Maximal weight contribution of Scattering Amplitudes

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Loops & Legs in QFT
April 25 — 30, 2022
Ettal, Germany

European Research Council
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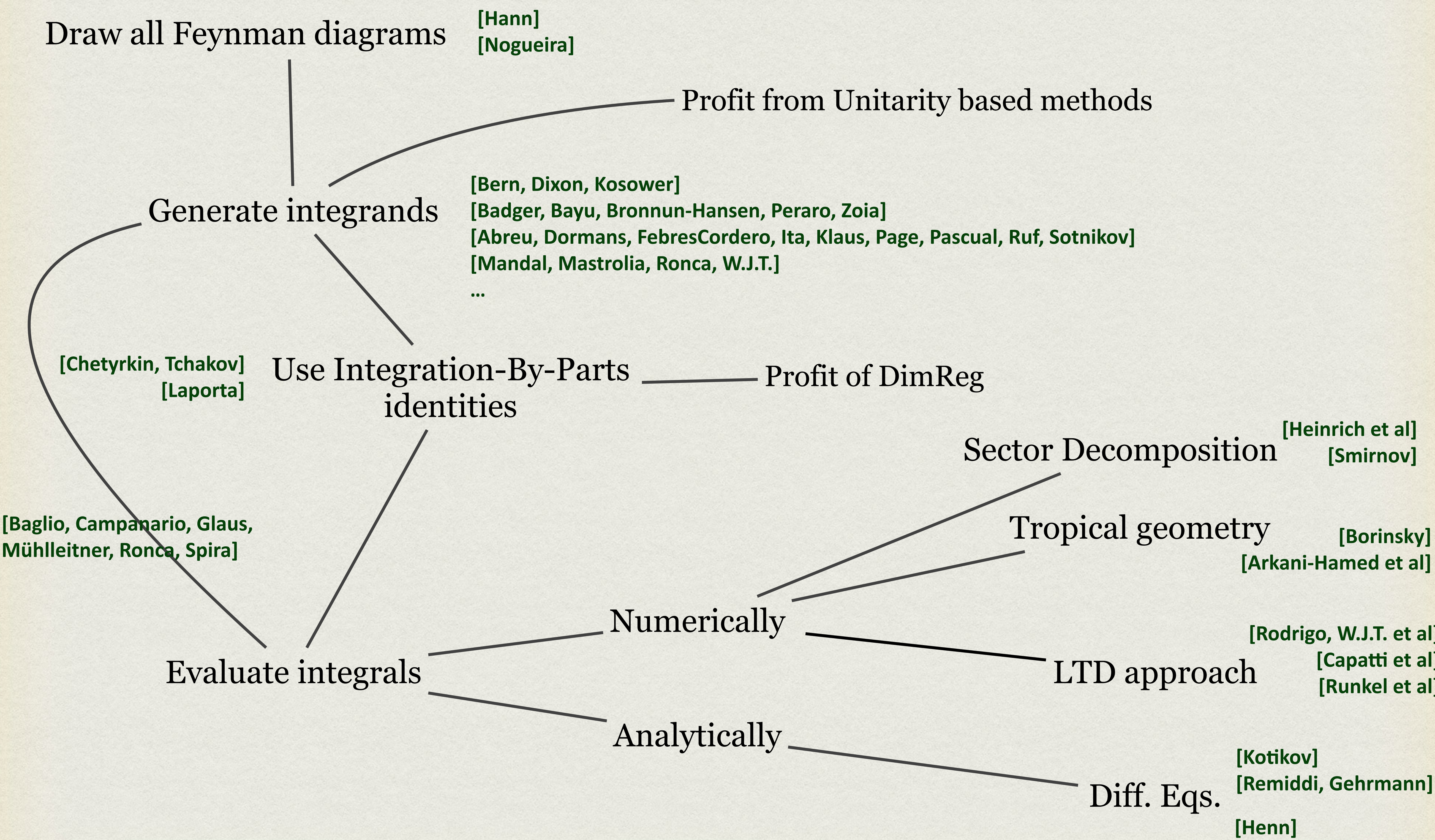
Standard approach @multi-loop level



Standard approach @multi-loop level

This talk: we propose a method that gives preliminary insights on the structure of Scattering Amplitude w/o involving heavy computational calculations

$$\mathscr{A}^{(L)} = \int \omega^{(L)} = \sum_{i=1}^{n} \tilde{c}_i \int \prod_{j=1}^{4L} d\log\left(\tau_{i,j}\right) + \dots$$

Maximal Weight Contribution
of Scattering Amplitudes

2112.08900 [hep-th]

Outline

- Constructing Dlog integrands
- Maximal Weight contribution of Scattering Amplitudes
- Application —> H-> gg @ two loops
- Evanescent terms
- Conclusions & Outlook

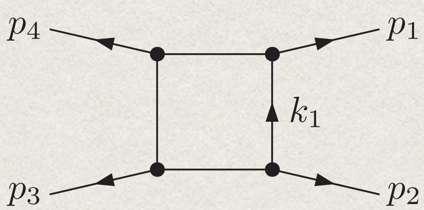
This talk is based on **2112.08900 [hep-th]**

Constructing D-Log integrands

Constructing D-Log integrands

- Rely on algorithm proposed by [Wasser (2018)] [Henn, Mistlberger, Smirnov, Wasser (2020)]
  from an Ansatz :: get integrands that admit a dlog representation

- Four-point one-loop integrand family

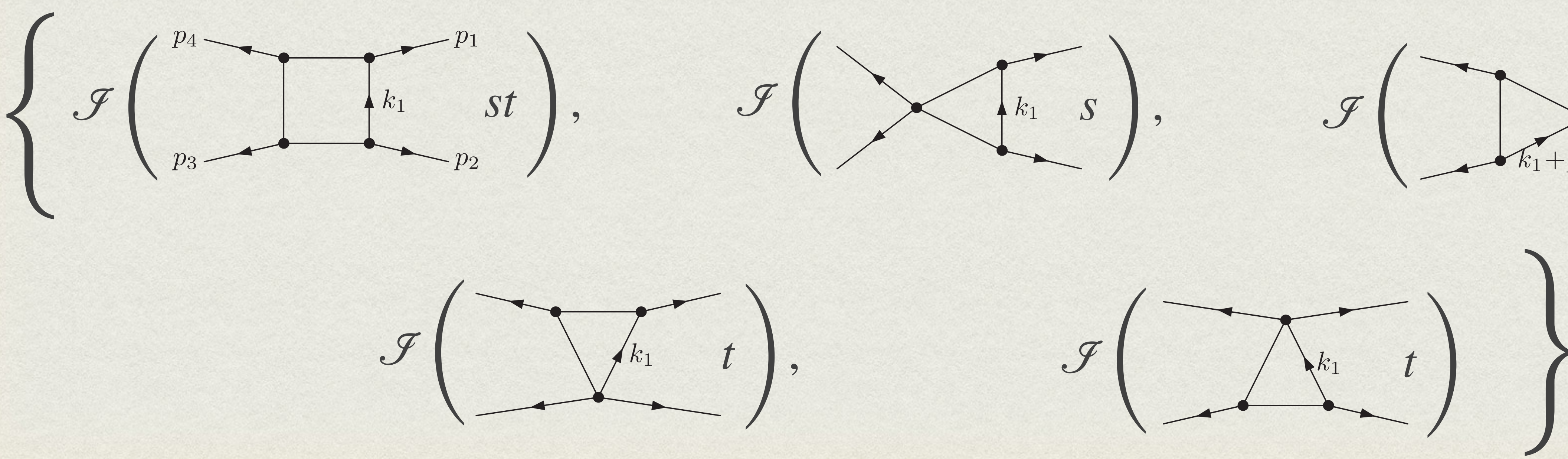
$$\mathcal{I}\Bigg(\ \ $$

$$\mathcal{N}\Bigg) = \frac{d^4k_1\,\mathcal{N}}{\left(k_1-p_1\right)^2 k_1^2 \left(k_1+p_2\right)^2 \left(k_1+p_2+p_3\right)^2} \to d\log\tau_1 \ldots d\log\tau_4$$

integrand family

$s = (p_1+p_2)^2\,,\ t = (p_2+p_3)^2$

- Mathematica implementation :: `DlogBasis`



Constructing D-Log integrands

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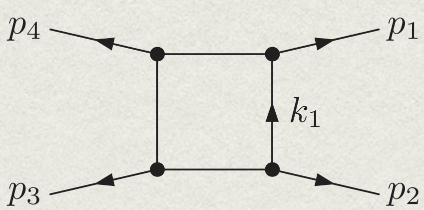

