mixing

$$\mathcal{O}_{1}^{(N),R}(\mu^2) = Z_{1\,i}^{(N)}(\mu^2)\,\mathcal{O}_{i}^{(N),\text{bare}}$$

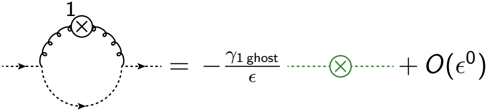


$$-\sum_{i>1}\frac{\gamma_{1\,i}^{(N)}}{\epsilon}$$

$$+\,O\left(\epsilon^0\right)$$

Alien operators: ghost and gluon terms

Divergent subdiagrams require counterterms $\propto$ ghost operators



- Basis of all Alien operators entering at 2 loops (Dixon, Taylor 1974)

$$\overline{O}_A = \overline{F}^{a\alpha}\overline{D}^{ab}_{\alpha}\partial^{m-2}\overline{A}^b - \bar{g}f^{abc}\overline{F}^a_{\alpha}\sum_{i=1}^{m-2}\frac{\kappa_i}{\eta}\partial^{\alpha}\left[(\partial^{i-1}\overline{A}^b)(\partial^{m-2-i}\overline{A}^c)\right] + \boxed{O(\bar{g}^2)} \quad (2.9)$$

$$\overline{O}_{\omega} = -\xi^a\partial^m\overline{\omega}^a - \bar{g}f^{abc}\bar{\xi}^a\sum_{i=1}^{m-2}\frac{\eta_i}{\eta}\partial\left[(\partial^{m-2-i}\overline{A}^b)(\partial^i\overline{\omega}^c)\right] + \boxed{O(\bar{g}^2)} \quad (2.10)$$

- Characterisation of Alien operators (Joglekar, Lee 1975)
  - BRST-exact operators
  - Operators proportional to the Equation of Motion

How do we construct Aliens beyond two loops?

The main problems are in the **gluonic sector**.

$\rightarrow$ Restrict to Yang Mills theory and OMEs with insertion of $\mathcal{O}_1^{(N)}$

Objective

- Find all remaining alien operators $\mathcal{O}_{i>1}^{(N)}$
- Compute renormalisation constants $Z_{1\,i}^{(N)}$ that give

$$\gamma_{\mathrm{g\,g}}^{(N)} = a\frac{\partial}{\partial a} Z_{1\,1}^{(N)}\Big|_{\frac{1}{\epsilon}}$$

Setup $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\mathrm{G.F.+G}}$

$$\mathcal{L}_0 = -\frac{1}{4} F^a_{\mu\nu} F^{a;\mu\nu}$$

$$\mathcal{L}_{\mathrm{GF+G}} = s\left[\bar{c}^a \left(\partial^\mu A^a_\mu - \frac{\xi_L}{2} b^a\right)\right] = \bar{s}\left[c^a \left(\frac{\xi_L}{2} b^a - \partial^\mu A^a_\mu\right)\right]$$

- Gauge invariance of $\mathcal{L}_0$ under

$$\delta_\omega A^a_\mu = D^{ab}_\mu \omega^b = \partial_\mu \omega^a + g f^{abc} A^b_\mu \omega^c$$

- BRST(s)/anti-BRST($\bar{s}$) invariance, s.t. $s^2 = \bar{s}^2 = \{s, \bar{s}\} = 0$

$$s\left(A^a_\mu\right) = D^{ab}_\mu c^b \qquad \bar{s}\left(A^a_\mu\right) = D^{ab}_\mu \bar{c}^b$$

$$s\left(c^a\right) = -\frac{g}{2} f^{abc} c^b c^c \qquad \bar{s}\left(\bar{c}^a\right) = -\frac{g}{2} f^{abc} \bar{c}^b \bar{c}^c$$

$$s\left(\bar{c}^a\right) = -b^a \qquad \bar{s}\left(c^a\right) = -\bar{b}^a$$

$$s\left(b^a\right) = 0 \qquad \bar{s}\left(\bar{b}^a\right) = 0$$

Equation of Motion Operators

- Variations of $S_0 = \int \mathcal{L}_0 \, d^d x$ under $A^a_\mu \to A^a_\mu + \mathcal{G}^a_\mu$

$$\mathcal{O}^{(N)}_{\text{EOM}} = \frac{\delta S_0}{\delta A^a_\mu} \mathcal{G}^a_\mu(A^{a_1}_{\mu_1}, \partial_{\mu_1} A^{a_1}_{\mu_2}, \dots) \underset{\substack{\text{Lead}\\\text{Twist}}}{=} (D^\nu F_\nu)^a \, \mathcal{G}^a(A^{a_1}, \partial A^{a_1} \dots)$$

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- Concrete ansatz with general couplings

