Loops and Legs in QFT 2022

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Computer Algebra and Hypergeometric Structures for Feynman Integrals

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joint with Johannes Blümlein and Marco Saragnese (DESY, arXiv:2111.15501)

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JOHANNES KEPLER UNIVERSITY LINZ

Der Wissenschaftsfonds.

SAGEX
Scattering Amplitudes: from Geometry to Experiment

Outline

1. Motivation: solving linear recurrences
   and differential equations
2. Extracting HYP structures from D-equations
3. Conclusion

1. Motivation

Example: A master integral from Ladder and V-topologies

[arXiv:1509.08324]

$$F(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+n+1} \underbrace{(1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

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The integrand is

- hyperexponential in x,y,z:

$$\frac{D_x f(\varepsilon,n,x,y,z)}{f(\varepsilon,n,x,y,z)}\in\mathbb{Q}(\varepsilon,n,x,y,z)$$

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The integrand is

- hyperexponential in x,y,z:
- hypergeometric in n:

$$\frac{f(\varepsilon,n+1,x,y,z)}{f(\varepsilon,n,x,y,z)}\in\mathbb{Q}(\varepsilon,n,x,y,z)$$

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Ablinger's MultiIntegrate.m $\downarrow$ (9 hours)

$$a_0(\varepsilon,n)F(\varepsilon,n)+a_1(\varepsilon,n)F(\varepsilon,n+1)+\cdots+a_5(\varepsilon,n)F(\varepsilon,n+5)=0$$

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$$\Big\downarrow \text{ Ablinger's MultiIntegrate.m (9 hours)}$$

$$a_0(\varepsilon, n) F(\varepsilon, n) + a_1(\varepsilon, n) F(\varepsilon, n+1) + \cdots + a_5(\varepsilon, n) F(\varepsilon, n+5) = 0$$

$$\Big\downarrow \text{ recurrence solver}$$

$F(\varepsilon, n) =$ expression in terms of special functions

A recurrence solver (Sigma.m)

GIVEN a recurrence

$a_0(n), \dots, a_\delta(n)$: polynomials in n
$h(n)$: expression in indefinite nested sums defined over hypergeometric products.

$$a_0(n)F(n) + \dots + a_\delta(n)F(n+\delta) = h(n);$$

together with initial values $F(0), \dots, F(\delta - 1) \in \mathbb{K}$

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Special cases of indefinite nested sums over hypergeometric products:

$$S_{2,1}(n)=\sum_{i=1}^{n}\frac{1}{i^2}\sum_{j=1}^{i}\frac{1}{j}$$ (harmonic sums)

J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018 [arXiv:hep-ph/9810241];
J.A.M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037 [arXiv:hep-ph/9806280].

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Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{k=1}^{n}\frac{2^k}{k}\sum_{i=1}^{k}\frac{2^{-i}}{i}\sum_{j=1}^{i}\frac{S_1(j)}{j}$$ (generalized harmonic sums)

S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083];
J. Ablinger, J. Blümlein and CS, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378].

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$$\sum_{k=1}^{n}\frac{1}{(1+2k)^2}\sum_{j=1}^{k}\frac{1}{j^2}\sum_{i=1}^{j}\frac{1}{1+2i}$$ (cyclotomic harmonic sums)

J. Ablinger, J. Blümlein and CS, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063].

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Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{j=1}^{n} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} \qquad \text{(binomial sums)}$$

J. Ablinger, J. Blümlein, C. G. Raab and CS, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

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Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{h=1}^{n} 2^{-2h} (1-\eta)^h \binom{2h}{h} \sum_{k=1}^{h} \frac{2^{2k}}{k^2 \binom{2k}{k}}$$ (generalized binomial sums)

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, CS, K. Schönwald. Nucl.Phys.B 932. 2018. [arXiv:1804.02226].
J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. Nucl.Phys.B 955. 2020. [arXiv:2004.08916]

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A more general example:

$$\sum_{k=1}^{n}\left(\prod_{i=1}^{k}\frac{1+i+i^2}{i+1}\right)\sum_{j=1}^{k}\frac{1}{j\binom{4j}{3j}^2}$$