$\mathcal{N}\Big) = \dfrac{d^4k_1\,\mathcal{N}}{\left(k_1-p_1\right)^2 k_1^2 \left(k_1+p_2\right)^2 \left(k_1+p_2+p_3\right)^2} \to d\log\tau_1 \;\ldots d\log\tau_4$

integrand family

$s=(p_1+p_2)^2\,,\; t=(p_2+p_3)^2$

- Mathematica implementation :: `DlogBasis`

Leading Logarithmic singularities

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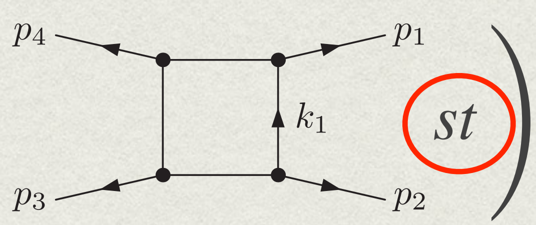

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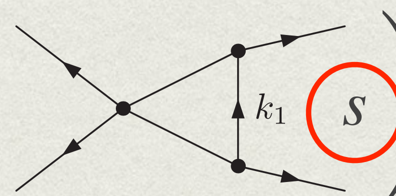

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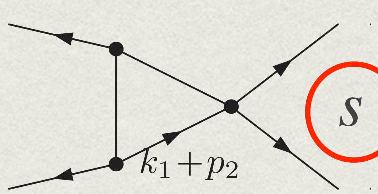

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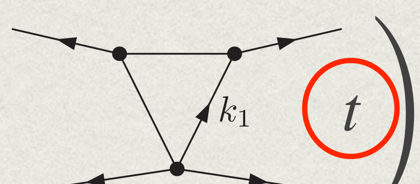

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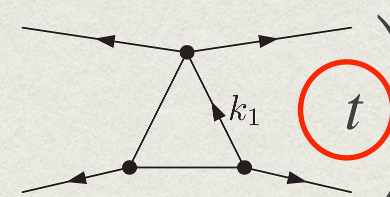


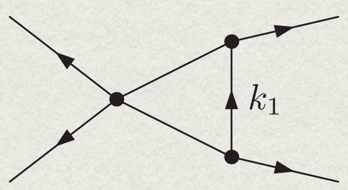
$\Big)\Big\}$

Constructing D-Log integrands

- Leading singularities —> one-loop triangle

Parametrise loop momentum in a four-dimensional basis

$$k_1^\mu = \alpha_1\, p_1^\mu + \alpha_2\, p_2^\mu + \alpha_3\, \epsilon_{12}^\mu + \alpha_4\, \epsilon_{21}^\mu$$

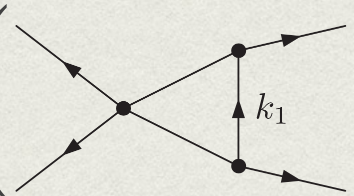
$\mathcal{I}\Big($ 

$\Big)$

$$= \frac{d^4 k_1}{(k_1 - p_1)^2\, k_1^2\, (k_1 + p_2)^2}$$

$$= \frac{d\alpha_1\, d\alpha_2\, d\alpha_3\, d\alpha_4}{s\underbrace{\left(\left(\alpha_1 - 1\right)\alpha_2 - \alpha_3\alpha_4\right)}_{\tau_1}\underbrace{\left(\alpha_1\alpha_2 - \alpha_3\alpha_4\right)}_{\tau_2}\underbrace{\left(\alpha_1\left(\alpha_2 + 1\right) - \alpha_3\alpha_4\right)}_{\tau_3}}$$

Change of variables

$\mathcal{I}\Big($ 

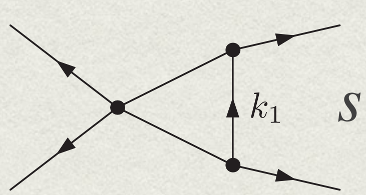
$\Big)$

$$= \frac{1}{s} d\log\tau_1\, d\log\tau_2\, d\log\tau_3\, d\log\alpha_4$$

$$= \frac{1}{s} d\log\left(k_1 - p_1\right)^2\, d\log k_1^2\, d\log\left(k_1 + p_2\right)^2\, d\log\left(2\, k_1\cdot e_3\right)$$

- Define dlog integrand

single term only!

$\mathcal{I}\Big($ 

$s\Big)$

$$= d\log\left(k_1 - p_1\right)^2\, d\log k_1^2\, d\log\left(k_1 + p_2\right)^2\, d\log\left(2\, k_1\cdot e_3\right)$$

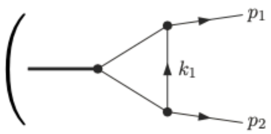
dlog integrands as products of dlog forms

- dlog form for integrands with Feynman propagators

[Henn, W.J.T. (2021)]

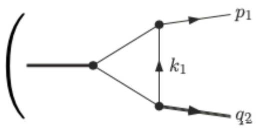
single terms only!

- one-mass triangle

$\mathcal{I}\Big($  $s\Big) \equiv \omega^{1\mathrm{m-tri}}\left(k_1; p_1, p_2\right)$

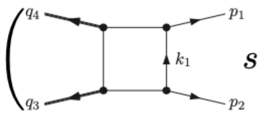
$$= d\log\left(k_1 - p_1\right)^2 \; d\log k_1^2 \, d\log\left(k_1 + p_2\right)^2 \; d\log\left(2\, k_1 \cdot e_3\right)$$

- two-mass triangle

$\omega^{2\mathrm{m-tri}}\left(k_1; p_1, q_2\right) \equiv \mathcal{I}\Big($  $\left(2\, p_1 \cdot q_2\right)\Big)$

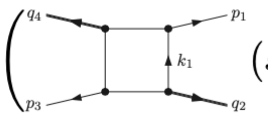
$$= d\log\left(k_1 - p_1\right)^2 \; d\log k_1^2 \, d\log\left(k_1 + q_2\right)^2 \; d\log\left(2 k_1 \cdot e_3\right)$$

- two-mass-hard box

$\omega^{2\mathrm{mh-box}}\left(k_1; p_1, p_2, q_3\right) \equiv \mathcal{I}\Big($  $s\, t\Big)$

$$= \mp d\log \frac{\left(k_1 - p_1\right)^2}{\left(k_1 - k_1^{\pm}\right)^2} \, d\log \frac{k_1^2}{\left(k_1 - k_1^{\pm}\right)^2} \, d\log \frac{\left(k_1 + p_2\right)^2}{\left(k_1 - k_1^{\pm}\right)^2} \, d\log \frac{\left(k_1 + p_2 + q_3\right)^2}{\left(k_1 - k_1^{\pm}\right)^2}$$

- two-mass-easy box

$\omega^{2\mathrm{me-box}}\left(k_1; p_1, q_2, p_3\right) \equiv \mathcal{I}\Big($  $\left(s\, t - q_2^2\, q_4^2\right)\Big)$

$$= \mp d\log \frac{\left(k_1 - p_1\right)^2}{\left(k_1 - k_1^{\pm}\right)^2} d\log \frac{k_1^2}{\left(k_1 - k_1^{\pm}\right)^2} d\log \frac{\left(k_1 + q_2\right)^2}{\left(k_1 - k_1^{\pm}\right)^2} d\log \frac{\left(k_1 + q_2 + p_3\right)^2}{\left(k_1 - k_1^{\pm}\right)^2}$$

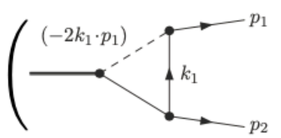
$k_1^{\pm}$ :: solutions on the maximal cut conditions

$$\left(k_1^{\pm} - p_1\right)^2 = \left(k_1^{\pm}\right)^2 = \left(k_1^{\pm} + q_2\right)^2 = \left(k_1^{\pm} + q_2 + q_3\right)^2 = 0$$

[Bern, Herrmann, Litsey, Stankowicz, Trnka (2014)]

dlog integrands as products of dlog forms

- dlog form for integrands with Feynman and eikonal propagators

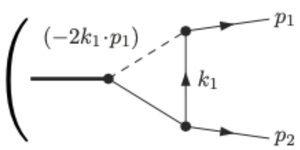


$$\mathcal{I}\left(\cdots \; s \right) = s\,\frac{d^4k_1}{(-2k_1\cdot p_1)k_1^2\,(k_1+p_2)^2}$$

[Henn, W.J.T. (2021)]

- one-mass triangle

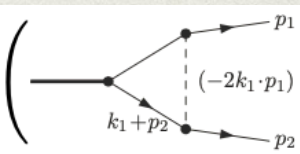
$$\omega^{\text{1m}-\text{tri}}_{\text{Eikonal }1}(k_1;p_1,p_2) \equiv \mathcal{I}\left(\cdots \; s \right)$$



$$= d\log\left(-2k_1\cdot p_1\right)\, d\log k_1^2\, d\log\left(k_1+p_2\right)^2\, d\log\left(2k_1\cdot e_3\right)$$