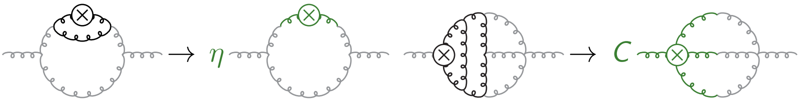
$$\mathcal{O}^{(N)}_{\text{EOM}} = (D^\mu F_\mu)^a \Big[\eta \, \partial^{N-2} A^a + g f^{a a_1 a_2} \sum_{i_1+i_2=N-3} \kappa_{i_1 i_2} (\partial^{i_1} A^{a_1})(\partial^{i_2} A^{a_2})$$

$$+ g^2 \sum_{\substack{i_1+i_2+i_3\\ N-4}} \underbrace{\left(\kappa^{(1)}_{i_1 i_2 i_3} f^{a a_1 z} f^{a_2 a_3 z} + \kappa^{(2)}_{i_1 i_2 i_3} d_4^{a a_1 a_2 a_3} + \kappa^{(3)}_{i_1 i_2 i_3} d_{\widehat{4ff}}^{a a_1 a_2 a_3} \right)}_{C^{a a_1 a_2 a_3}_{i_1 i_2 i_3}} (\partial^{i_1} A^{a_1})..(\partial^{i_3} A^{a_3})$$

$$+ g^3 \sum_{\substack{i_1+..+i_4\\ N-5}} \underbrace{\left(\kappa^{(1)}_{i_1 \dots i_4} (f\,f\,f)^{a a_1 a_2 a_3 a_4} + \kappa^{(2)}_{i_1 \dots i_4} d_{4f}^{a a_1 a_2 a_3 a_4} \right)}_{C^{a a_1 \dots a_4}_{i_1 \dots i_4}} (\partial^{i_1} A^{a_1})..(\partial^{i_4} A^{a_4}) + O(g^4) \Big]$$

Diagrammatic interpretation

- $\mathcal{O}^{(N)}_{\text{EOM}}$ cancels subdivergences in **gluonic** diagrams at 4 loops



Coefficients (e.g. $C^{a\ldots a_3}_{i_1..i_3}$) have colour structure of n-point subgraphs.

- Impose symmetry constraints on $\kappa_{i_1\ldots i_{n-1}}$

$$\kappa_{ij} + \kappa_{ji} = 0, \qquad \kappa^{(1)}_{ijk} + \kappa^{(1)}_{ikj} = 0,$$
$$\kappa^{(1)}_{ijkl} + \kappa^{(1)}_{ijlk} = 0, \qquad \kappa^{(1)}_{ijk} + \kappa^{(1)}_{jki} + \kappa^{(1)}_{kij} = 0$$
$$\kappa^{(1)}_{ijkl} + \kappa^{(1)}_{iklj} + \kappa^{(1)}_{iljk} = 0, \qquad \kappa^{(1)}_{ijkl} + \kappa^{(1)}_{jilk} + \kappa^{(1)}_{lkji} + \kappa^{(1)}_{klij} = 0$$
$$\ldots$$

Generalised Gauge Invariance

Define $\mathcal{L}_{\text{EGI}}$ to include all purely *gluonic* terms

$$S_{\text{EGI}} = \int \mathcal{L}_{\text{EGI}}\, d^d x = \int \underbrace{\mathcal{L}_0 + \mathcal{C}_1\, \mathcal{O}_1^{(N)}}_{\text{Gauge invariant}} + \underbrace{\mathcal{O}_{\text{EOM}}^{(N)}}_{\sim \delta S_0}\, d^d x$$

$\mathcal{C}_1 \to$ coupling of $\mathcal{O}_1^{(N)}$, analogous to η, κ_{ij}, $\kappa_{ijk}^{(1)}$, ...

$\mathcal{O}_{\text{EOM}}^{(N)}$ gauge variant $\to$ Generalised gauge transf. cancels $\delta_\omega \mathcal{O}_{\text{EOM}}^{(N)}$

$$\delta A_\mu^a = \delta_\omega A_\mu^a + \delta_\omega^\Delta A_\mu^a \qquad \text{such that} \qquad \delta S_{\text{EGI}} = 0$$

$$\boxed{\delta_\omega^\Delta A_\mu^a = -\delta_\omega \mathcal{G}_\mu^a + g\, f^{abc}\, \mathcal{G}_\mu^b\, \omega^c}$$

Generalised BRST transformations

Generalised gauge transf. $\rightarrow$ Generalised BRST transf.

$$\delta A^a_\mu = \delta_\omega A^a_\mu + \delta^\Delta_\omega A^a_\mu \quad \underset{\omega^a \to c^a}{\longrightarrow} \quad s'(A^a_\mu) = s(A^a_\mu) + s_\Delta(A^a_\mu)$$

From the expression of $\delta^\Delta_\omega A^a_\mu$ we get

$$s_\Delta A^a_\mu = -s\left(\mathcal{G}^a_\mu\right) + g\, f^{abc}\, \mathcal{G}^b_\mu\, c^c \qquad (1)$$