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recurrence solver $\big\downarrow$

$F(\varepsilon,n)=$ expression in terms of special functions

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$$F(\varepsilon,n)=\int_0^1 dx\int_0^1 dy\int_0^1 dz \underbrace{x^{\varepsilon/2}y^{\varepsilon/2}(1-z)^{-\frac{3\varepsilon}{2}-2}z^{\frac{\varepsilon}{2}+n+1}(1-xz)^{\varepsilon/2}\times(1-yz)^{\varepsilon/2}(x+y-1)^n}_{f(\varepsilon,n,x,y,z)}$$

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`Sigma.m` $\downarrow$

$$F(\varepsilon,n)=F_{-3}(n)\varepsilon^{-3}+F_{-2}(n)\varepsilon^{-2}+\dots$$

A refined recurrence solver (Sigma.m)

GIVEN a recurrence

$a_0(\varepsilon,n),\dots,a_\delta(\varepsilon,n)$: polynomials in ε,n
$h_l(n),h_{l+1}(n),\dots,h_\lambda(n)$:
expressions in indefinite nested sums
defined over hypergeometric products.

$$a_0(\varepsilon,n)F(\varepsilon,n)+\dots+a_\delta(\varepsilon,n)F(\varepsilon,n+\delta)$$
$$=h_l(n)\varepsilon^l+h_{l+1}(n)\varepsilon^{l+1}+\dots h_\lambda(n)\varepsilon^r+O(\varepsilon^{r+1});$$

together with ε-expansions of $F(0),\dots,F(\delta-1)$ up to a certain order.

A refined recurrence solver (Sigma.m)

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together with ε-expansions of $F(0),\dots,F(\delta-1)$ up to a certain order.

DECIDE constructively if the coefficients $F_i(n)$ of

$$F(n)=F_l(n)\,\varepsilon^l+F_{l+1}(n)\,\varepsilon^{l+1}+\dots+F_\lambda(n)\,\varepsilon^r+O(\varepsilon^{r+1})$$

can be given in terms of indefinite nested sums defined over hypergeometric products.

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656[cs.SC]
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

$$F(N) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+N+1} (1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^N$$

↓ (package `MultiIntegrate.m`)

$$a_0(\varepsilon,n)F(n) + a_1(\varepsilon,n)F(n+1) + \cdots + a_5(\varepsilon,n)F(n+5) = 0$$

$$F(2) = \frac{20}{27\varepsilon^3} - \frac{40}{27\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{1393}{486} + \frac{5\zeta_2}{18}\right) + \dots$$

$$\vdots$$

$$F(6) = \frac{22}{147\varepsilon^3} - \frac{535}{2058\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{630043}{1234800} + \frac{11\zeta_2}{196}\right) + \dots$$

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↓ (summation package `Sigma.m`)

$$F(n)=F_{-3}(n)\,\varepsilon^{-3}+F_{-2}(n)\,\varepsilon^{-2}+F_{-1}(n)\,\varepsilon^{-1}+\dots$$

We get
$F_{-3}(n) = \frac{8(-1)^n}{3(n+1)(n+2)} + \frac{8(2n+3)}{3(n+1)^2(n+2)}$

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$$F_{-2}(n) = -\frac{4(-1)^n\left(3n^3+18n^2+31n+18\right)}{3(n+1)^3(n+2)^2} - \frac{4\left(6n^3+32n^2+51n+26\right)}{3(n+1)^3(n+2)^2}$$

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$$\begin{aligned} F_{-1}(n) = {} & (-1)^n\left(\frac{2\big(9n^5+81n^4+295n^3+533n^2+500n+204\big)}{3(n+1)^4(n+2)^3} + \frac{\zeta_2}{(n+1)(n+2)}\right) \\ & + \frac{2\big(18n^5+150n^4+490n^3+755n^2+536n+132\big)}{3(n+1)^4(n+2)^3} + \frac{(2n+3)\zeta_2}{(n+1)^2(n+2)} \\ & + \left(-\frac{4}{(n+1)^2(n+2)} + \frac{4(-1)^n}{(n+1)(n+2)}\right) S_2(n) \\ & + \left(\frac{4(-1)^n}{3(n+1)(n+2)} - \frac{4(n+9)}{3(n+1)^2(n+2)}\right) S_{-2}(n) \end{aligned}$$

We calculated it up to ε^4 in about 2 hours.