- two-mass triangle

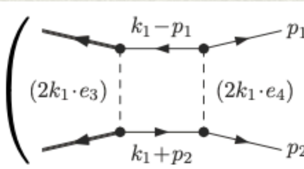
$$\omega^{\text{1m}-\text{tri}}_{\text{Eikonal }2}(k_1;p_1,p_2) \equiv \mathcal{I}\left(\cdots \; s \right) \qquad (2$$



$$= d\log\left(-2k_1\cdot p_1\right)\, d\log\left(k_1-p_1\right)^2\, d\log\left(k_1+p_2\right)^2\, d\log\left(2k_1\cdot e_3\right)$$

- two-mass-hard box

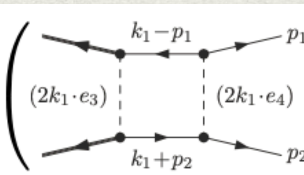
$$\omega^{\text{Box}}_{\text{Eikonal }1}(k_1;p_1,p_2) \equiv \mathcal{I}\left(\cdots \; s^2 \right)$$



$$= d\log\frac{2k_1\cdot e_3}{k_1^2}\, d\log\frac{2k_1\cdot e_4}{k_1^2}\, d\log\frac{(k_1-p_1)^2}{k_1^2}\, d\log\frac{(k_1+p_2)^2}{k_1^2}$$

- two-mass-easy box

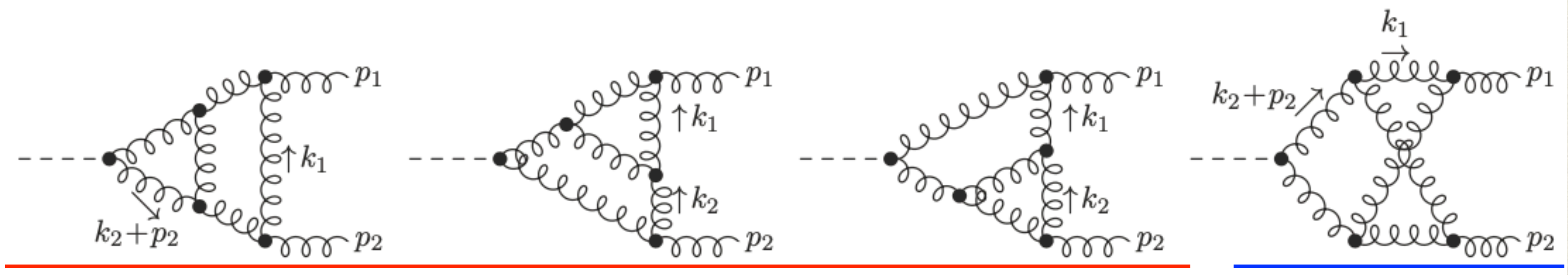
$$\omega^{\text{Box}}_{\text{Eikonal }2}(k_1;p_1,p_2) \equiv \mathcal{I}\left(\cdots \; s\,k_1^2 \right)$$



$$= d\log\frac{(k_1-p_1)^2}{2k_1\cdot e_4}\, d\log\frac{(k_1+p_2)^2}{2k_1\cdot e_3}\, d\log\frac{2k_1\cdot p_1}{2k_1\cdot e_4}\, d\log\left(2k_1\cdot e_3\right)$$

Two-loop dlog integrands

- Let's focus on a particular two-loop application, H—> gg



Planar

Non-planar

$$\mathcal{I}_{\rm p}^{(2)} = \frac{d^4k_1\, d^4k_2\, \mathcal{N}}{k_1^2 k_2^2\, (k_1-k_2)^2\, (k_1-p_1)^2\, (k_2-p_1)^2\, (k_1+p_2)^2\, (k_2+p_2)^2}$$

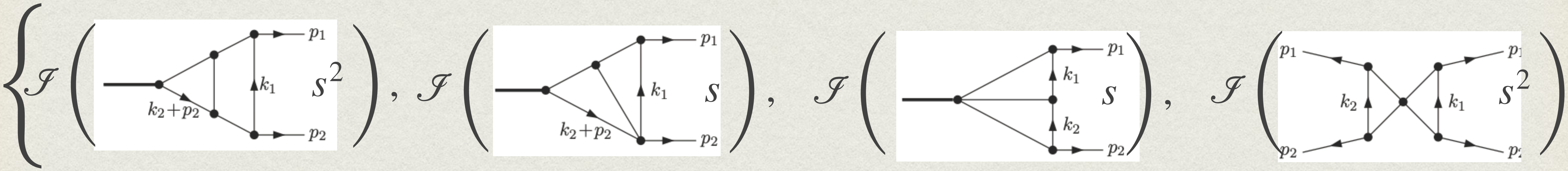
Organisation of two-loop Feynman diagrams at integrand level!

$$\mathcal{I}_{\rm np}^{(2)} = \frac{d^4k_1\, d^4k_2\, \mathcal{N}}{k_1^2\, (k_1-p_1)^2\, (k_2-p_1)^2\, (k_1-k_2)^2\, (k_1-k_2-p_2)^2\, (k_2+p_2)^2}$$