Rest of BRST transf. **unchanged** i.e. $s_\Delta(c^a) = s_\Delta(\bar{c}^a) = s_\Delta(b^a) = 0$.
Automatically nilpotent by eq.(1)

$$s'^2(A^a_\mu) = \{s, s_\Delta\}\, A^a_\mu = 0$$

Ghost operators

Define complete Lagrangian invariant under generalised BRST

$$\widetilde{\mathcal{L}} = \underbrace{\mathcal{L}_0 + \mathcal{C}_1\, \mathcal{O}_1^{(N)} + \mathcal{O}_{\text{EOM}}^{(N)}}_{\mathcal{L}_{\text{EGI}}} + s' \underbrace{\left[\bar{c}^a \left(\partial^\mu A_\mu^a - \frac{\xi_L}{2} b^a\right)\right]}_{\text{Original ancestor for gauge fix.}}$$

Ghost operators given by s_Δ, once $\mathcal{G}_\mu^a$ is known

$$\mathcal{O}_G^{(N)} = -\bar{c}^a \partial^\mu \left(s_\Delta A_\mu^a\right) = \bar{c}^a \partial^\mu \left(s\left(\mathcal{G}_\mu^a\right) - g\, f^{abc}\, \mathcal{G}_\mu^b\, c^c\right)$$

For twist-2 we already have the **most general expression**

$$\mathcal{G}_\mu^a = \Delta_\mu\, \mathcal{G}^a(A^{a_i}, \partial A^{a_i}, \partial^2 A^{a_i}, \dots)$$

Examples

Spin $N = 2$

$$\mathcal{G}^a = \eta\, A^a$$

$$\mathcal{O}^{(2)}_{\text{EOM}} = (D^\nu F_\nu)^a\, \mathcal{G}^a = \eta\, (D^\nu F_\nu)^a A^a, \qquad \mathcal{O}^{(2)}_{G} = \eta\, \bar{c}^a \partial^2 c^a$$

Spin $N = 4$

$$\mathcal{G}^a = \eta\, \partial^2 A^a + 2\, g \kappa_{01}\, f^{a a_1 a_2}\, A^{a_1} \partial A^{a_2} + g^2\, \kappa^{(2)}_{000}\, d^{a a_1 a_2 a_3} A^{a_1} A^{a_2} A^{a_3}$$

$$\begin{aligned}\mathcal{O}^{(4)}_{G} = &- \eta\, \partial \bar{c}^a \Big[\partial^3 c^a + g\, f^{abc} \Big(2\, \partial A^b \partial c^c + A^b \partial^2 c^c\Big)\Big] \\ &- 2\, g \kappa_{01} \partial \bar{c}^a \Big[f^{abc} \Big(A^b \partial^2 c^c - \partial A^b \partial c^c\Big) + g\, f^{abz} f^{cdz}\, A^b A^c \partial c^d\Big] \\ &- 3 g^2\, \kappa^{(2)}_{000} d_4^{abcd}\, \partial \bar{c}^a A^b A^c \partial c^d\end{aligned}$$

Generalised anti-BRST invariance

$$\widetilde{\mathcal{L}} = \mathcal{L}_0 + \mathcal{C}_1\, \mathcal{O}_1^{(N)} + \mathcal{O}_{\text{EOM}}^{(N)} + \bar{s}' \left[c^a \left(\frac{\xi_L}{2} b^a - \partial^\mu A_\mu^a \right) \right]$$

where

$$\bar{s}' A_\mu^a = \bar{s} A_\mu^a + \bar{s}_\Delta A_\mu^a, \qquad \bar{s}_\Delta A_\mu^a = -\bar{s}\left(\mathcal{G}_\mu^a\right) + g\, f^{abc}\, \mathcal{G}_\mu^b\, \bar{c}^c$$

New expression for **ghost operators** → new non-trivial relations

$$\mathcal{O}_G^{(N)} = \underbrace{-\bar{c}^a\, \partial^\mu \left(s_\Delta A_\mu^a \right)}_{\text{from BRST}} = \underbrace{c^a\, \partial^\mu \left(\bar{s}_\Delta A_\mu^a \right)}_{\text{from anti-BRST}}$$

For instance, at $N = 4$, this implies $2\kappa_{01} = \eta$.