Find a recurrence and solve it for the integral/sum

$$D_\varepsilon(n) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, n, x_1, x_2, \ldots, x_7) dx_1 dx_2 \ldots dx_7$$
$$\stackrel{?}{=} F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \ldots$$

ε-recurrence solver

multivariate
Almquist/Zeilberger
(Jakob Ablinger)

$$a_0(\varepsilon, n) D_\varepsilon(n) + \ldots + a_d(\varepsilon, n) D_\varepsilon(n+d) = h(\varepsilon, n)$$

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ε-recurrence solver

multivariate
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$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \ldots, i_7)$$

MultiSum Package
(K. Wegschaider)

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ε-recurrence solver

multivariate
Almquist/Zeilberger
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$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, n, i_1, i_2, \ldots, i_7)$$

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Holonomic/difference ring approach

$$a_0(\varepsilon, n) D_\varepsilon(n) + \ldots + a_d(\varepsilon, n) D_\varepsilon(n + d) = h(\varepsilon, n)$$

Motivation: solving recurrences and D-equations

[coming, e.g., from IBP methods]

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x), \dots, \hat{R}_\lambda(x)$ (in terms of special functions)
Determine $\hat{I}_1(x), \dots, \hat{I}_\lambda(x)$ (for given initial values) s.t.

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} = A(x) \begin{pmatrix} \hat{I}_1(x) \\ \dots \\ \hat{I}_\lambda(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \dots \\ \hat{R}_\lambda(x) \end{pmatrix}$$

given

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$\downarrow$ uncoupling algorithms
(Zürcher,Abramov/Zima,Gauss,...)

1. $\hat{I}_1(x)$ is a solution of

$$b_0(x)\hat{I}_1(x) + b_1(x)D_x\hat{I}_1(x) + \dots + b_\lambda(x)D_x^\lambda\hat{I}_1(x) = \hat{r}(x)$$

Given invert. $A(x) \in \mathbb{K}(x)^{\lambda\times\lambda}$ and $\hat{R}_1(x), \ldots, \hat{R}_\lambda(x)$ (in terms of special functions)
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2. For $i = 2, \ldots, r$ we get

$$\hat{I}_i(x) = \text{LinCom}(\hat{I}_1(x), \ldots, D_x^{\lambda-1}\hat{I}_1(x)) + \text{LinCom}(\ldots, D^i\hat{R}_i(x), \ldots)$$

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DE-solver

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DE-solver REC-solver

2. Extracting HYP structures from linear D-equations

I. A differential equation solver (`HarmonicSums.m`)

GIVEN a linear differential equation $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$

$$b_0(x)f(x) + \cdots + b_\lambda(x)D^\lambda f(x) = 0;$$

together with initial values $f(0), \dots, D^{\lambda-1}f(x)|_{x=0} \in \mathbb{K}$

I. A differential equation solver (`HarmonicSums.m`)

GIVEN a linear differential equation $b_0(x), \dots, b_\lambda(x) \in \mathbb{K}[x]$

$$b_0(x)f(x) + \cdots + b_\lambda(x)D^\lambda f(x) = 0;$$

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Special cases of iterated integrals over hyperexponential functions:

$$H_{1,-1}(x) = \int_0^x \frac{1}{1-\tau_1} \int_0^{\tau_1} \frac{1}{1+\tau_2} d\tau_2 d\tau_1$$ (harmonic polylogarithm)

E. Remiddi, E. and J.A.M. Vermaseren, Int. J. Mod. Phys. **A15** (2000) [arXiv:hep-ph/9905237]

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S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083];
J. Ablinger, J. Blümlein and CS, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378].