Two-loop dlog integrands

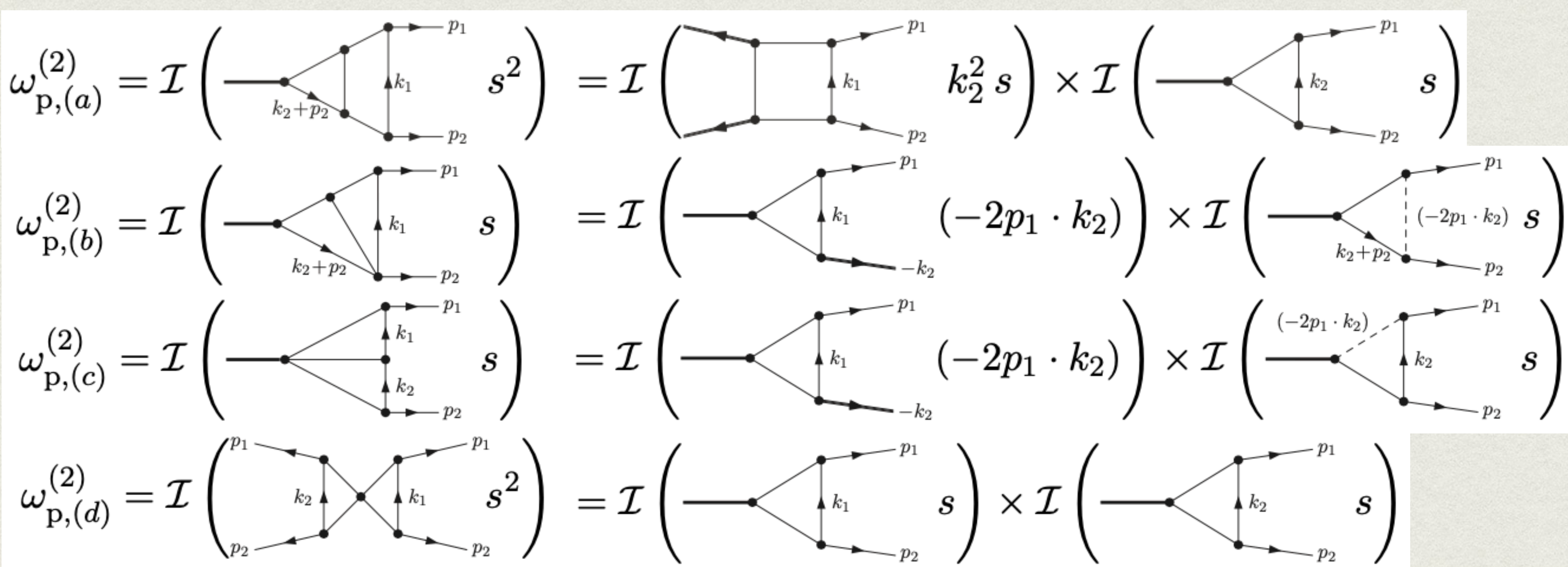
- Planar two-loop dlog integrands

from `DlogBasis` —> 9 dlog integrands



$+ \text{ flipping } k_1 \leftrightarrow k_2 \Big\}$

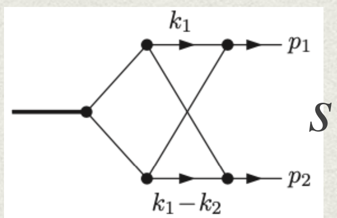
- loop-by-loop approach

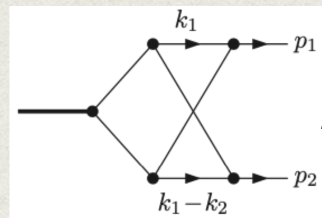


Two-loop dlog integrands

- Non-planar two-loop dlog integrands

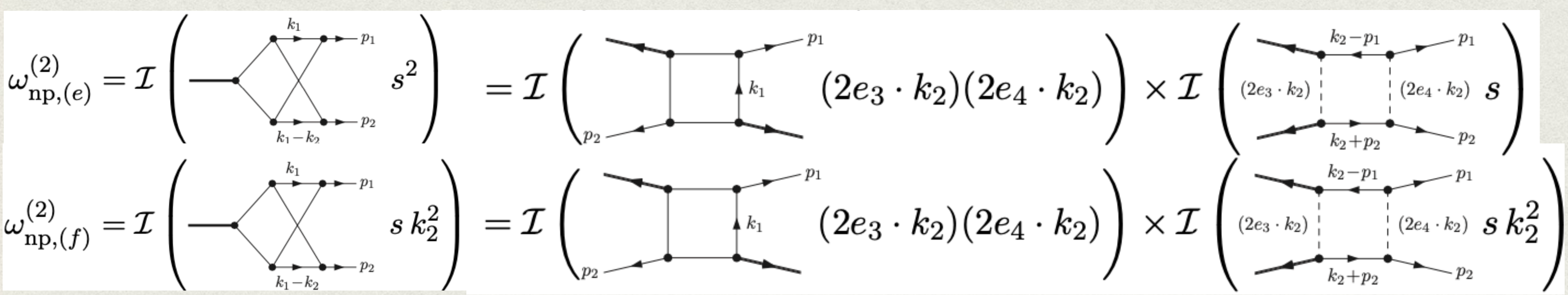
from `DlogBasis` —> 8 dlog integrands

$\Bigg\{ \mathcal{I}\Bigg($ 

$s^2 \Bigg), \quad \mathcal{I}\Bigg($ 

$k_2^2\, s \Bigg), \quad + \text{planar sub-topologies} \Bigg\}$

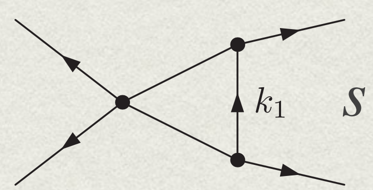
- loop-by-loop approach



Recap :: constructing Dlog integrands

- Get basis of Dlog integrands from `DlogBasis`

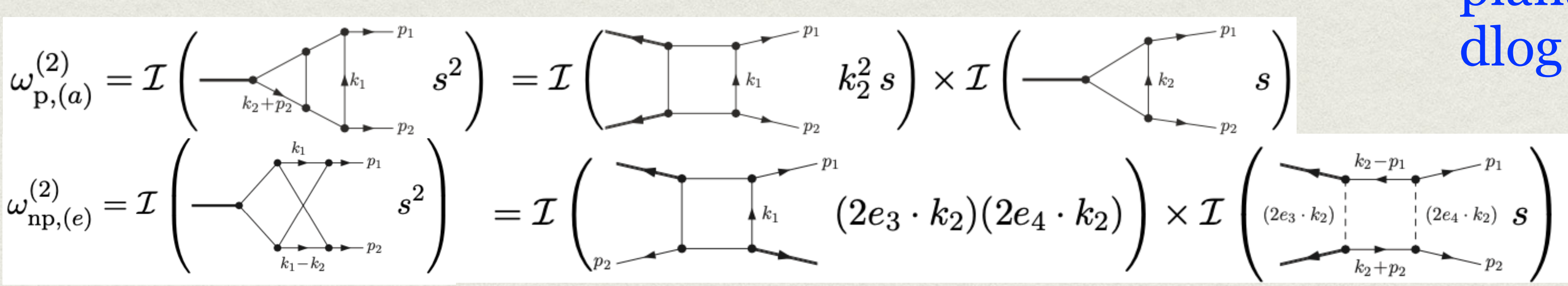
- Provide algebraic expression for one-loop Dlog integrands as product of dlog forms



$$\mathcal{I}\left(\cdots \; s \right) = d\log\left(k_1 - p_1\right)^2 \, d\log k_1^2 \, d\log\left(k_1 + p_2\right)^2 \, d\log\left(2\,k_1 \cdot e_3\right)$$

- Loop-by-loop approach for two-loop dlog integrands

planar & non-planar dlog integrands



Maximal weight contribution

Maximal weight contribution

- Get a Dlog integrand basis
  + algebraic expression as product of Dlog forms

$$\{\mathcal{J}_i^{(L)}\}\,, \quad i \in \{1,\ldots,m\}$$

$$\mathcal{J}_i^{(L)} = \sum_j b_{ij} \prod_{k=1}^{4L} d\log \alpha_{ijk}$$

for convenience ***j=1***

- Imagine an integrand decomposition of the scattering amplitude

$$\mathscr{A}^{(L)} = \int \omega^{(L)} \longrightarrow \omega^{(L)} = \sum_{i=1}^{m} c_i \mathcal{J}_i^{(L)} + \ldots$$

terms w/ (at least) a double pole

- Define a projection operator at integrand level

$$\mathscr{P}\left(\omega^{(L)}\right) = \sum_{i=1}^{n} \tilde{c}_i \prod_{j=1}^{4L} d\log\left(\tau_{i,j}\right)$$

rational functions depending on loop components α_i of loop parametrisation

- Extract coefficients $\tilde{c}_i$ :: related to generalised unitarity

$$\tilde{c}_i = \oint_{\tau_{i,1}=0} \cdots \oint_{\tau_{i,4L}=0} \omega^{(L)}\,.$$

Make use of Multi-variate partial fractioning —> Leinartas' decomposition

[Leinartas (1978)]
[Abreu et al (2019)]
[Boehm et al (2020)]
[Heller and Manteuffel (2021)]

Maximal weight contribution

- The simplest application: Beta function

$$I_{a,b} = \int_0^1 \omega_{a,b} = \frac{\Gamma(-a+\epsilon)\Gamma\left(-b+\epsilon\right)}{\Gamma\left(-a-b+2\epsilon\right)}$$

$$\omega_{a,b} = dz\, z^{-1-b+\epsilon}\,(1-z)^{-1-a+\epsilon}$$

two forms: $\left\{d\log z, -d\log(1-z)\right\}$.

$$\omega_{a,b} = c_0\,\left[d\log z\right] + c_1\left[-d\log(1-z)\right] + \ldots\,,$$

$z \leftrightarrow 1-z$ symmetry

$$I_{0,-1} = I_{-1,0} = \frac{1}{\epsilon} - \frac{\pi^2}{6}\epsilon + 2\,\zeta_3\,\epsilon^2 + \mathcal{O}\left(\epsilon^3\right)$$

uniform transcendental weight 1

$$[\epsilon] \to -1\,,\ [\zeta_n] \to n\,,\ \log^n[x] \to n\,, \ldots$$

Extract coefficients :: Cauchy residue thm

$$c_0 = \oint_{z=0} \omega_{a,b}\big|_{\epsilon=0} = \oint_{z=0} \frac{dz}{z^{a+1}\,(1-z)^{b+1}} = \binom{a+b}{a}$$

Maximal weight projection

$$I_{a,b} = \binom{a+b}{a}\left(\frac{2}{\epsilon} - \frac{\pi^2}{3}\epsilon + 4\,\zeta_3\,\epsilon^2\right) + \mathcal{O}\left(\epsilon^3\right) + \text{weight drop terms}\,.$$

Maximal weight contribution

- The simplest application: Beta function

$$I_{a,b} = \int_0^1 \omega_{a,b} = \frac{\Gamma(-a+\epsilon)\Gamma\left(-b+\epsilon\right)}{\Gamma\left(-a-b+2\epsilon\right)}$$

$$\omega_{a,b} = dz\, z^{-1-b+\epsilon}\,(1-z)^{-1-a+\epsilon}$$

two forms: $\left\{d\log z, -d\log(1-z)\right\}$.

$$\omega_{a,b} = c_0\,\left[d\log z\right] + c_1\left[-d\log(1-z)\right] + \ldots\,,$$

$z \leftrightarrow 1-z$ symmetry

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Extract coefficients :: Cauchy residue thm