Summary

- Complete set of gluonic gauge invariant + Aliens

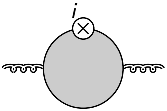
$$\widetilde{\mathcal{L}} = \mathcal{L}_0 - \frac{\left(\partial^\mu A^a_\mu\right)^2}{2\xi_L} - \bar{c}^a \partial^\mu D^{ab}_\mu c^b + \underbrace{\mathcal{C}_1\, \mathcal{O}^{(N)}_1}_{\text{Gauge inv.}} + \underbrace{\sum_{i>1} \mathcal{C}_i\, \mathcal{O}^{(N)}_i}_{\mathcal{O}^{(N)}_{\text{EOM}} + \mathcal{O}^{(N)}_G}$$

- $\mathcal{O}^{(N)}_{\text{EOM}} = (D^\mu F_\mu)^a \, \mathcal{G}^a$, where $\mathcal{G}^a$ has known ansatz.
- $\mathcal{O}^{(N)}_G = -(\partial \bar{c}^a)\, \left(s\,(\mathcal{G}^a) - g\, f^{abc}\, \mathcal{G}^b\, c^c\right)$, **automatically**.
- Consistency with anti-BRST $\to \{\mathcal{C}_{i>2}\}$ restricted to minimal set. E.g. for $N = 4$,

$$\{\mathcal{C}_i\} = \{\eta, \kappa^{(2)}_{000}\} \text{ or } \{\mathcal{C}_i\} = \{\kappa_{01}, \kappa^{(2)}_{000}\}.$$

Background field correlators

$$\Gamma^{(N)}_{i;BB}(g,\xi) =$$



$$= \begin{cases} \Gamma^{(N),0}_{i;BB} + \delta\Gamma^{(N)}_{i;BB}(g,\xi) & i = 1 \\ \delta\Gamma^{(N)}_{i;BB}(g,\xi) & i \neq 1 \end{cases}$$

Extraction of the physical renormalisation constant $Z^{(N)}_{1\,1} = 1 + \delta Z^{(N)}_{1\,1}$

$$\delta Z^{(N)}_{1\,1} = -\frac{1}{Z_B\,\Gamma^{(N),0}_{1;BB}} K_\epsilon \left[Z_B \sum_{i\geq 1} Z^{(N)}_{1\,i}\,\delta\Gamma^{(N)}_{i;BB}(g_B,\xi_B) \right]$$

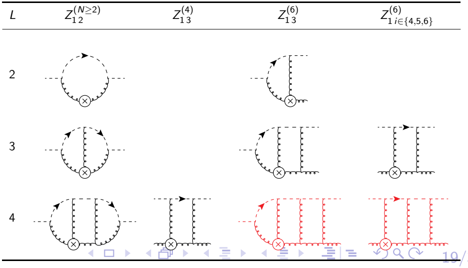
- $Z_B = Z_a^{-1}$ B-field renormalisation

  K_ϵ extracts poles in ϵ
- $Z^{(N)}_{1\,i>1}$ cancel subdivergences

  $\rightarrow$ 2-, 3-pt subgraphs up to 3 loops

  $\rightarrow$ 4-pt subgraphs up to 2 loops

  $\rightarrow$ 5-pt subgraphs up to 1 loop



Renormalising operators with $N = 2$

$$\mathcal{O}_1^{(2)} = \frac{1}{2} F_\mu^a F^{a;\mu}, \qquad \mathcal{O}_2^{(2)} = (D^\nu F_\nu)^a A^a + \bar{c}^a \partial^2 c^a$$

- Renormalisation constants: mixing

$$\delta Z_{12}^{(2)} = -a \frac{C_A}{2\epsilon} + a^2 C_A^2 \left[\frac{19}{24\epsilon^2} + \frac{5}{48} \frac{\xi}{\epsilon} - \frac{35}{48\epsilon} \right] + a^3 C_A^3 \Big[- \frac{779}{432\epsilon^3} + \frac{1}{\epsilon^2} \Big(\frac{2807}{864} - \frac{35\xi}{216} + \frac{5\xi^2}{288} \Big) + \frac{1}{\epsilon} \Big(- \frac{16759}{7776} - \frac{11\zeta_3}{72} + \frac{377\xi}{1728} + \frac{5\zeta_3 \xi}{72} - \frac{65\xi^2}{1728} \Big) \Big] + O(a^4)$$

- Renormalisation constants: gauge invariant operator

$$\boxed{Z_{11}^{(2)} = 1 + O(a^5)}$$

Agreement with general theorem (Freedman, Muzinich, Weinberg 1974) and explicit calculations (Moch, Ruijl, Ueda, Vermaseren, Vogt 2021).