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J. Ablinger, J. Blümlein and CS, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063].

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J. Ablinger, J. Blümlein, C. G. Raab and CS, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

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(generalized radical integrals)

J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, CS, K. Schönwald. Nucl.Phys.B 932. 2018. [arXiv:1804.02226].
J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, CS, K. Schönwald. Nucl.Phys.B 955. 2020. [arXiv:2004.08916]

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A more general example:

$$\int_0^x e^{\int_1^{\tau_1} \frac{1}{1+y+y^2}dy} \int_0^{\tau_1} \frac{1}{1+\tau_2} d\tau_2 d\tau_1$$

HarmonicSums can also deal with Liouvillian solutions (i.e., it contains Kovacic's algorithm):

$$(11+20x)f'(x)+(1+x)(35+134x)f''(x) + 3(1+x)^2(4+37x)f^{(3)}(x)+18x(1+x)^3f^{(4)}(x)=0$$

$$\downarrow$$

$$\Big\{c_1+c_2\int_0^x\frac{1}{1+\tau_1}\,d\tau_1+c_3\int_0^x\frac{1}{1+\tau_1}\int_0^{\tau_1}\frac{\sqrt[3]{1+\sqrt{1+\tau_2}}}{1+\tau_2}d\tau_2 d\tau_1 + c_4\int_0^x\frac{1}{1+\tau_1}\int_0^{\tau_1}\frac{\sqrt[3]{1-\sqrt{1+\tau_2}}}{1+\tau_2}d\tau_2 d\tau_1 \mid c_1,c_2,c_3,c_4\in\mathbb{K}\Big\}$$

Connection: DE $\longleftrightarrow$ REC

Let

$$f(x) = \sum_{n=0}^{\infty} F(n) x^n$$

be a (formal) power series. Then:

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$$\Updownarrow$$

There exist $a_0(x), \ldots, a_\delta(x) \in \mathbb{K}[x]$ with

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Example 1: Find a power series solution

$$f(x) = \sum_{n=0}^{\infty} F(n) x^n$$

for

$$-\left(x^4 - 64x^3\right) f^{(4)}(x) - 2\left(5x^3 - 144x^2\right) f^{(3)}(x)$$
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for

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for further transormations see [arXiv:1706.01299]

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$$f(x) = \sum_{n=0}^{\infty} F(n) x^n$$

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$$f(x) = c_1 \cdot {}_3F_2\left[\begin{matrix}1,1,1\\ \frac{1}{2},\frac{1}{2}\end{matrix};\frac{x}{16}\right] + c_2 \sum_{n=0}^{\infty} \frac{S_1(n)}{\binom{2n}{n}^2} x^n$$

for

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II. A linear DE-solver for hypergeometric structures

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such that

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$F(n) =$ expression in terms of indefinite nested sums over hypergeometric products

III. A partial linear DE-solver

Find a power series solution

$$f(x_1,\dots,x_r)=\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}F(n_1,\dots,n_r)x_1^{n_1}\dots x_r^{n_r}$$

for

$$\sum_{(s_1,\dots,s_r)\in T}\underbrace{b_{(s_1,\dots,s_r)}(x_1,\dots,x_r)}_{\in\mathbb{K}[x_1,\dots,x_r]}D_{x_1}^{s_1}\dots D_{x_r}^{s_r}f(x_1,\dots,x_r)=0 \quad \underset{\text{finite}}{T\subset\mathbb{N}^r}$$

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Note:

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[otherwise Hilbert's 10th problem would be algorithmically decidable...]