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uniform transcendental weight 1

$$[\epsilon] \to -1\,,\ [\zeta_n] \to n\,,\ \log^n[x] \to n\,, \ldots$$

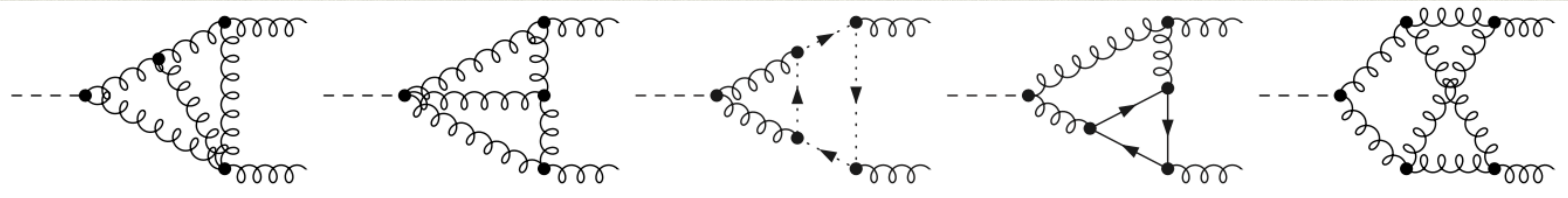
Maximal weight projection for $a = 5, b = 3$

weight drop terms

$$I_{5,3} = \frac{112}{\epsilon} - \frac{56}{3}\,\pi^2\,\epsilon + 224\,\zeta_3\,\epsilon^2 - \frac{2216}{15} - \frac{19468}{225}\epsilon + \epsilon^2\left(-\frac{234554}{3375} + \frac{1108}{45}\pi^2\right) + \mathcal{O}\left(\epsilon^3\right)\,.$$

Maximal weight contribution

- Application $H \to gg$ @2L



$$\mathscr{A}^{(2)}_{Hgg} = g_S^4\, g_{\rm EFT} \left(g^{\mu_1\mu_2} - \frac{2p_1^{\mu_2} p_2^{\mu_2}}{s_{12}} \right) \delta^{a_1 a_2}\, \varepsilon^{\mu_1}_{1,a_1}\, \varepsilon^{\mu_2}_{2,a_2}\, A_1^{(2)}$$

Maximal weight projection

Decomposition into planar and non-planar dlog integrands

$$\mathscr{P}\left(A_1^{(2)}\right) = \sum_{i=1}^{9} c_{{\rm p},i}\, \omega^{(2)}_{{\rm p},i} + \sum_{i=1}^{8} c_{{\rm np},i}\, \omega^{(2)}_{{\rm np},i}\,.$$

main task :: extract coefficients

Maximal weight contribution

Recall structure of $\omega_{\mathrm{p},(a)}$

$$\omega_{\mathrm{p},a} = \mp\, d\log\frac{(k_1-p_1)^2}{(k_1-k_\pm)^2}\; d\log\frac{k_1^2}{(k_1-k_\pm)^2}\; d\log\frac{(k_1+p_2)^2}{(k_1-k_\pm)^2}\; d\log\frac{(k_1-k_2)^2}{(k_1-k_\pm)^2}$$

$$\times\, d\log(k_2-p_1)^2\; d\log k_2^2\; d\log(k_2+p_2)^2\; d\log(2k_2\cdot e_3)$$

and loop momentum parametrisation

$$k_i^\mu = \alpha_{i,1}\, p_1^\mu + \alpha_{i,2}\, p_2^\mu + \alpha_{i,3}\, \epsilon_{12}^\mu + \alpha_{i,4}\, \epsilon_{21}^\mu$$

Maximal weight contribution

Recall structure of $\omega_{\mathrm{p},(a)}$

$$\omega_{\mathrm{p},a} = \mp\, d\log\underbrace{\frac{(k_1-p_1)^2}{(k_1-k_\pm)^2}}_{\tau_1}\, d\log\underbrace{\frac{k_1^2}{(k_1-k_\pm)^2}}_{\tau_2}\, d\log\underbrace{\frac{(k_1+p_2)^2}{(k_1-k_\pm)^2}}_{\tau_3}\, d\log\underbrace{\frac{(k_1-k_2)^2}{(k_1-k_\pm)^2}}_{\tau_4}$$

$$\times\, d\log\underbrace{(k_2-p_1)^2}_{\tau_5}\, d\log\underbrace{k_2^2}_{\tau_6}\, d\log\underbrace{(k_2+p_2)^2}_{\tau_7}\, d\log\underbrace{(2k_2\cdot e_3)}_{\tau_8}$$

and loop momentum parametrisation

$$k_i^\mu = \alpha_{i,1}\, p_1^\mu + \alpha_{i,2}\, p_2^\mu + \alpha_{i,3}\, \epsilon_{12}^\mu + \alpha_{i,4}\, \epsilon_{21}^\mu$$

change of variables

$\{\alpha_{1,1}, \ldots, \alpha_{1,4}, \alpha_{2,1}, \ldots, \alpha_{2,4},\} \to \{\tau_1, \ldots, \tau_8\}$

Maximal weight contribution

Recall structure of $\omega_{\mathrm{p},(a)}$

$$\omega_{\mathrm{p},a} = \mp\, d\log\underbrace{\frac{(k_1-p_1)^2}{(k_1-k_\pm)^2}}_{\tau_1}\, d\log\underbrace{\frac{k_1^2}{(k_1-k_\pm)^2}}_{\tau_2}\, d\log\underbrace{\frac{(k_1+p_2)^2}{(k_1-k_\pm)^2}}_{\tau_3}\, d\log\underbrace{\frac{(k_1-k_2)^2}{(k_1-k_\pm)^2}}_{\tau_4}$$

$$\times\, d\log\underbrace{(k_2-p_1)^2}_{\tau_5}\, d\log\underbrace{k_2^2}_{\tau_6}\, d\log\underbrace{(k_2+p_2)^2}_{\tau_7}\, d\log\underbrace{(2k_2\cdot e_3)}_{\tau_8}$$

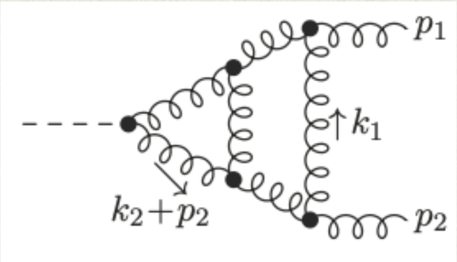
and loop momentum parametrisation

$$k_i^\mu = \alpha_{i,1}\, p_1^\mu + \alpha_{i,2}\, p_2^\mu + \alpha_{i,3}\, \epsilon_{12}^\mu + \alpha_{i,4}\, \epsilon_{21}^\mu$$

change of variables
$\{\alpha_{1,1},\ldots,\alpha_{1,4},\alpha_{2,1},\ldots,\alpha_{2,4},\} \to \{\tau_1,\ldots,\tau_8\}$

Express integrand in terms of τ_i variables, according to $\omega_{\mathrm{p},(i)}$, e.g.

Read off $\tilde{c}_i$ —> Projection operation



$$= \mathcal{I}(\tau_i) = \frac{\tilde{c}_i}{\tau_1\cdots\tau_8} + R$$

partial fractioning —> Leinartas' decomposition :: `MultivariateApart`

[Heller and Manteuffel (2021)]

Maximal weight contribution

Recall structure of $\omega_{\mathrm{p},(a)}$

$$\omega_{\mathrm{p},a} = \mp\, d\log\underbrace{\frac{(k_1-p_1)^2}{(k_1-k_\pm)^2}}_{\tau_1}\, d\log\underbrace{\frac{k_1^2}{(k_1-k_\pm)^2}}_{\tau_2}\, d\log\underbrace{\frac{(k_1+p_2)^2}{(k_1-k_\pm)^2}}_{\tau_3}\, d\log\underbrace{\frac{(k_1-k_2)^2}{(k_1-k_\pm)^2}}_{\tau_4}$$

$$\times\, d\log\underbrace{(k_2-p_1)^2}_{\tau_5}\, d\log\underbrace{k_2^2}_{\tau_6}\, d\log\underbrace{(k_2+p_2)^2}_{\tau_7}\, d\log\underbrace{(2k_2\cdot e_3)}_{\tau_8}$$

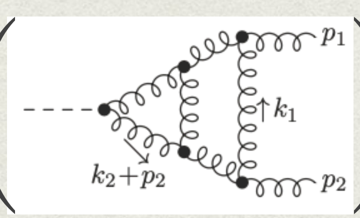
and loop momentum parametrisation

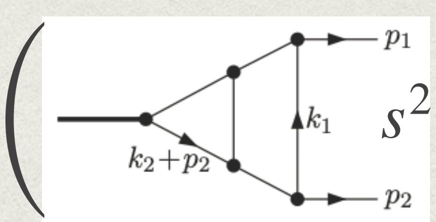
$$k_i^\mu = \alpha_{i,1}\, p_1^\mu + \alpha_{i,2}\, p_2^\mu + \alpha_{i,3}\, \epsilon_{12}^\mu + \alpha_{i,4}\, \epsilon_{21}^\mu$$

change of variables

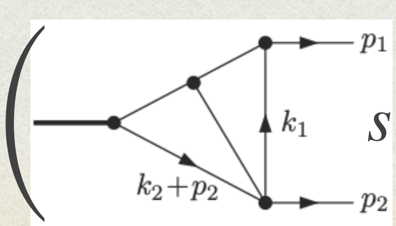
$\{\alpha_{1,1},\ldots,\alpha_{1,4},\alpha_{2,1},\ldots,\alpha_{2,4},\} \to \{\tau_1,\ldots,\tau_8\}$