Renormalising operators with $N=4$

- Operator basis and mixing

$$\mathcal{O}_1^{(4)} = \frac{1}{2}\mathrm{Tr}\left[F_\nu D^2 F^\nu\right],$$
$$\mathcal{O}_2^{(4)} = (D.F)^a\left[\partial^2 A^a + g f^{abc} A^b \partial A^c\right] - \partial\bar{c}^a\, \partial^3 c^a$$
$$- g f^{abc}\, \partial\bar{c}^a \left[2A^b \partial^2 c^c + \partial A^b\, \partial c^c\right]$$
$$- g^2 f^{abe} f^{cde}\, \partial\bar{c}^a\, A^b A^c\, \partial c^d,$$
$$\mathcal{O}_3^{(4)} = d^{abcd}\left[(D.F)^a A^b A^c A^d - 3\partial\bar{c}^a\, A^b A^c\, \partial c^d\right]$$

$$\delta Z_{1\,2}^{(4)} = -\frac{aC_A}{12\epsilon} - a^2 C_A^2\left[\frac{97}{1440\epsilon^2} - \frac{\xi}{320\epsilon} + \frac{8641}{86400\epsilon}\right]$$
$$+ a^3 C_A^3\left[\frac{9437}{86400\epsilon^3} - \frac{1}{\epsilon^2}\left(\frac{1520341}{15552000} - \frac{853\xi}{86400}\right)\right.$$
$$\left. - \frac{1}{\epsilon}\left(\frac{166178237}{466560000} + \frac{\zeta_3}{2400} - \frac{37199\xi}{648000} - \frac{37\zeta_3\,\xi}{9600}\right)\right]$$
$$\delta Z_{1\,3}^{(4)} = \frac{a\,C_A}{24\epsilon}$$

- Gauge invariant operator renormalisation agrees with (Moch, Ruijl, Ueda, Vermaseren, Vogt 2021)

$$\delta Z_{1\,1}^{(4)} = a\frac{21 C_A}{5\epsilon} + a^2 C_A^2\left(\frac{28}{25\epsilon^2} + \frac{7121}{1000\epsilon}\right) - a^3 C_A^3\left(\frac{1316}{1125\epsilon^3} + \frac{151441}{45000\epsilon^2} - \frac{103309639}{4050000\epsilon}\right)$$
$$+ a^4\left\{C_A^4\left[\frac{11186}{5625\epsilon^4} + \frac{1512989}{450000\epsilon^3} - \frac{5437269017}{162000000\epsilon^2} + \frac{1}{\epsilon}\left(\frac{1502628149}{13500000} + \frac{1146397\zeta_3}{45000}\right.\right.\right.$$
$$\left.\left.\left. - \frac{126\zeta_5}{5}\right)\right] + \frac{d_{AA}}{N_A}\left(\frac{21623}{600\epsilon} + \frac{3899\,\zeta_3}{15\epsilon} - \frac{1512\,\zeta_5}{5\epsilon}\right)\right\}$$

Operator basis for $N = 6$

Alien operators $\mathcal{O}_2^{(6)} \ldots \mathcal{O}_6^{(6)}$

$$\mathcal{O}_2^{(6)} = (D.F)^a \partial^4 A^a - \partial\bar{c}^a\, \partial^5 c^a + g f^{abc} \Big[\frac{5}{3}(D.F)^a A^b \partial^3 A^c - \partial\bar{c}^a \Big(\frac{8}{3} A^b \partial^4 c^c + 4\partial A^b \partial^3 c^c + 6\partial^2 A^b \partial^2 c^c + \frac{7}{3}\partial^3 A^b \partial c^c \Big) \Big] + g^2 (ff)^{abcd} \Big[\frac{5}{3}(D.F)^a \partial A^b \partial A^c A^d - \frac{5}{3}\partial\bar{c}^a \Big(A^b A^c \partial^3 c^d + 4A^b \partial A^c \partial^2 c^d + 3A^b \partial^2 A^c \partial c^d + \partial A^b \partial A^c \partial c^d - 2\partial A^b A^c \partial^2 c^d \Big) \Big] - g^3 (fff)^{abcde} \Big[\frac{2}{3}(D.F)^a A^b A^c A^d \partial A^e + \frac{1}{3}\partial\bar{c}^a \Big(2A^b A^c A^d \partial^2 c^e + 6A^b A^c \partial A^d \partial c^e - A^b \partial A^c A^d \partial c^e - 8\partial A^b A^c A^d \partial c^e \Big) \Big]$$

Simplest vertex $\rightarrow$ 2-point. Higher-point interactions have the same coupling.

$$\mathcal{O}_3^{(6)} = g\, f^{abc} \Big[(D.F)^a \Big(2\partial A^b\, \partial^2 A^c + \frac{4}{3} A^b\, \partial^3 A^c \Big) - \frac{4}{3}\partial\bar{c}^a \Big(A^b \partial^4 c^c + \frac{3}{2}\partial A^b\, \partial^2 c^c - \frac{3}{2}\partial^2 A^b\, \partial^2 c^c - \partial^3 A^b \partial c^c \Big) \Big] + g^2 (ff)^{abcd} \Big[-\frac{10}{3}(D.F)^a \partial A^b A^c \partial A^d \ldots$$

$\mathcal{O}_3^{(6)}$ has a leading 3-point vertex. $\mathcal{O}_4^{(6)} \ldots \mathcal{O}_6^{(6)}$ start with a 4-point.