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$$\sum_{(s_1,\dots,s_r)\in S}\underbrace{a_{(s_1,\dots,s_r)}(n_1,\dots,n_r)}_{\in\mathbb{K}[n_1,\dots,n_r]}F(n_1+s_1,\dots,n_r+s_r)=0 \qquad S\subset\mathbb{Z}^r \text{ finite}$$

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But: there are methods to hunt for solutions based on

M. Kauers, CS, *Partial denominator bounds for partial linear difference equations*, in: Proc. ISSAC'10 (2010)

M. Kauers, CS, *A refined denominator bounding algorithm for multivariate linear difference equations*, in: Proc. ISSAC'11 (2011)

J. Blümlein, M. Saragnese, CS, *Hypergeometric Structures in Feynman Integrals*, arXiv:2111.15501 [math-ph]

$$
\begin{aligned}
&(n+1)^2\left(k+n^2+2\right)\left(3kn^2-4k^2-5kn-12k+2n^3+2n^2-8n-8\right)F(n,k+1)\\
&+(n+1)^2\left(k+n^2+3\right)\left(2k^2-2kn^2+2kn+6k-n^3-n^2+4n+4\right)F(n,k+2)\\
&\quad+(n+1)^2(k+n+1)\left(2k-n^2+n+4\right)\left(k+n^2+1\right)F(n,k)\\
&\quad-(k+1)n^2(n+2)^2\left(k+n^2+2n+2\right)F(n+1,k)\\
&\quad+kn^2(n+2)^2\left(k+n^2+2n+3\right)F(n+1,k+1)=0
\end{aligned}
$$

$$
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\end{aligned}
$$

$$\Big\downarrow\; W=\{S_1(k),S_1(n+k),S_{2,1}(n+k)\}$$

degree bound 5

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degree bound 5

37 solutions $\dfrac{p}{(1+n)^2(1+k+n^2)}$ with

$$
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&\quad+(n+1)^2(k+n+1)\left(2k-n^2+n+4\right)\left(k+n^2+1\right)F(n,k)\\
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$$\Big\downarrow\; W=\{S_1(k),S_1(n+k),S_{2,1}(n+k)\}$$

degree bound 5

37 solutions $\dfrac{p}{(1+n)^2(1+k+n^2)}$ with

$$
\begin{aligned}
p\in\Big\{&1+\tfrac{1}{2}nS_1(k+n),k,n,kn,kn^2,kn^3,kn^4,kS_1(n),knS_1(n),kn^2S_1(n),kn^3S_1(n),kS_1(n)^2,\\
&knS_1(n)^2,kn^2S_1(n)^2,kS_1(n)^3,knS_1(n)^3,kS_1(n)^4,kS_{2,1}(n),knS_{2,1}(n),kn^2S_{2,1}(n),kn^3S_{2,}\\
&kS_1(n)S_{2,1}(n),knS_1(n)S_{2,1}(n),kn^2S_1(n)S_{2,1}(n),kS_1(n)^2S_{2,1}(n),knS_1(n)^2S_{2,1}(n),\\
&kS_1(n)^3S_{2,1}(n),kS_{2,1}(n)^2,knS_{2,1}(n)^2,kn^2S_{2,1}(n)^2,kS_1(n)S_{2,1}(n)^2,knS_1(n)S_{2,1}(n)^2,\\
&kS_1(n)^2S_{2,1}(n)^2,kS_{2,1}(n)^3,knS_{2,1}(n)^3,kS_1(n)S_{2,1}(n)^3,kS_{2,1}(n)^4\Big\}
\end{aligned}
$$

IV. A partial linear coupled DE-solver (arXiv:2111.15501)

Find a power series solution

$$f(x,y)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}F(n,m)x^n y^m$$

for

$$\begin{aligned}(x-1)yD_{xy}f(x,y)+(x(2\varepsilon+\tfrac{7}{2})-\varepsilon+1)D_x f(x,y)\\ +(x-1)xD_x^2 f(x,y)+y(2\varepsilon+1)D_y f(x,y)+\tfrac{3}{2}(2\varepsilon+1)f(x,y)=0,\end{aligned}$$

$$\begin{aligned}x(y-1)D_{xy}f(x,y)+x(4-\varepsilon)D_x f(x,y)+(y-1)yD_y^2 f(x,y)\\ +(y(\tfrac{13}{2}-\varepsilon)-\varepsilon+1)D_y f(x,y)+\tfrac{3(4-\varepsilon)}{2}f(x,y)=0.\end{aligned}$$

