Epsilon Factorized Differential Equations for Elliptic Feynman Integrals

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Niels Bohr International Academy (NBIA), University of Copenhagen.

April 27, 2022

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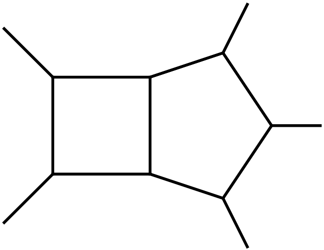

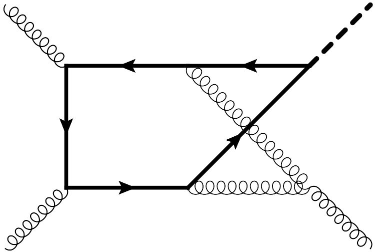

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see the literature in the near future

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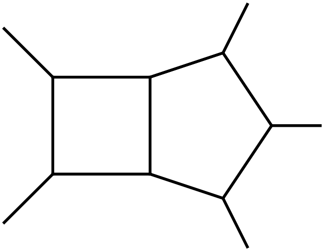

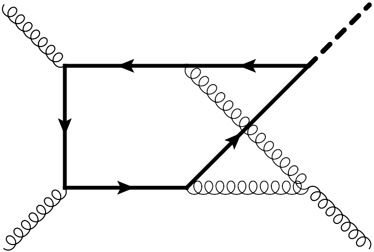

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see the literature in the near future

see however *Seva Chestnov* on Friday and *Henrik Munch* later today
on intersection related stuff

Published for SISSA by Springer

Received: *November 9, 2021*
Revised: *February 24, 2022*
Accepted: *February 25, 2022*
Published: *March 11, 2022*

On epsilon factorized differential equations for elliptic Feynman integrals

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Abstract: In this paper we develop and demonstrate a method to obtain epsilon factorized differential equations for elliptic Feynman integrals. This method works by choosing an integral basis with the property that the period matrix obtained by integrating the basis over a complete set of integration cycles is diagonal. The method is a generalization of a similar method known to work for polylogarithmic Feynman integrals. We demonstrate the method explicitly for a number of Feynman integral families with an elliptic highest sector.

Keywords: Perturbative QCD, Scattering Amplitudes

ArXiv ePrint: 2110.07968

JHEP03(2022)079

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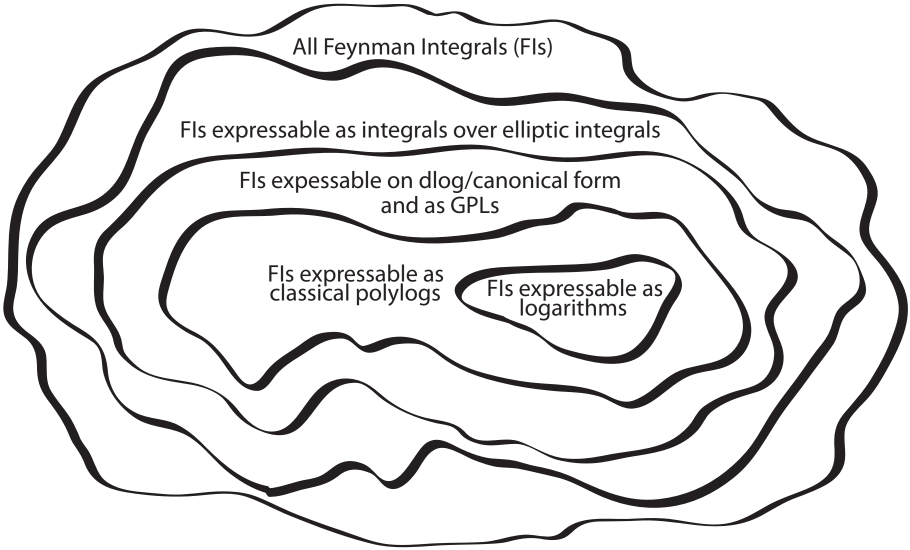
All Feynman Integrals (FIs)
FIs expressable as integrals over elliptic integrals
FIs expessable on dlog/canonical form and as GPLs
FIs expressable as classical polylogs
FIs expressable as logarithms

The metod of differential equations is the most fruitful approach to the computation of Feynman integrals

In general the equation system $\partial_s \tilde{J} = \tilde{A}^{(s)} \tilde{J}$ will be hard to solve.