Renormalisation constants for $N = 6$

- Mixing with alien operators

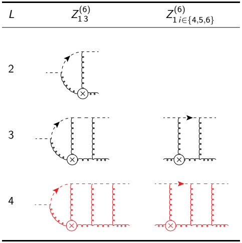
$$\delta Z^{(6)}_{1\,2} = -a\,\frac{C_A}{30\epsilon} - a^2\,C_A^2\left[\frac{653}{10080\epsilon^2} + \frac{19\xi}{20160\epsilon} + \frac{185093}{4233600\epsilon}\right] + O(a^3)$$

$$\delta Z^{(6)}_{1\,3} = -a\,\frac{C_A}{48\epsilon} - a^2\,C_A^2\left[\frac{2021}{40320\epsilon^2} + \frac{235813}{8467200\epsilon} + O(\xi)\right] + O(a^3),$$

$$\delta Z^{(6)}_{1\,4} = -a\,\frac{C_A}{32\epsilon} + O(a^2),$$

$$\delta Z^{(6)}_{1\,5} = a\,\frac{C_A}{24\epsilon} + O(a^2),$$

$$\delta Z^{(6)}_{1\,6} = O(a^2).$$



- Gauge invariant operator renormalisation agrees with (Larin, Nogueira, van Ritbergen, Vermaseren 1996)

$$\delta Z^{(6)}_{1\,1} = a\,\frac{83\,C_A}{14\epsilon} + a^2\,C_A^2\left(\frac{7885}{1176\epsilon^2} + \frac{1506899}{148176\epsilon}\right) + a^3\,C_A^3\left(-\frac{465215}{148176\epsilon^3} + \frac{243375989}{18670176\epsilon^2} + \frac{96390174479}{2613824640\epsilon}\right) + O(a^4),$$

Conclusion

- Renormalisation of Operator Matrix Elements is a promising method to compute N3LO PDF evolution.
- Conceptual obstacle: mixing with alien operators.
  Severe problem with **gluon operators**.
- We construct all aliens mixing with gluon operators for all N up to 4 loops
  - Most general ansatz for Equation of Motion operators
  - BRST-construction of ghost operators
  - Reduction to minimal basis with anti-BRST identities
- We check our procedure by renormalising operators $N = 2$ and $N = 4$ up to 4 loops and $N = 6$ to 3 loops.

Outlook

Next steps include

- Include the treatment of fermions $\rightarrow$ complete OME methodology to compute **singlet** splitting functions in QCD.
    - Note: mixing is much simpler in the fermionic sector and it starts one loop order higher.
- Improve tools to compute subdivergences (R^*, massive tadpoles ...)
    - 3-point functions to 3 loops
    - 4-point functions to 2 loops

  with insertion of operators of high dimension.

Opening the way to compute N3LO PDF evolution and **composite operator renormalisation** for more applications (e.g. EFTs ...)

Thank you for the attention!

Backup slides

Comparison with Joglekar and Lee

$\mathcal{O}^{(N)}_{\text{EOM}}$ generated by *classical* EOM. $\mathcal{O}^{(N)}_{G} = s_\Delta\left(\bar{c}\partial^\mu A^a_\mu\right)$ *generalised-BRST-exact*. But

$$\mathcal{O}^{(N)}_{\text{EOM}} = \frac{\delta S_0}{\delta A^a_\mu}\mathcal{G}^a_\mu, \qquad \mathcal{O}^{(N)}_{G} = -s\left(\bar{c}^a\partial^\mu\mathcal{G}^a_\mu\right) + \underbrace{\left[\frac{1}{\xi_L}\partial^\mu\left(\partial^\nu A^a_\nu\right) + g\, f^{abc}(\partial^\mu\bar{c}^b)c^c\right]}_{\text{Gauge fix. and ghost terms in EOM of} A^a_\mu}\mathcal{G}^b_\mu$$

$$\boxed{\mathcal{O}^{(N)}_{\text{EOM}} + \mathcal{O}^{(N)}_{G} = s\left((\partial^\mu\bar{c}^a)\mathcal{G}^a_\mu\right) + \frac{\delta S}{\delta A^a_\mu}\mathcal{G}^a_\mu}$$

Now we compare with Joglekar and Lee operators of class I and II

$$\mathcal{O}_{\text{I}} = s\left(F(A,c,\bar{c})\right) + \frac{\delta S}{\delta A^a_\mu}\frac{\partial F(A,c,\bar{c})}{\partial(\partial_\mu\bar{c}^a)} \qquad \Rightarrow F = \left(\partial^\mu\bar{c}^a\right)\mathcal{G}^a_\mu$$

$$\mathcal{O}_{\text{II}} = \frac{\delta S}{\delta c^a}X^a(A,c,\bar{c}) \qquad \Rightarrow X^a = 0 \quad \frac{\delta S}{\delta c^a} \text{ is twist-3!}$$