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$$f(x,y)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}F(n,m)x^ny^m$$

for

$$(x-1)yD_{xy}f(x,y)+(x(2\varepsilon+\tfrac{7}{2})-\varepsilon+1)D_xf(x,y)$$
$$+(x-1)xD_x^2f(x,y)+y(2\varepsilon+1)D_yf(x,y)+\tfrac{3}{2}(2\varepsilon+1)f(x,y)=0,$$

$$x(y-1)D_{xy}f(x,y)+x(4-\varepsilon)D_xf(x,y)+(y-1)yD_y^2f(x,y)$$
$$+(y(\tfrac{13}{2}-\varepsilon)-\varepsilon+1)D_yf(x,y)+\tfrac{3(4-\varepsilon)}{2}f(x,y)=0.$$

$$\downarrow$$

$$\tfrac{3}{2}(2\varepsilon+1)F(n,m)-n(\varepsilon-1)F(n+1,m)=0,$$
$$-\tfrac{3}{2}(\varepsilon-4)F(n,m)-m(\varepsilon-1)F(n,m+1)=0.$$

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$$\begin{aligned}\tfrac{3}{2}(2\varepsilon+1)F(n,m)-n(\varepsilon-1)F(n+1,m)=0,\\-\tfrac{3}{2}(\varepsilon-4)F(n,m)-m(\varepsilon-1)F(n,m+1)=0.\end{aligned}$$

$\downarrow$simplified Ore-Sato Theorem

$$F(n,m)=\Big(\prod_{i=1}^{n}\frac{(1+2i)(3+i-\varepsilon)}{2i(-2+i+\varepsilon)}\Big)\prod_{i=1}^{m}\frac{(1+2i+2n)(i+2\varepsilon)}{2i(-2+i+n+\varepsilon)}$$

IV. A partial linear coupled DE-solver (arXiv:2111.15501)

Find a power series solution

$$f(x,y)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}F(n,m)x^n y^m=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(\frac{3}{2}\right)_{m+n}(4-\varepsilon)_n(1+2\varepsilon)_m}{m!n!(-1+\varepsilon)_{m+n}}$$

for

$$(x-1)yD_{xy}f(x,y)+(x(2\varepsilon+\tfrac{7}{2})-\varepsilon+1)D_x f(x,y)$$
$$+(x-1)xD_x^2 f(x,y)+y(2\varepsilon+1)D_y f(x,y)+\tfrac{3}{2}(2\varepsilon+1)f(x,y)=0,$$

$$x(y-1)D_{xy}f(x,y)+x(4-\varepsilon)D_x f(x,y)+(y-1)yD_y^2 f(x,y)$$
$$+(y(\tfrac{13}{2}-\varepsilon)-\varepsilon+1)D_y f(x,y)+\tfrac{3(4-\varepsilon)}{2}f(x,y)=0.$$

$$\downarrow$$

$$\tfrac{3}{2}(2\varepsilon+1)F(n,m)-n(\varepsilon-1)F(n+1,m)=0,$$
$$-\tfrac{3}{2}(\varepsilon-4)F(n,m)-m(\varepsilon-1)F(n,m+1)=0.$$

$\downarrow$ simplified Ore-Sato Theorem

$$F(n,m)=\frac{\left(\frac{3}{2}\right)_{m+n}(4-\varepsilon)_n(1+2\varepsilon)_m}{m!n!(-1+\varepsilon)_{m+n}}$$

$$f(x,y)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\underbrace{\frac{\left(\frac{3}{2}\right)_{m+n}(4-\varepsilon)_n(1+2\varepsilon)_m}{m!n!(-1+\varepsilon)_{m+n}}}_{F_{-1}(n,m)\varepsilon^{-1}+F_0(n,m)\varepsilon^0+\dots}$$

$$f(x,y)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\underbrace{\frac{\left(\frac{3}{2}\right)_{m+n}(4-\varepsilon)_n(1+2\varepsilon)_m}{m!n!(-1+\varepsilon)_{m+n}}}_{F_{-1}(n,m)\varepsilon^{-1}+F_0(n,m)\varepsilon^0+\ldots}$$