Differential equations in *canonical form* [Henn (2013)]

$$\partial_s \bar{J} = \epsilon A^{(s)} \bar{J} \tag{1}$$

A is free of epsilon dependence, and additionally

$$A^{(s)} = \sum_i B_i \partial_s \log(f_i(s)) \tag{2}$$

In many cases, this can be trivially integrated order by order in ϵ to give

$$J_i = \sum_j G_{ij} \epsilon^j$$

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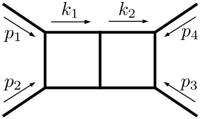
Eq. (2) does not generalize beyond GPLs. But how about eq. (1)?
Let us go through how to obtain the canonical form in a way that generalizes.

Let us start by a non-elliptic example from [Henn (2013)], to motivate our method.

Massless double box:



This integral family has eight master integrals - only two in the highest sector

$$I_{\{a\}} = \int \frac{u\, x_8^{-a_8}\, \mathrm{d}^8 x}{x_1^{a_1} \cdots x_7^{a_7}} \quad \to \quad I_{7\times\text{cut}} = \int_{\mathcal{C}} u_{7\times\text{cut}}\, \hat{\phi}\, \mathrm{d}z \qquad u_{7\times\text{cut}} = s^{d-6} z^{\frac{d}{2}-3} (z+s)^{2-\frac{d}{2}} (z-t)^{d-5}$$

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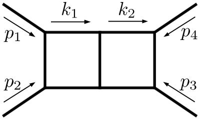

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It is known that $J_1 = s^2 t I_{1111111;0}$, $J_2 = s^2 I_{1111111;-1}$ gives canonical form.

$$\partial_s \bar{J} = \epsilon A^{(s)} \bar{J} \qquad \text{with} \qquad A^{(s)} = \begin{bmatrix} \frac{2}{s+t} - \frac{2}{s} & \frac{2}{s} - \frac{2}{s+t} \\ \frac{1}{s+t} & \frac{-2}{s} - \frac{1}{s+t} \end{bmatrix}$$

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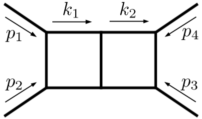

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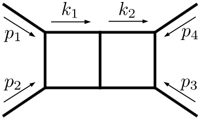

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$a = 0$:	one pole in $z = 0$ of $1/(s^2 t)$	and	one pole in $z = t$ of $-1/(s^2 t)$
$a = -1$:	one pole in $z = t$ of $-1/(s^2)$	and	one pole in $z = \infty$ of $1/(s^2)$

The above prefactors make the integrals pure.

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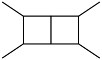
Motivation

Let us do  again in a different way

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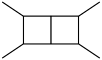
Let us do  again in a different way

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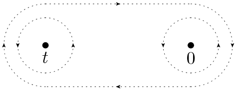
Possible integration contours:



$J_1 = f_1 I_{1111111;0}$, $J_2 = f_2 I_{1111111;-1}$ and also $\gamma_1 = \mathcal{C}_0, \gamma_2 = \mathcal{C}_\infty$

We write down the *period matrix* $P_{ij} = \int_{\gamma_j} \hat{\Phi}_i \mathrm{d}z$

$$\hat{\Phi}_i = \frac{-z^{i-1}}{s^2 z(z-t)} \quad \Rightarrow \quad P = 2\pi i \begin{bmatrix} \frac{f_1}{s^2 t} & 0 \\ 0 & \frac{f_2}{s^2} \end{bmatrix}$$

$$P = 2\pi i I \Rightarrow f_1 = s^2 t, f_2 = s^2$$

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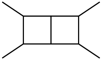
Let us do  again in a different way

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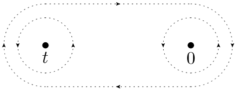
Possible integration contours: 

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$$J_i = f_{i1} I_{1111111;0} + f_{i2} I_{1111111;-1} \quad \text{gives} \quad P = 2\pi i \begin{bmatrix} \frac{f_{11}}{s^2 t} & \frac{f_{12}}{s^2} \\ \frac{f_{21}}{s^2 t} & \frac{f_{22}}{s^2} \end{bmatrix}$$

$$P = 2\pi i I \Rightarrow f_{11} = s^2 t, f_{12} = 0, f_{21} = 0, f_{22} = s^2$$