Space of alien operators

Spin N	2	4	6	8	10	12	14	16
w aBRST	1	2	5	12	25	50	87	140
w/o aBRST	1	3	11	30	66	126	215	339

Generalised Gauge transformation (I)

Expression of EOM operators

$$\mathcal{G}^a = \sum_{k=1}^{\infty} g^{k-1} \sum_{\substack{i_1+\cdots+i_k \\ =N-k-1}} C^{a;a_1\ldots a_k}_{i_1\ldots i_k} \left(\partial^{i_1} A^{a_1}\right)\ldots\left(\partial^{i_k} A^{a_k}\right)$$

$C^{a;a_1\ldots a_k}_{i_1\ldots i_k}$ given up to 4 loops. Then $\delta^{\Delta}_{\omega} A^a_{\mu} = -\Delta_{\mu}\left(\delta_{\omega}\mathcal{G}^a - g\, f^{abc}\mathcal{G}^b\omega^c\right)$ reads

$$\begin{aligned}
\delta^{\Delta}_{\omega} A^a_{\mu} &= -\Delta_{\mu} \sum_{k=1}^{\infty} \sum_{\substack{i_1+\cdots+i_k \\ N-k-1}} \left(\partial^{i_1} A^{a_1}\right)\ldots\left(\partial^{i_k+1}\omega^{a_k}\right) \sum_{\sigma\in Z_k} C^{a;a_{\sigma(1)}\cdots a_{\sigma(k)}}_{i_{\sigma(1)}\cdots i_{\sigma(k)}} \\
&+ g\Delta_{\mu} \sum_{k=1}^{\infty} \sum_{\substack{i_1+\cdots+i_{k+1} \\ N-k-2}} \left(\partial^{i_1} A^{a_1}\right)\ldots\left(\partial^{i_k} A^{a_k}\right)\left(\partial^{i_{k+1}+1}\omega^{a_{k+1}}\right) \sum_{m=1}^{k} \binom{i_m + i_{k+1} + 1}{i_m} \\
&\qquad \times C^{a;a_1\cdots a_{m-1}\, b a_{m+1}\cdots a_k}_{i_1\ldots i_m+i_{k+1}+1\ldots i_k} f^{b\, a_m\, a_{k+1}} \\
&= -\Delta_{\mu} \sum_{k=1}^{\infty} g^{k-1} \sum_{\substack{i_1+\cdots+i_k \\ =N-k-1}} \widetilde{C}^{a;a_1\ldots a_k}_{i_1\ldots i_k} \left(\partial^{i_1} A^{a_1}\right)\ldots\left(\partial^{i_{k-1}} A^{a_{k-1}}\right)\left(\partial^{i_k+1}\omega^{a_k}\right)
\end{aligned}$$

Generalised Gauge transformation (II)

Explicit coefficients up to 4 loops

$$\widetilde{C}^{a;a_1\,a_2}_{i_1\,i_2} = \eta^{(1)}_{i_1 i_2}\, f^{a;a_1\,a_2},$$

$$\widetilde{C}^{a;a_1 a_2 a_3}_{i_1 i_2 i_3} = \eta^{(1)}_{i_1 i_2 i_3}\,(ff)^{aa_1 a_2 a_3} + \eta^{(2)}_{i_1 i_2 i_3}\, d^{aa_1 a_2 a_3} + \eta^{(3)}_{i_1 i_2 i_3}\, d^{aa_1 a_2 a_3}_{\widehat{4ff}},$$

$$\widetilde{C}^{a;a_1 a_2 a_3 a_4}_{i_1 i_2 i_3 i_4} = \eta^{(1)}_{i_1 i_2 i_3 i_4}\,(fff)^{aa_1 a_2 a_3 a_4} + \eta^{(2a)}_{i_1 i_2 i_3 i_4}\, d^{aa_1 a_2 a_3 a_4}_{4f} + \eta^{(2b)}_{i_1 i_2 i_3 i_4}\, d^{aa_4 a_1 a_2 a_3}_{4f}.$$

The η coefficients are given by

$$\eta^{(1)}_{i_1 i_2 i_3} = 2\kappa_{i_1 (i_2+i_3+1)} \begin{pmatrix} i_2+i_3+1 \\ i_2 \end{pmatrix} + 2\left[\kappa^{(1)}_{i_1 i_2 i_3} + \kappa^{(1)}_{i_3 i_2 i_1}\right],$$

$$\eta^{(1)}_{i_1 i_2 i_3 i_4} = 2\left[\kappa^{(1)}_{i_1 i_2 (i_3+i_4+1)} + \kappa^{(1)}_{(i_3+i_4+1) i_2 i_1}\right] \begin{pmatrix} i_3+i_4+1 \\ i_3 \end{pmatrix} + 2\left[\kappa^{(1)}_{i_1 i_2 i_3 i_4} + \kappa^{(1)}_{i_1 i_4 i_3 i_2} + \kappa^{(1)}_{i_4 i_1 i_3 i_2} + \kappa^{(1)}_{i_4 i_3 i_1 i_2}\right],$$