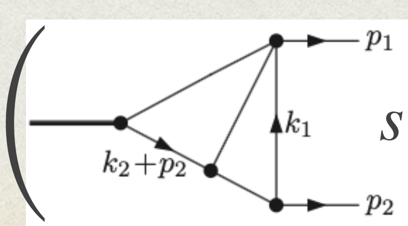
Perform projection operation

$\mathscr{P}\Big($ 

$\Big) = \Big\{ -2\,\mathscr{I}\Big($ 

$s^2\Big)$

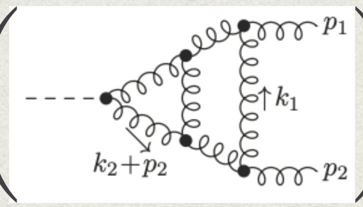
$+\dfrac{3}{2}\Big[\mathscr{I}\Big($ 

$\Big) + \mathscr{I}\Big($ 

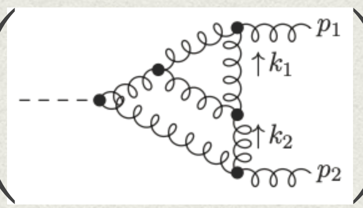
$\Big)\Big]\Big\}\; s$

Maximal weight contribution

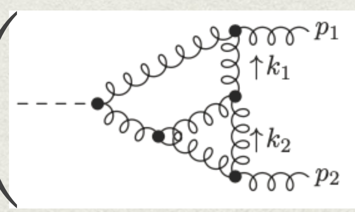
- Application $H \to gg$ @2L :: planar contribution

$\mathscr{P}\Big($ 

$\Big) = \left[-2\omega_{\mathrm{p},(a)} + \frac{3}{2}\left(\omega_{\mathrm{p},(b)} + \omega'_{\mathrm{p},(b)}\right)\right] s$

$\mathscr{P}\Big($ 

$\Big) = \left[-\frac{3}{4}\omega_{\mathrm{p},(b)} - \frac{1}{2}\omega_{\mathrm{p},(c)}\right] s\,,$

$\mathscr{P}\Big($ 

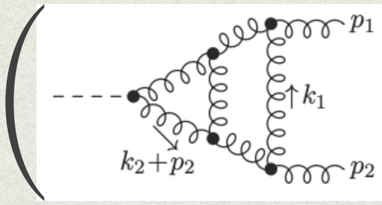
$\Big) = \left[-\frac{3}{4}\omega'_{\mathrm{p},(b)} - \frac{1}{2}\omega_{\mathrm{p},(c)}\right] s\,,$

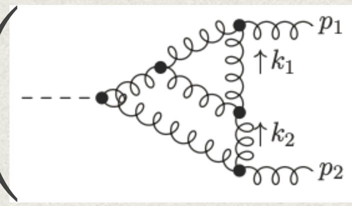
add all contributions up

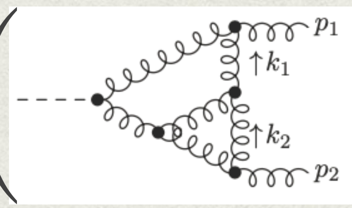
$$\mathscr{P}\left(A_1^{(2)}\right)\Big|_{\mathrm{planar}} = \left[-2\,\omega_{\mathrm{p},(a)} + \frac{3}{4}\left(\omega_{\mathrm{p},(b)} + \omega'_{\mathrm{p},(b)}\right) - \omega_{\mathrm{p},(c)}\right] s\,.$$

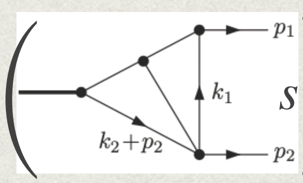
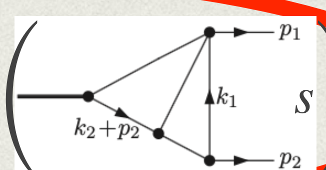
Maximal weight contribution

- Application $H \to gg$ @2L :: planar contribution

$\mathcal{P}\Big($  $\Big) = \left[-2\omega_{\mathrm{p},(a)} + \frac{3}{2}\left(\omega_{\mathrm{p},(b)} + \omega'_{\mathrm{p},(b)}\right)\right] s$

$\mathcal{P}\Big($  $\Big) = \left[-\frac{3}{4}\omega_{\mathrm{p},(b)} - \frac{1}{2}\omega_{\mathrm{p},(c)}\right] s\,,$

$\mathcal{P}\Big($  $\Big) = \left[-\frac{3}{4}\omega'_{\mathrm{p},(b)} - \frac{1}{2}\omega_{\mathrm{p},(c)}\right] s\,,$

$\int \mathcal{I}\Big($  $\Big) = \int \mathcal{I}\Big($  $\Big)$

@ integral level

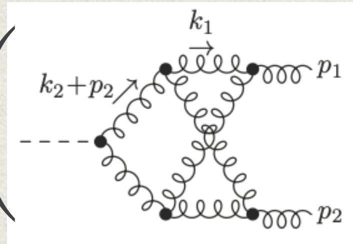
add all contributions up

$$\mathcal{P}\left(A_1^{(2)}\right)\Big|_{\mathrm{planar}} = \left[-2\,\omega_{\mathrm{p},(a)} + \frac{3}{4}\left(\omega_{\mathrm{p},(b)} + \omega'_{\mathrm{p},(b)}\right) - \omega_{\mathrm{p},(c)}\right] s\,.$$

Maximal weight contribution

- Application $H \to gg$ @2L :: non-planar contribution

$$\mathscr{P}\left(\vphantom{\int} \right.$$



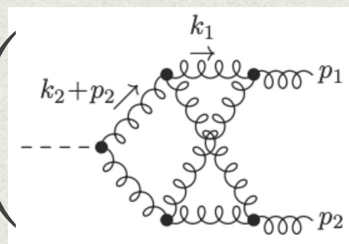
$$\left. \vphantom{\int} \right) = \left[-\frac{1}{2}\omega_{\mathrm{np},(e)} - \frac{3}{8}\left(\omega_{\mathrm{np},(b);1} + \omega_{\mathrm{np},(b);2} + \omega_{\mathrm{np},(b);4} + \omega_{\mathrm{np},(b);5}\right) + \frac{1}{2}\left(\omega_{\mathrm{np},(c);3} + \omega_{\mathrm{np},(c);6}\right)\right] s\,.$$

add all contributions up

$$\mathscr{P}\left(A_1^{(2)}\right) = \left[-2\omega_{\mathrm{p},(a)} - \frac{1}{2}\omega_{\mathrm{np},(e)} - \frac{3}{8}\left(\omega_{\mathrm{np},(b);1} + \omega_{\mathrm{np},(b);2} + \omega_{\mathrm{np},(b);4} + \omega_{\mathrm{np},(b);5} - 2\omega_{\mathrm{p},(b)} - 2\omega'_{\mathrm{p},(b)}\right) + \frac{1}{2}\left(\omega_{\mathrm{np},(c);3} + \omega_{\mathrm{np},(c);6} - 2\omega_{\mathrm{p},(c)}\right)\right] s$$

Maximal weight contribution

- Application $H \to gg$ @2L :: non-planar contribution



$$\mathscr{P}\left(\cdots \right) = \Big[-\frac{1}{2}\omega_{\mathrm{np},(e)} - \frac{3}{8}\left(\omega_{\mathrm{np},(b);1} + \omega_{\mathrm{np},(b);2} + \omega_{\mathrm{np},(b);4} + \omega_{\mathrm{np},(b);5}\right) + \frac{1}{2}\left(\omega_{\mathrm{np},(c);3} + \omega_{\mathrm{np},(c);6}\right)\Big] s \, .$$

add all contributions up

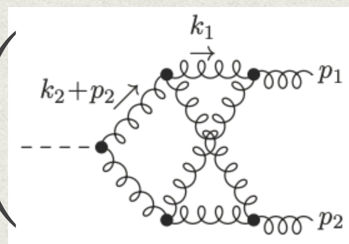
@ integral level

$$\mathscr{P}\left(A_1^{(2)}\right) = \Big[-2\omega_{\mathrm{p},(a)} - \frac{1}{2}\omega_{\mathrm{np},(e)} - \frac{3}{8}\underbrace{\left(\omega_{\mathrm{np},(b);1} + \omega_{\mathrm{np},(b);2} + \omega_{\mathrm{np},(b);4} + \omega_{\mathrm{np},(b);5} - 2\omega_{\mathrm{p},(b)} - 2\omega'_{\mathrm{p},(b)}\right)}_{=0} + \frac{1}{2}\underbrace{\left(\omega_{\mathrm{np},(c);3} + \omega_{\mathrm{np},(c);6} - 2\omega_{\mathrm{p},(c)}\right)}_{=0}\Big] s$$

