Feynman Integrals and GKZ Hypergeometric Systems

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Loops and Legs in Quantum Field theory

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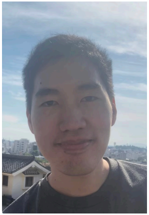

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Outline

Feynman integrals as A-hypergeometric functions

Pfaffian systems

Macaulay matrix

Simple example

Linear relations

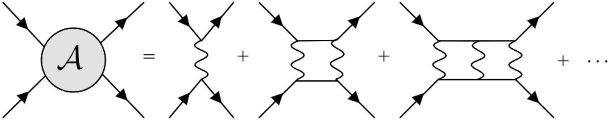
Conclusions

Feynman integrals as
A-hypergeometric functions

Motivation for studying Feynman integrals

- Precision calculations in QFT (and beyond) require evaluation of Feynman diagrams at **loop level**

- For instance : scattering amplitudes



- **IBP** reduction and evaluation of Feynman integrals by **DEQs** can be very demanding

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- ↳ Seek novel **mathematical frameworks** for understanding and manipulating Feynman integrals

- ↳ Mapping out the **space of functions** to which Feynman integrals belong

Feynman integrals and special functions

Space of functions for Feynman integrals is extremely rich:

- GPLs
- Elliptics
- Modular forms
- Integrals over Calabi-Yau varieties

Is there an upper bound on the ”complexity” of a Feynman integral?

Yes. Any Feynman integral is a special case of an *A*-**hypergeometric function**.

[**Gel’fand, Kapranov, Zelevinsky ’89**] [Nasrollahpoursamami ’16] [de la Cruz ’19] [Vanhove ’19] [Klausen ’20 & ’22] [Feng, Chang, Chen, Zhang ’20] [Tellander, Helmer ’21]

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with systems of the form (0.2) there are the integrals $\int \Pi P_i (t_1, \ldots, t_n)^{\alpha_i} t_1^{\beta_1} \ldots t_n^{\beta_n} dt_1 \ldots dt_n$, where P_i are polynomials, i.e., practically all integrals which arise in quantum field theory.

[GKZ, *Hypergometric functions and toral manifolds*, ’89]

A-hypergeometric functions

GKZ data:

Parameters	$\beta := (\beta_0, \dots, \beta_n) \in \mathbb{C}^{n+1}$
Variables	$z := (z_1, \dots, z_N) \in \mathbb{C}^N$
Vectors	$a_1, \dots, a_N \in \mathbb{Z}^{n+1}$

↳ Construct $(n+1) \times N$ matrix of integers $A = (a_1 \ \dots \ a_N)$

$F_\beta(z)$ is an A-**hypergeometric function** if it satisfies two sets of PDEs :

- $(n+1)$ equations

$$E_j \bullet F_\beta(z) := \left[\sum_{i=1}^{N} (a_j)_i \, z_i \frac{\partial}{\partial z_i} - \beta_j \right] \bullet F_\beta(z) = 0$$

- For all integer vectors u satisfying $A \cdot u = 0$

$$\Box_u \bullet F_\beta(z) := \left[\prod_{u_i > 0}^{N} \left(\frac{\partial}{\partial z_i} \right)^{u_i} - \prod_{u_i < 0}^{N} \left(\frac{\partial}{\partial z_i} \right)^{-u_i} \right] \bullet F_\beta(z) = 0$$

A-hypergeometric functions as integrals

A-hypergeometric functions enjoy an integral representation [GKZ '90]

$$F_\beta(z) = \int_{\mathcal{C}} g(z,x)^{\beta_0} x_1^{-\beta_1} \cdots x_n^{-\beta_n} \frac{\mathrm{d}x}{x} \quad , \quad \frac{\mathrm{d}x}{x} := \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n}$$

- $g(z,x)$: Laurent polynomial in x with *independent, indeterminate* coefficients z

$$g(z,x) = \sum_{i=1}^{N} z_i\, x^{\alpha_i} \quad , \quad x^{\alpha_i} := x_1{}^{\alpha_{i,1}} \cdots x_n{}^{\alpha_{i,n}} \quad , \quad \alpha_i \in \mathbb{Z}^n$$

- Encode exponents of x in $a_i := \left(1 \,, \alpha_i \right)^T \in \mathbb{Z}^{n+1} \;\rightarrow\; A$-matrix