$$\eta^{(2a)}_{i_1 i_2 i_3 i_4} = 3\kappa^{(2)}_{i_1 i_2 (i_3+i_4+1)} \begin{pmatrix} i_3+i_4+1 \\ i_3 \end{pmatrix} + 2\,\kappa^{(2)}_{i_1 i_2 i_3 i_4},$$

$$\eta^{(1)}_{i_1 i_2} = 2\,\kappa_{i_1\, i_2} + \eta \begin{pmatrix} i_1+i_2+1 \\ i_1 \end{pmatrix},$$

$$\eta^{(2)}_{i_1 i_2 i_3} = 3\kappa^{(2)}_{i_1 i_2 i_3},$$

$$\eta^{(3)}_{i_1 i_2 i_3} = 2\left[\kappa^{(3)}_{i_1 i_2 i_3} - \kappa^{(3)}_{i_3 i_2 i_1}\right],$$

$$\eta^{(2b)}_{i_1 i_2 i_3 i_4} = 2\,\kappa^{(2)}_{i_4 i_1 i_2 i_3}.$$

Anti-BRST identities

Consistency with anti-BRST symmetry requires

$$-\bar{c}^a \partial^\mu \left(s_\Delta A^a_\mu \right) = c^a \partial^\mu \left(\bar{s}_\Delta A^a_\mu \right)$$

which implies constraints on the η coefficients in $s_\Delta A^a_\mu$ and $\bar{s}_\Delta A^a_\mu$

$$\eta^{(1)}_{i_1 i_2} = - \sum_{s_1=0}^{i_1} (-1)^{s_1+i_2} \begin{pmatrix} s_1 + i_2 \\ s_1 \end{pmatrix} \eta^{(1)}_{(i_1-s_1)(i_2+s_1)},$$

$$\eta^{(1)}_{i_1 i_2 i_3} = \sum_{s_1=0}^{i_1} \sum_{s_2=0}^{i_2} \frac{(s_1+s_2+i_3)!}{s_1! s_2! i_3!} (-1)^{s_1+s_2+i_3} \eta^{(1)}_{(i_2-s_2)(i_1-s_1)(i_3+s_1+s_2)},$$

$$\eta^{(2)}_{i_1 i_2 i_3} = \sum_{s_1=0}^{i_1} \sum_{s_2=0}^{i_2} \frac{(s_1+s_2+i_3)!}{s_1! s_2! i_3!} (-1)^{s_1+s_2+i_3} \eta^{(2)}_{(i_1-s_1)(i_2-s_2)(i_3+s_1+s_2)},$$

$$\eta^{(3)}_{i_1 i_2 i_3} = \sum_{s_1=0}^{i_1} \sum_{s_2=0}^{i_2} \frac{(s_1+s_2+i_3)!}{s_1! s_2! i_3!} (-1)^{s_1+s_2+i_3} \eta^{(3)}_{(i_2-s_2)(i_1-s_1)(i_3+s_1+s_2)}$$

Anti-BRST identities (continued)

$$\eta^{(1)}_{i_1 i_2 i_3 i_4} = -\sum_{s_1=0}^{i_1}\sum_{s_2=0}^{i_2}\sum_{s_3=0}^{i_3} \frac{(s_1+s_2+s_3+i_4)!}{s_1! s_2! s_3! i_4!} \times (-1)^{s_1+s_2+s_3+i_4}\, \eta^{(3)}_{(i_3-s_3)(i_2-s_2)(i_1-s_1)(i_4+s_1+s_2+s_3)},$$

$$\eta^{(2a)}_{i_1 i_2 i_3 i_4} = -\sum_{s_1=0}^{i_1}\sum_{s_2=0}^{i_2}\sum_{s_3=0}^{i_3} \frac{(s_1+s_2+s_3+i_4)!}{s_1! s_2! s_3! i_4!} \times (-1)^{s_1+s_2+s_3+i_4}\, \eta^{(2a)}_{(i_1-s_1)(i_2-s_2)(i_3-s_3)(i_4+s_1+s_2+s_3)},$$

$$\eta^{(2b)}_{i_1 i_2 i_3 i_4} = \eta^{(2a)}_{i_1 i_3 i_2 i_4} - \eta^{(2a)}_{i_1 i_2 i_3 i_4} + \sum_{s_1=0}^{i_1}\sum_{s_2=0}^{i_2}\sum_{s_3=0}^{i_3} \frac{(s_1+s_2+s_3+i_4)!}{s_1! s_2! s_3! i_4!} \times (-1)^{s_1+s_2+s_3+i_4}\, \eta^{(2b)}_{(i_1-s_1)(i_2-s_2)(i_3-s_3)(i_4+s_1+s_2+s_4)}.$$