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The algorithm

For $J_i = \int_{\mathcal{C}} u\hat{\varphi}_i \mathrm{d}^n x$ write $u\hat{\varphi}_i = \sigma\hat{\Phi}_i$

where σ is *pure* and $\hat{\Phi}$ free of ϵ exponents

(In our previous example

$$s^{-2(1+\epsilon)} z^{-1-\epsilon} (s+z)^{\epsilon} (s-z)^{-1-2\epsilon} \hat{\varphi} \;=\; s^{-2\epsilon} z^{-\epsilon} (z+s)^{\epsilon} (z-t)^{-2\epsilon} \times \frac{\hat{\varphi}}{s^2 z (z-t)} \Bigg)$$

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then I claim: The set of J_i will have epsilon factorized diff-eqs if

$$P = (2\pi i)^n I \qquad \text{where} \qquad P_{ij} = \int_{\gamma_j} \hat{\Phi}_i \mathrm{d}^n x$$

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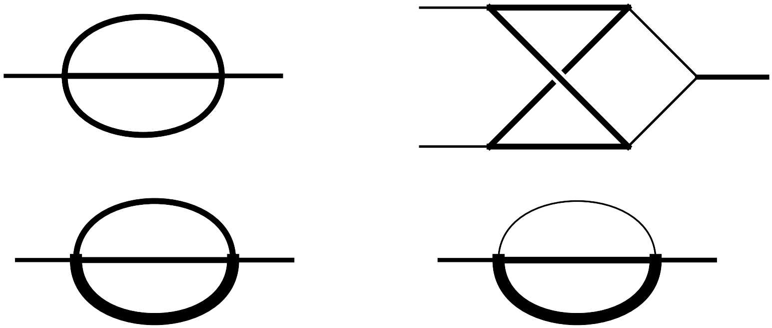
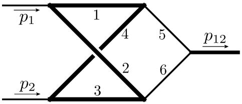
In practice: $\hat{\varphi}_i = \sum_l f_{il}\hat{\phi}_l$ where the $\hat{\phi}_l$ are an *intermediate basis*.

$P = (2\pi I)^n I$ gives ν^2 constraints, fixes all f_{il} uniquely.

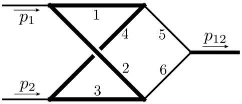
I did four elliptic examples in the paper:



First elliptic example: 

The non-planar double triangle

Again two integrals in the highest, elliptic sector.

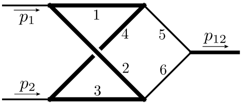
First elliptic example:  The non-planar double triangle

Again two integrals in the highest, elliptic sector. Around $d = 4$

$$u|_{6\times\text{cut}} = s^{-1+2\epsilon}\big(z(z+s)(z^2+sz-4m^2s)\big)^{-\frac{1}{2}-\epsilon}$$

Factorizing out the pure part we get integrals of the form

$$\int_{\mathcal{C}} \frac{\hat{\phi}\mathrm{d}z}{Y} \qquad \text{with} \qquad Y = \sqrt{z(z+s)(z^2+sz-4m^2s)}$$

First elliptic example: 

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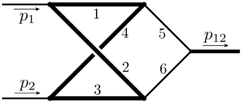

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$$Y^2 = (z-r_{\text{i}})(z-r_{\text{ii}})(z-r_{\text{iii}})(z-r_{\text{iv}}) \qquad \text{with}$$

$$r_{\text{i}} = -\tfrac{1}{2}\sqrt{s}(\sqrt{s}+\sqrt{16m^2+s})\,, \quad r_{\text{ii}} = -s\,, \quad r_{\text{iii}} = 0\,, \quad r_{\text{iv}} = -\tfrac{1}{2}\sqrt{s}(\sqrt{s}-\sqrt{16m^2+s})$$

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Examples: npt

What are the independent contours for

$$\int_{\mathcal{C}} \frac{\hat{\phi}\mathrm{d}z}{Y} \qquad \text{with} \qquad Y = \sqrt{(z-r_{\mathsf{i}})(z-r_{\mathsf{ii}})(z-r_{\mathsf{iii}})(z-r_{\mathsf{iv}})}$$