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Example. $n = 2$ integration variables, $N = 3$ monomials :

$$g(z,x) = z_1 x_1 + z_2 x_2 + z_3 x_1 x_2 = z_1\, x_1^1\, x_2^0 + z_2\, x_1^0\, x_2^1 + z_3\, x_1^1\, x_2^1$$

$A = (a_1 \;\, a_2 \;\, a_3) =$

1	1	1	
1	0	1	x_1
0	1	1	x_2
z_1	z_2	z_3	

Feynman integrals are A-hypergeometric

A-hypergeometric function

$$F_\beta(z) = \int_{\mathcal{C}} g(z,x)^{\beta_0} x_1^{-\beta_1} \cdots x_n^{-\beta_n} \frac{\mathrm{d}x}{x}$$

Generalized Feynman integral (GFI):

$$\mathcal{I}^{(d_0)}(\nu) := c^{(d_0)}(\nu) F_\beta(z) \quad , \quad \beta = (\epsilon, -\epsilon\delta, \ldots, -\epsilon\delta) - (d_0/2, \nu_1, \ldots, \nu_n)$$

$$c^{(d_0)}(\nu) := \frac{\Gamma(d_0/2-\epsilon)}{\Gamma\left((L+1)(d_0/2-\epsilon) - \sum_{i=1}^n \nu_i - n\epsilon\delta\right) \prod_{i=1}^n \Gamma(\nu_i + \epsilon\delta)}$$

- Identify $\mathcal{C} \to (0,\infty)^n \quad , \quad \delta \to 0 \quad , \quad z \to \mathbb{N}$ or $(m^2, p^2, p_i \cdot p_j)$
- ↳ Lee-Pomeransky representation of L-loop Feynman integral in $d = d_0 - 2\epsilon$ dimensions with propagator powers ν_i [Lee,Pomeransky '13]
- $g \to \mathcal{G} = \mathcal{U} + \mathcal{F}$ built from Symanzik polynomials $\mathcal{U}, \mathcal{F}$

Example. Massless bubble: $\mathcal{G}(z,x) = z_1 x_1 + z_2 x_2 + z_3 x_1 x_2 \;,\; z = (1, 1, -p^2)$

Motivation for GKZ

Crucial difference:

- **Generalized** Feynman integral (GFI): z_i are *independent*
- Standard parametric integral: z_i are *dependent* (say $z_1 = z_2 = m^2$)

Motivation for this generalization:

- Wealth of mathematical structure and symmetry
- Combinatorics, algebraic geometry, $\mathcal{D}$-modules, intersection theory . . .
- Algorithms for Landau singularities, series expansions, *d*-shift . . .
- Strong CASs: **`asir`**, `polymake`, `IntegrableConnections`, . . .
- Study IBPs/DEQs derived from *external* variables (connection to Lorentz-invariance relations?)

Punchlines of this talk

1. GFIs can be represented by **differential operators**
   - The operators behave like elements of a **Weyl algebra**, $[\partial, z] = 1$

2. Given an operator basis *e* (master integrals), can derive **Pfaffian system**

$$\boxed{\frac{\partial}{\partial z_i} e = P_i \cdot e} \quad , \quad P_i = \text{ Pfaffian matrix}$$

   - System of DEQs obeyed by MIs derived without IBPs
   - Derived from novel algorithm based on the **Macaulay matrix**

3. Pfaffian systems lead to **recurrence relations**
   - **IBP**-like relations for GFIs. DEQs $\rightarrow$ IBPs.

Pfaffian systems

Pfaffian systems

- Function $f = f(z)$, operators $D_i = \sum_{k\in K} q_k(z)\,\partial^k$, $K \subset \mathbb{N}_0^N$, $\partial^k := \partial_1^{k_1}\cdots\partial_N^{k_N}$

  $\hookrightarrow\ e := (D_1 \bullet f, \dots, D_r \bullet f)$

- **Pfaffian system** : Rational matrices $P_i \in \mathbb{Q}^{r\times r}(z)$, $i = 1,\dots,N$

$$\boxed{\partial_i\, e = P_i \cdot e} \quad , \quad \text{Integrability}:\ \partial_i P_j + P_j\cdot P_i = \partial_j P_i + P_i \cdot P_j$$

How to write a GFI in the form $e := (D_1 \bullet f, \dots, D_r \bullet f)$?