Maximal weight contribution

- Application $H \to gg$ @2L :: non-planar contribution

$$\mathcal{P}\Bigg($$



$$\Bigg) = \Bigg[-\frac{1}{2}\omega_{\mathrm{np},(e)} - \frac{3}{8}\left(\omega_{\mathrm{np},(b);1} + \omega_{\mathrm{np},(b);2} + \omega_{\mathrm{np},(b);4} + \omega_{\mathrm{np},(b);5}\right) + \frac{1}{2}\left(\omega_{\mathrm{np},(c);3} + \omega_{\mathrm{np},(c);6}\right)\Bigg] s\,.$$

add all contributions up

$$\int_{k_1,k_2} \mathcal{P}\left(A_1^{(2)}\right) = \int_{k_1,k_2} \left[-2\omega_{\mathrm{p},(a)} - \frac{1}{2}\omega_{\mathrm{np},(e)}\right] s$$

$$= \left(-\frac{1}{\epsilon^4} + \frac{\pi^2}{12\epsilon^2} + \frac{25\zeta_3}{6\epsilon} + \frac{7\pi^4}{120}\right) s + \mathcal{O}(\epsilon)$$

Plug expressions for MIs —> in agreement w/ [Harlander (2000)]

Recap :: maximal weight contribution

- Strategy to extract the maximal weight contribution from the knowledge of Dlog basis

- Multivariate residue ::
  Use of partial fractioning —> Leinartas' decomposition

- Proof-of-concept application :: H—> gg @ 2L

Analysis of four-dimensional vanishing numerators

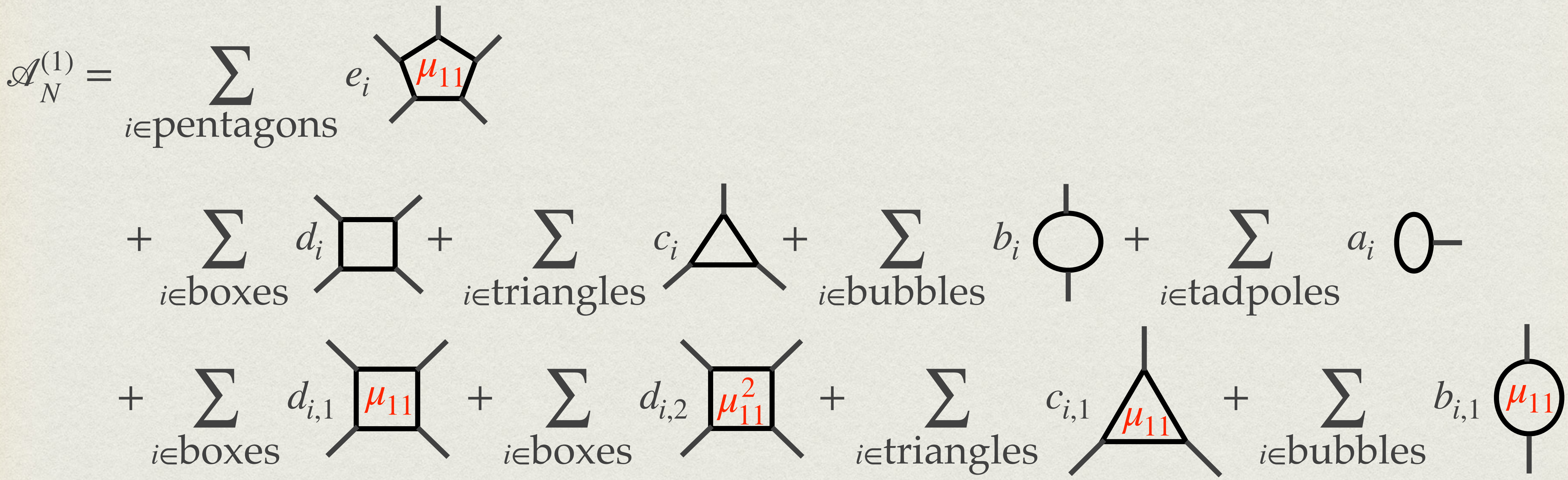
One-loop evanescent numerators

Extend loop parametrisation to $D = 4 - 2\epsilon$ dimensions

$$k^{\alpha}_{i,[D]} = k^{\alpha}_{i} + k^{\alpha}_{i,[D-4]} \qquad \begin{aligned} k_i \cdot k_{j,[D-4]} &= 0 \\ k_{i,[D-4]} \cdot k_{j,[D-4]} &= -\mu_{ij} \end{aligned}$$

Evanescent terms
(similar analysis for Gram determinants)

Any N-point one-loop amplitude in (renormalisable theories)



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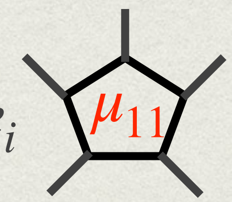
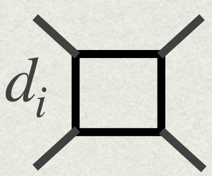
Evanescent terms
(similar analysis for
Gram determinants)

Any N-point one-loop amplitude in (renormalisable theories)

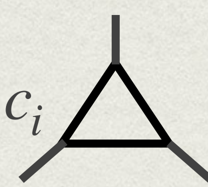
$$\mathscr{A}_N^{(1)} = \sum_{i \in \text{pentagons}} e_i$$



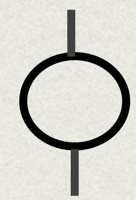
$$+ \sum_{i \in \text{boxes}} d_i$$



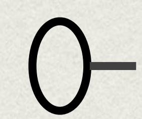
$$+ \sum_{i \in \text{triangles}} c_i$$



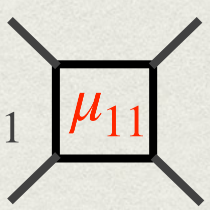
$$+ \sum_{i \in \text{bubbles}} b_i$$



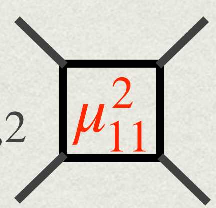
$$+ \sum_{i \in \text{tadpoles}} a_i$$



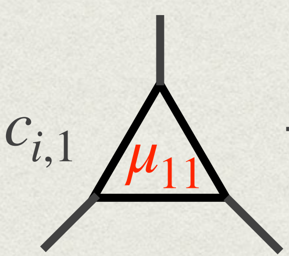
$$+ \sum_{i \in \text{boxes}} d_{i,1}$$



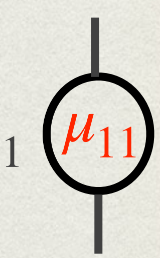
$$+ \sum_{i \in \text{boxes}} d_{i,2}$$



$$+ \sum_{i \in \text{triangles}} c_{i,1}$$



$$+ \sum_{i \in \text{bubbles}} b_{i,1}$$



Evanescent terms —> dimension shift **[Bern & Morgan (1995)]**

$$I_{n,i}^{(1),4-2\epsilon}\left[\mu_{11}\right] = -\,\epsilon\, I_{n,i}^{(1),6-2\epsilon}\left[1\right]$$

- $n > 4$ —> Transcendental weight of $I_n^{(1),D=6-2\epsilon}$ = three
- $n \leq 4$ —> Transcendental weight of $I_n^{(1),D=6-2\epsilon}$ = two

One-loop evanescent numerators

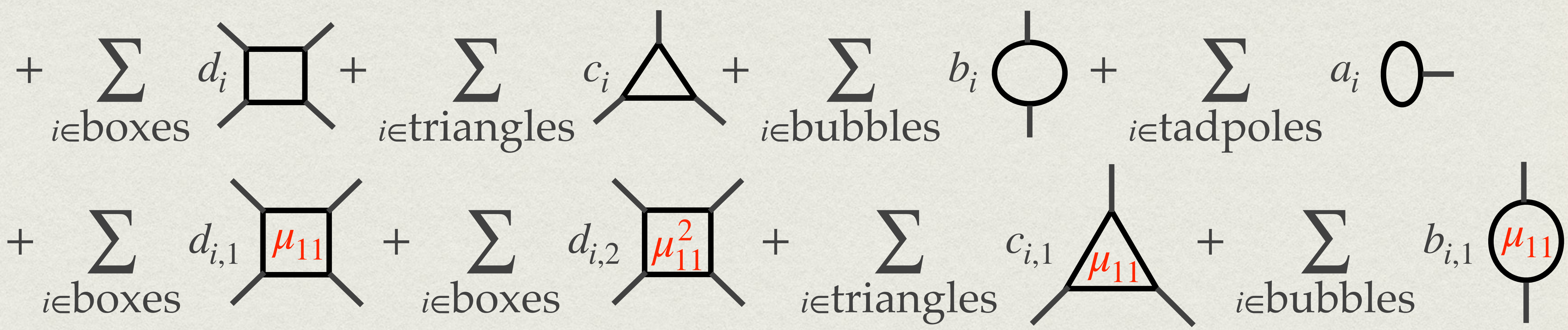
Extend loop parametrisation to $D = 4 - 2\epsilon$ dimensions

$$k^{\alpha}_{i,[D]} = k^{\alpha}_{i} + k^{\alpha}_{i,[D-4]} \qquad \begin{aligned} k_i \cdot k_{j,[D-4]} &= 0 \\ k_{i,[D-4]} \cdot k_{j,[D-4]} &= -\mu_{ij} \end{aligned}$$