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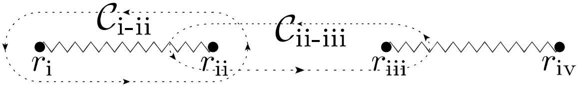

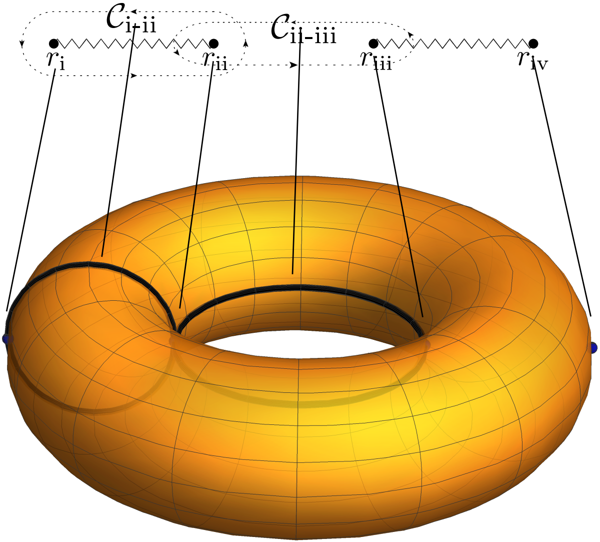

$\mathcal{C}_{\text{i-ii}}$
$\mathcal{C}_{\text{ii-iii}}$
r_{i}
r_{ii}
r_{iii}
r_{iv}

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We have

$$\hat{\phi}_1 = \frac{1}{s}\,, \quad \hat{\phi}_2 = \frac{(1+2\epsilon)(z+s)}{s\,(z^2+sz-4m^2 s)}\,, \quad \gamma_1 = \mathcal{C}_{\text{ii-iii}}\,, \quad \gamma_2 = \mathcal{C}_{\text{i-ii}}\,,$$

and we want

$$P_{ij} = \int_{\gamma_j} \frac{(f_{i1}\hat{\phi}_1 + f_{i2}\hat{\phi}_2)\mathrm{d}z}{Y} = f_{il}\,g_{lj} \qquad \text{with} \qquad Y = \sqrt{z(z+s)(z^2+sz-4m^2 s)}$$

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$$g_{11} = \int_{\gamma_1} \frac{\hat{\phi}_1 \mathrm{d}z}{Y} = \frac{8K(k^2)}{s^{3/2}(\sqrt{16m^2+s}+\sqrt{s})} \quad \text{where} \quad k^2 = \frac{4\sqrt{s}\sqrt{16m^2+s}}{(\sqrt{16m^2+s}+\sqrt{s})^2}$$

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$$g_{21} = \frac{-8(1+2\epsilon)}{s^{3/2}(\sqrt{16m^2+s}+\sqrt{s})}\left(K(k^2) + \frac{\sqrt{16m^2+s}+\sqrt{s}}{\sqrt{16m^2+s}-\sqrt{s}}E(k^2)\right)$$

$$g_{22} = \frac{i(1+2\epsilon)}{s^{3/2}m^2}\left(\frac{K(1-k^2)}{\sqrt{16m^2+s}} - \frac{\sqrt{16m^2+s}+\sqrt{s}}{2(16m^2+s)}E(1-k^2)\right)$$

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$$\hat{\phi}_1 = \frac{1}{s}\,, \quad \hat{\phi}_2 = \frac{(1+2\epsilon)(z+s)}{s\,(z^2+sz-4m^2s)}\,, \quad \gamma_1 = \mathcal{C}_{\text{ii-iii}}\,, \quad \gamma_2 = \mathcal{C}_{\text{i-ii}}\,,$$