$$F_{-1}(n,m)=-\frac{1}{6}\frac{x^m y^n(3+n)!\left(\frac{3}{2}\right)_{m+n}}{n!(-2+m+n)!}$$

$$F_0(n,m)=\left[\ldots 6S_1(n)+6S_1(m+n)-12S_1(m)\right]\frac{x^m y^n(3+n)!\left(\frac{3}{2}\right)_{m+n}}{n!(-2+m+n)!}$$

$$f(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \underbrace{\frac{\left(\frac{3}{2}\right)_{m+n} (4-\varepsilon)_n (1+2\varepsilon)_m}{m! n! (-1+\varepsilon)_{m+n}}}_{F_{-1}(n,m)\varepsilon^{-1} + F_0(n,m)\varepsilon^0 + \ldots}$$

$$= \varepsilon^{-1} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} F_{-1}(n,m) + \varepsilon^0 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} F_0(n,m) + \ldots$$

$$F_{-1}(n,m) = -\frac{1}{6} \frac{x^m y^n (3+n)! \left(\frac{3}{2}\right)_{m+n}}{n!(-2+m+n)!}$$

$$F_0(n,m) = \left[\ldots 6S_1(n) + 6S_1(m+n) - 12S_1(m)\right] \frac{x^m y^n (3+n)! \left(\frac{3}{2}\right)_{m+n}}{n!(-2+m+n)!}$$

$$f(x,y)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\underbrace{\frac{\left(\frac{3}{2}\right)_{m+n}(4-\varepsilon)_n(1+2\varepsilon)_m}{m!n!(-1+\varepsilon)_{m+n}}}_{F_{-1}(n,m)\varepsilon^{-1}+F_0(n,m)\varepsilon^0+\dots}$$

$$=\varepsilon^{-1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}F_{-1}(n,m)+\varepsilon^{0}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}F_0(n,m)+\dots$$

$||$ `Sigma.m`

$$\varepsilon^{-1}\Big[-\frac{P_1(x,y)}{64(-1+x)^2(-1+y)^5(x-y)^3}-\frac{15x^6}{4(-1+x)^2(x-y)^4}\sum_{i=1}^{\infty}\frac{x^i\left(\frac{3}{2}\right)_i}{i!}$$

$$+\frac{P_2(x,y)}{64(-1+y)^5(x-y)^4}\sum_{i=1}^{\infty}\frac{y^i\left(\frac{3}{2}\right)_i}{i!}\Big]+\varepsilon^0\Big[\dots\Big]+\dots$$

$$f(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \underbrace{\frac{\left(\frac{3}{2}\right)_{m+n}(4-\varepsilon)_n(1+2\varepsilon)_m}{m!n!(-1+\varepsilon)_{m+n}}}_{F_{-1}(n,m)\varepsilon^{-1}+F_0(n,m)\varepsilon^0+\dots}$$

$$= \varepsilon^{-1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} F_{-1}(n,m) + \varepsilon^{0}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} F_0(n,m) + \dots$$

$||$ `Sigma.m`

$$\varepsilon^{-1}\Big[-\frac{P_1(x,y)}{64(-1+x)^2(-1+y)^5(x-y)^3} - \frac{15x^6}{4(-1+x)^2(x-y)^4}\sum_{i=1}^{\infty}\frac{x^i\left(\frac{3}{2}\right)_i}{i!}$$

$$+\frac{P_2(x,y)}{64(-1+y)^5(x-y)^4}\sum_{i=1}^{\infty}\frac{y^i\left(\frac{3}{2}\right)_i}{i!}\Big] + \varepsilon^0\Big[\dots\Big] + \dots$$

$||$

$$\varepsilon^{-1}\Big[-\frac{15x^6}{4(x-y)^4(1-x)^{7/2}} - \frac{15y^3Q(x,y)}{64(x-y)^4(1-y)^{13/2}}\Big] + \varepsilon^0\Big[\dots\Big] + \dots$$