Evanescent terms
(similar analysis for
Gram determinants)

Any N-point one-loop amplitude in (renormalisable theories)

Evanescent terms lead to maximal weight terms starting at $\mathcal{O}(\epsilon)$ for $n > 4$



Evanescent terms —> dimension shift **[Bern & Morgan (1995)]**

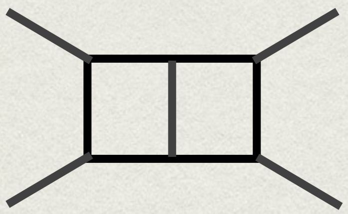
$$I^{(1),4-2\epsilon}_{n,i}\left[\mu_{11}\right] = -\,\epsilon\, I^{(1),6-2\epsilon}_{n,i}\left[1\right]$$

weight drop

- $n > 4$ —> Transcendental weight of $I^{(1),D=6-2\epsilon}_{n}$ = three
- $n \leq 4$ —> Transcendental weight of $I^{(1),D=6-2\epsilon}_{n}$ = two

Two-loop evanescent numerators

- Evanescent two-loop four-point terms have non-maximal weight



Construct most general integrand that could appear in a massless YM theory

↓

Extract the most generic vanishing four-dimensional integrand →

$$G(k_1, k_2, p_1, p_2, p_4)$$

$$G(k_1, p_1, p_2, p_4)G(k_2, p_1, p_2, p_4) - G^2\begin{pmatrix} k_1, p_1, p_2, p_4 \\ k_2, p_1, p_2, p_4 \end{pmatrix}$$

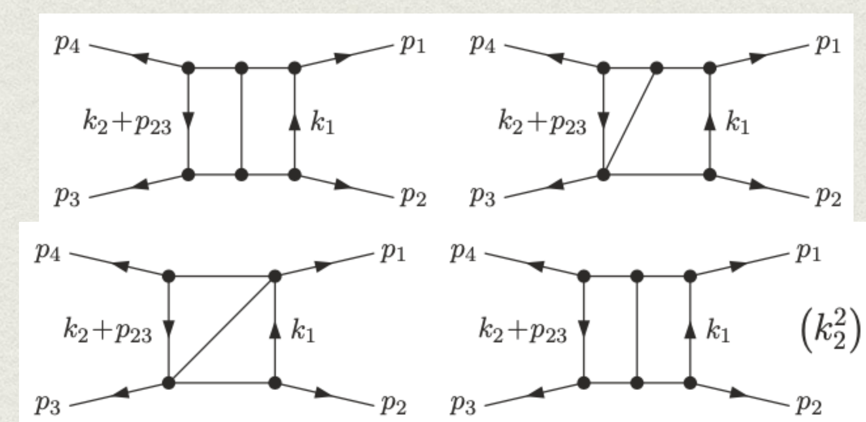
Particular cases **[Gluza, Kajda, Kosower (2010)]**

↓

Make use of integration-by-part id's

↓

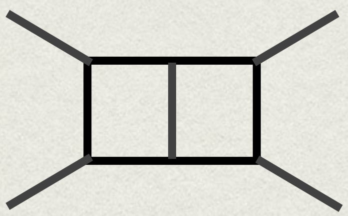
MIs —> 3-pt and 4-pt Dlog integrals



[Henn (2013)]

Two-loop evanescent numerators

- Evanescent two-loop four-point terms have non-maximal weight



Construct most general integrand that could appear in a massless YM theory

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$$G(k_1,p_1,p_2,p_4)G(k_2,p_1,p_2,p_4) - G^2\begin{pmatrix} k_1,p_1,p_2,p_4 \\ k_2,p_1,p_2,p_4 \end{pmatrix}$$

Particular cases **[Gluza, Kajda, Kosower (2010)]**

↓

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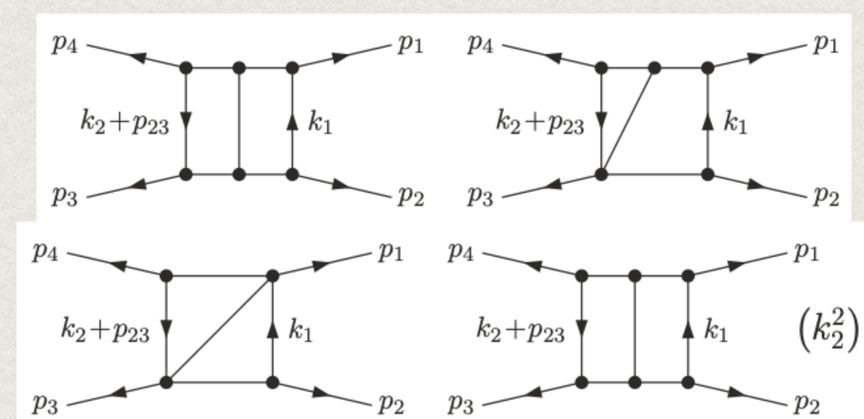
MIs —> 3-pt and 4-pt Dlog integrals



[Henn (2013)]

$$\int_{k_1,k_2} \frac{N_4}{\text{double box propagators}} = \epsilon \sum_{i=1}^{8} b_i(\epsilon)\, \omega_i^{(2)}$$

weight drop

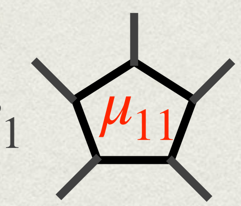
Two-loop evanescent numerators

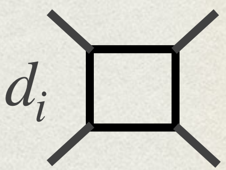
- Two-loop five-pt amplitude in N=4sYM

Evidence of no ambiguity in our method

express the amplitude as

$$A_5 = A_5^{\text{tree}} M_5 \longrightarrow M_5 = 1 + g^2 M_5^{(1)} + g^4 M_5^{(2)} + \ldots$$

$$M_5^{(1)} = e_1$$ 

$$+ \sum_{i \in \text{boxes}} d_i$$ 

parity odd

parity even

duality between Amplitudes & Wilson loops

[Alday, Maldacena (2007)]
[Brandhuber, Heslop, Travaglini (2008)]
[Drummond, Henn, Korchemsky, Sokatchev (2010)]

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Two-loop evanescent numerators

- Two-loop five-pt amplitude in N=4sYM

Evidence of no ambiguity in our method

express the amplitude as

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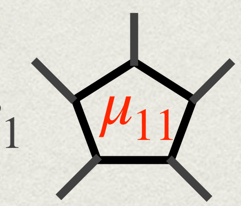
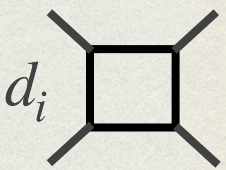
$$M_5^{(1)} = e_1$$ 

$$+ \sum_{i \in \text{boxes}} d_i$$ 

parity odd | parity even

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parity even

At order g^4

$$M_5^{(2)} - \frac{1}{2}\left(M_5^{(1)}\right)^2 + \mathcal{O}(\epsilon)$$

Recap :: four-dimensional vanishing numerators

@1L

- 4-pt —> projection method valid at all orders in ϵ
- 5-pt (and beyond) —> projection method valid up-to $\mathcal{O}(1)$
  maximal weight terms start at $\mathcal{O}(\epsilon)$

@2L

- 4-pt —> evanescent terms lead to weight drop
- 5-pt —> Evidence that N=4sYM amplitudes are free of ambiguities

Conclusions & Outlook

We have reached:

- ☑ Integrand treatment of multi-loop scattering amplitudes.
- ☑ Dlog integrands in terms of products of Dlog forms.
- ☑ Decomposition of Scattering Amplitudes in terms of Dlog forms.
- ☑ systematic procedure to obtain a maximal weight —> check on ttbar.
- ☑ proof-of-concept application :: H -> gg @2L.

[Mandal, Mastrolia, Ronca, W.J.T. (2022)]

We are working on:

- ☐ Extend procedure to extract next-to Maximal weight terms.
- ☐ Extend from Dlog to elliptic integrands.
- ☐ Elaborate on other representations, e.g. Baikov, Causal. [W.J.T. (2021)]
- ☐ Provide a geometrical interpretation, e.g. negative geometries.

[Arkani-Hamed, Henn, Trnka (2021)]

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Thanks