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$$P_{ij} = \int_{\gamma_j} \frac{(f_{i1}\hat{\phi}_1 + f_{i2}\hat{\phi}_2)\mathrm{d}z}{Y} = f_{il}g_{lj} \qquad \text{with} \qquad Y = \sqrt{z(z+s)(z^2+sz-4m^2s)}$$

$$g_{11} = \frac{8K(k^2)}{s^{3/2}(\sqrt{16m^2+s}+\sqrt{s})} \qquad g_{12} = \frac{-8iK(1-k^2)}{s^{3/2}(\sqrt{16m^2+s}+\sqrt{s})}$$

$$g_{21} = \frac{-8(1+2\epsilon)}{s^{3/2}(\sqrt{16m^2+s}+\sqrt{s})}\left(K(k^2) + \frac{\sqrt{16m^2+s}+\sqrt{s}}{\sqrt{16m^2+s}-\sqrt{s}}E(k^2)\right)$$

$$g_{22} = \frac{i(1+2\epsilon)}{s^{3/2}m^2}\left(\frac{K(1-k^2)}{\sqrt{16m^2+s}} - \frac{\sqrt{16m^2+s}+\sqrt{s}}{2(16m^2+s)}E(1-k^2)\right)$$

Imposing $P = 2\pi i I$ fixes the f_{il} uniquely, for instance

$$f_{11} = \tfrac{1}{2}is^{3/2}(\sqrt{16m^2+s}+\sqrt{s})E(1-k^2) - is^{3/2}\sqrt{16m^2+s}\,K(1-k^2)$$

$\ldots$

Examples: npt

So now we have $J_i = f_{i1} I_{111111;0} + f_{i2} I_{211111;0}$ with f_{il} fixed.

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So now we have $J_i = f_{i1} I_{111111;0} + f_{i2} I_{211111;0}$ with f_{il} fixed.

We get $\partial_s \bar{J} = \epsilon A \bar{J}$ with

$$A_{11} = \frac{8(12m^2+s)K(k^2)K(1-k^2)}{\pi\sqrt{s}(16m^2+s)(\sqrt{16m^2+s}+\sqrt{s})} + \frac{2}{\pi s}\left(1-\frac{8m^2}{16m^2+s}+\frac{\sqrt{s}}{\sqrt{16m^2+s}}\right)E(k^2)E(1-k^2) + \frac{-4(12m^2+s)K(k^2)E(1-k^2)}{\pi s(16m^2+s)} + \frac{-2(\sqrt{16m^2+s}+\sqrt{s})E(k^2)K(1-k^2)}{\pi\sqrt{s}(16m^2+s)}$$

$$A_{12} = \frac{-64im^2K(1-k^2)^2}{\pi\sqrt{s}\sqrt{16m^2+s}(\sqrt{16m^2+s}+\sqrt{s})} + \frac{i(\sqrt{16m^2+s}+\sqrt{s})^2E(1-k^2)^2}{\pi s(16m^2+s)} + \frac{4i}{\pi s}\left(\frac{\sqrt{s}}{\sqrt{16m^2+s}} - \frac{8m^2}{16m^2+s}\right)K(1-k^2)E(1-k^2)$$

$$A_{21} = \frac{i(12m^2+s)(\sqrt{16m^2+s}-\sqrt{s})^2K(k^2)^2}{4m^2\pi s(16m^2+s)} + \frac{i(\sqrt{16m^2+s}-\sqrt{s})^2E(k^2)^2}{\pi s(16m^2+s)} + \frac{-4iK(k^2)E(k^2)}{\pi s}$$

$$A_{22} = \frac{2(\sqrt{16m^2+s}-\sqrt{s})^2K(k^2)K(1-k^2)}{\pi s\sqrt{16m^2+s}(\sqrt{16m^2+s}+\sqrt{s})} + \frac{-2}{s\pi}\left(1-\frac{8m^2}{16m^2+s}-\frac{\sqrt{s}}{\sqrt{16m^2+s}}\right)E(k^2)E(1-k^2) + \frac{16m^2K(k^2)E(1-k^2)}{\pi s(16m^2+s)} + \frac{-32m^2E(k^2)K(1-k^2)}{\pi s\sqrt{16m^2+s}(\sqrt{16m^2+s}+\sqrt{s})}$$

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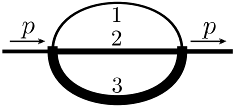
Examples: s3m

Next example: 

The three mass elliptic sunrise

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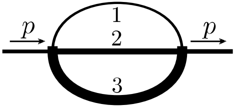
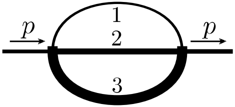