- *Claim*: $\exists$ operator D such that $D \bullet \mathcal{I}^{(d_0=0)}(0,\dots,0) = \mathcal{I}^{(d_0)}(\nu_1,\dots,\nu_n)$

- **Example.** (Generalized) massless bubble :

$$\partial_1\partial_3 \bullet \int (z_1x_1 + z_2x_2 + z_3x_1x_2)^{\epsilon}(x_1x_2)^{\epsilon\delta}\,\frac{\mathrm{d}x}{x} =$$

$$(\epsilon-1)\epsilon \int (z_1x_1 + z_2x_2 + z_3x_1x_2)^{\epsilon-2}\,(x_1x_2)^{\epsilon\delta}\, x_1\, x_1x_2\,\frac{\mathrm{d}x}{x}$$

$$\iff \frac{\partial_1\partial_3}{(\epsilon-1)\epsilon} \bullet \mathcal{I}^{(0)}(0,0) = \mathcal{I}^{(4)}(2,1)$$

Twisted cohomology

- Is there a general formula for D in $D \bullet \mathcal{I}^{(0)}(0) = \mathcal{I}^{(d_0)}(\nu)$?
- Yes, due to isomorphism: **GKZ** $\cong$ **twisted cohomology** [See talk by Seva Chestnov] [Cho, Matsumoto '95]
- $r = \#\{\text{solutions to GKZ system}\} = \dim(\text{cohomology}) = \#\{\text{master integrals}\}$

- $\mathcal{I} =$ pairing between *n*-form and contour $\mathcal{C}$ [Mastrolia, Mizera '18] [Frellesvig et al. '19 & '20]

$$\mathcal{I}^{(d_0)}(\nu) = \left\langle \varphi^{(d_0)}(\nu) \,\middle|\, \mathcal{C} \right]$$

- For this talk: $\mathcal{I}^{(d_0)}(\nu)$ represented as

$$\mathcal{I}^{(d_0)}(\nu) \longleftrightarrow \varphi^{(d_0)}(\nu) := \mathcal{G}(z,x)^{-d_0/2} x^{\nu} \frac{\mathrm{d}x}{x}$$

- In particular,

$$\mathcal{I}^{(0)}(0) \longleftrightarrow \varphi^{(0)}(0) = \frac{\mathrm{d}x}{x}$$

GFIs as operators

- D constructed such that [Matsubara-Heo, Takayama '20]

$$D \bullet \frac{\mathrm{d}x}{x} = \varphi^{(d_0)}(\nu)$$

Given (d_0, ν), find $R = (r_1, \ldots, r_N)^T \in \mathbb{Z}^N$ s.t $A \cdot R = (d_0/2, \nu)^T$

$$\boxed{D^{(d_0)}(\nu) = \prod_{r_i<0} U_i^{-r_i} \prod_{r_i>0} \# \partial_i^{r_i}} \quad , \quad U_i \, , \, \partial_i \sim \text{ shift } \beta \text{ by } \pm a_i$$

Change of perspective: Let $D^{(d_0)}(\nu)$ *represent* $\mathcal{I}^{(d_0)}(\nu)$

- Master integrals: $e = \left(D_1 \bullet \mathcal{I}^{(0)}(0), \ldots, D_r \bullet \mathcal{I}^{(0)}(0)\right) \quad \rightarrow \quad e = (D_1, \ldots, D_r)$
- The Pfaffian system $\partial_i e = P_i \cdot e$ now lives in a

$$\textbf{Weyl algebra}: \quad [\partial_i, z_j] = \delta_{ij}$$

Building Pfaffian systems

- Systems of DEQs for conventional Feynman integrals : Built via **IBPs**

  [Barucchi, Ponzano '72] [Chetyrkin, Tkachov '81] [Kotikov '90] [Remiddi '97] [Laporta '00] [Gehrmann, Remiddi '00] [Henn '13] [Papadopoulos '14]

- Pfaffian systems in algebraic geometry : Built via **Gröbner bases**

  [Saito, Sturmfels, Takayama '00] [Takayama '13, ch. 6]

- Pfaffians in this work : Built via **Macaulay matrices:**

  - Linear system whose solution expresses *higher*-order derivatives in terms of *lower*-order ones