V. Advanced expansion and summation

Consider the innocent DE:

$$(1-x)xf''(x) - (4\varepsilon + 2x - 3)f'(x) - (\varepsilon + 1)f(x) = 0$$

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$$\downarrow$$

$$F(n) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{n} (\varepsilon + 1 - i + i^2)}{n!(3-4\varepsilon)_n}$$

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$$\downarrow$$
$$F(n)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{n}(\varepsilon+1-i+i^2)}{n!(3-4\varepsilon)_n}\overset{\texttt{Sigma.m}}{=}F_0+\varepsilon F_1+O(\varepsilon^2)$$

with

$$F_0=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{n}(1-i+i^2)}{n!(3)_n}$$

$$F_1=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{n}(1-i+i^2)}{n!(3)_n}\left(-\frac{2n(3n+5)}{(n+1)(n+2)}+4S_1(n)+\sum_{i=1}^{n}\frac{1}{1-i+i^2}\right)$$

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$$F(n)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{n}(\varepsilon+1-i+i^2)}{n!(3-4\varepsilon)_n}\overset{\texttt{Sigma.m}}{=}F_0+\varepsilon\,F_1+O(\varepsilon^2)$$

$$F_0=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{n}(1-i+i^2)}{n!(3)_n}$$

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$$F_0=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{n}(1-i+i^2)}{n!(3)_n}=\lim_{N\to\infty}\sum_{n=0}^{N}\frac{\prod_{i=1}^{n}(1-i+i^2)}{n!(3)_n}$$

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$$\overset{\texttt{Sigma.m}}{=}\lim_{N\to\infty}\frac{(3+N)\left(1+N+N^2\right)}{3N!(3)_N}\prod_{i=1}^{n}\left(1-i+i^2\right)$$

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$$\overset{\texttt{Sigma.m}}{=}\lim_{N\to\infty}\frac{(3+N)\left(1+N+N^2\right)}{3N!(3)_N}\prod_{i=1}^{n}\left(1-i+i^2\right)$$
$$=\frac{2\cosh(\frac{\sqrt{3}\pi}{2})}{3\pi}$$

V. Advanced expansion and summation

$$F_1 = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{n}(1-i+i^2)}{n!(3)_n}\left(-\frac{2n(3n+5)}{(n+1)(n+2)} + 4S_1(n) + \sum_{i=1}^{n}\frac{1}{1-i+i^2}\right)$$

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$$\overset{\texttt{Sigma.m}}{=} -\frac{8}{3} + \frac{8C}{3} - \frac{20\cosh\left(\frac{\sqrt{3}\pi}{2}\right)}{9\pi} + \frac{2\cosh\left(\frac{\sqrt{3}\pi}{2}\right)}{3\pi}\sum_{i=1}^{\infty}\frac{1}{1-i+i^2}$$

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with the extra constant

$$C = \sum_{k=1}^{\infty}\left(\frac{1}{\pi k}\cosh\left[\frac{\sqrt{3}\pi}{2}\right] - \frac{1}{(k!)^2}\prod_{l=1}^{k}(1-l+l^2)\right)$$

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with the extra constant

$$C = \sum_{k=1}^{\infty} \left(\frac{1}{\pi k} \cosh\left[\frac{\sqrt{3}\pi}{2}\right] - \frac{1}{(k!)^2} \prod_{l=1}^{k}(1-l+l^2) \right)$$

$$= 1 + \frac{2\cosh\left[\frac{\sqrt{3}\pi}{2}\right] \left\{ \Re\left[\psi\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\right] + \gamma_E \right\}}{\pi}$$

with γ_E the Euler–Mascheroni constant and $\psi(x)$ the digamma function.

Conclusion

1. Up-to-date solver for linear recurrences and DEs (within `Sigma.m` and `HarmonicSums.m`)
2. Interplay: DE solver $\longleftrightarrow$ RE solver
3. Finding (generalized) hypergeometric structures from DEs
4. A first prototype method to solve partial linear DE/RE equations in QCD
5. A tactic to find Apell-like structures of coupled DE-systems
6. Advance expansion and summation methods for the simplification of the found solutions