The three mass elliptic sunrise

$$u|_{3\times\mathsf{cut}} = z^{\epsilon}\left(z^2-2(m_1^2+m_2^2)z+(m_1^2-m_2^2)^2\right)^{-\frac{1}{2}-\epsilon}\left(z^2-2(m_3^2+s)z+(m_3^2-s)^2\right)^{-\frac{1}{2}-\epsilon}$$

There are four MIs. We pick intermediate basis $I_{111;00}$, $I_{211;00}$, $I_{111;\text{-}10}$, $I_{111;0\text{-}1}$

$$\hat{\phi}_1 = 1\,, \quad \hat{\phi}_2 = \frac{(1+2\epsilon)(z+m_2^2+m_3^2)}{z^2-2(m_1^2+m_2^2)z+(m_1^2-m_2^2)^2}\,, \quad \hat{\phi}_3 = z\,, \quad \hat{\phi}_4 \sim \frac{1}{z}$$

Next example:  The three mass elliptic sunrise

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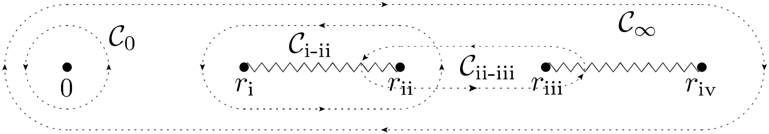

$$\gamma_1 = \mathcal{C}_{\text{ii-iii}}\,, \quad \gamma_2 = \mathcal{C}_{\text{i-ii}}\,, \quad \gamma_3 = \mathcal{C}_\infty\,, \quad \gamma_4 = \mathcal{C}_0$$

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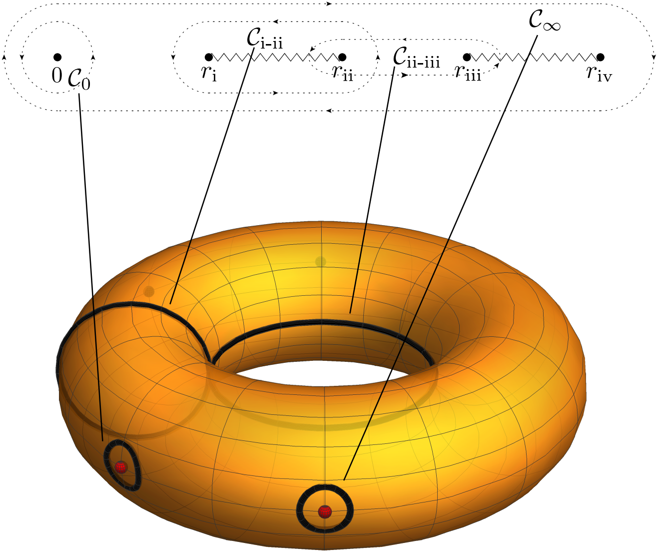
Examples: s3m



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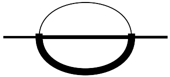
The three mass sunrise: 

Integrands $\hat{\phi}_1 = 1, \;\; \hat{\phi}_2 = \frac{(1+2\epsilon)(z+m_2^2+m_3^2)}{z^2-2(m_1^2+m_2^2)z+(m_1^2-m_2^2)^2}, \;\; \hat{\phi}_3 = z, \;\; \hat{\phi}_4 \sim \frac{1}{z}$

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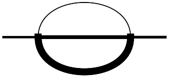
The three mass sunrise: 

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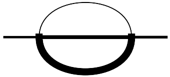

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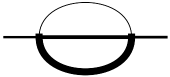
The three mass sunrise: 

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$g_{13} = 0, \quad g_{33} = -2\pi i, \quad \cdots$ We also see $\Pi(\tilde{n}^2, k^2)$

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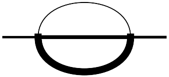

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We may then impose $P = 2\pi i I$. 16 constraints fixes the f_{il} uniquely.

$\partial_s \bar{J} = \epsilon A^{(s)} \bar{J}$. The expressions are too big to be written here ...

Discussion + Conclusion

Integration to elliptic GPLs not straight forward.

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We have (1) but not (2)

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I hope my algorithm and expressions can be a step in the generalization of canonical forms to the elliptic case and beyond.

Thank you for inviting me
and thank you for listening!

Hjalte Frellesvig

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