- **Example**. Basis $e = (\partial_1 \, , \, \partial_2 \, , \, 1)^T$

- Pfaffian $P_1 \in \mathbb{Q}^{3\times 3}(z)$ in direction z_1

$$\partial_1 e = (\partial_1^2 \, , \, \partial_1\partial_2 \, , \, \partial_1)^T = P_1 \cdot (\partial_1 \, , \, \partial_2 \, , \, 1)^T$$

- First row : $\partial_1^2 = a\partial_1 + b\partial_2 + c1$

- Coefficients a, b, c from a linear system: The Macaulay matrix

Macaulay matrix

Building the Macaulay matrix

Recall GKZ PDEs: $E_j \bullet F_\beta(z) \;=\; \Box_u \bullet F_\beta(z) \;=\; 0$,

$$E_j = \sum_{i=1}^{N} (a_j)_i \, z_i \, \partial_i - \beta_j \quad , \quad \Box_u = \prod_{u_i>0} \partial^{u_i} - \prod_{u_i<0} \partial^{-u_i}$$

- Choose seeds $\mathrm{Der}_d = \{\partial^k \,|\, k_1 + \ldots + k_N \leq d\}$ (typically $d = 1, 2$)
- Macaulay matrix of degree d: Act with $\partial^k \in \mathrm{Der}_d$ on GKZ operators

$$\{\partial^k E_j \, , \, \partial^k \Box_u\}_{\partial^k \in \mathrm{Der}_d} = M \cdot \mathtt{Mons}_d$$

- $\mathtt{Mons}_d$: All monomials (in ∂_i) appearing on the LHS
- Used $[\partial_i, z_j] = \delta_{ij}$ to commute ∂_i's to the right. E.g.

$$\partial_1 z_1 \partial_2 = (1 + z_1 \partial_1)\partial_2 = \partial_2 + z_1 \partial_1 \partial_2$$

Pfaffian from Macaulay

- **'Standard' monomial basis**: $e = \texttt{Std} = \{\partial^m\}$. Split

$$\partial_i \, \texttt{Std} = C_{\texttt{Ext}} \cdot \texttt{Ext} + C_{\texttt{Std}} \cdot \texttt{Std}$$

- $\texttt{Ext}$ (exterior): Monomials from $\partial_i \, \texttt{Std}$ *not* belonging to $\texttt{Std}$

- Macaulay matrix blocks: $M = \left(M_{\texttt{Ext}} \,\middle|\, M_{\texttt{Std}} \right)$

- Can show that $\partial_i \, \texttt{Std} = P_i \cdot \texttt{Std}$ is equivalent to

$$C_{\texttt{Ext}} - C \cdot M_{\texttt{Ext}} = 0 \qquad (1)$$
$$C_{\texttt{Std}} - C \cdot M_{\texttt{Std}} = P_i \qquad (2)$$

for *unknown* matrix C

Pfaffian from Macaulay:

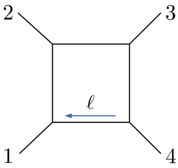
Solve (1) for $C \rightarrow$ insert C into (2) to get P_i

- Solving (1): fast with rational reconstruction over finite fields

[FiniteFlow: Peraro '19] [Firefly: Klappert, Lange '19] [Kira: Klappert,Lange,Maierhöfer,Usovitsch '20]

Simple example

Box: Setup



- **Kinematics**: $p_1^2 = \cdots = p_4^2 = 0$, $s = 2p_1 \cdot p_2$, $t = 2p_2 \cdot p_3$

- **GFI**: $n = 4$ integration variables, $N = 6$ monomials

$$\mathcal{I}^{(d_0)}(\nu) = c^{(d_0)}(\nu) \int_{\mathcal{C}} \mathcal{G}(z,x)^{\epsilon - d_0/2}\, x_1^{\nu_1 + \epsilon\delta} \cdots x_4^{\nu_4 + \epsilon\delta}\, \frac{\mathrm{d}x}{x}$$

$$\mathcal{G}(z,x) = z_1x_1 + z_2x_2 + z_3x_3 + z_4x_4 + z_5x_1x_3 + z_6x_2x_4 \;\;,\;\; A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

- **GKZ homogeneity**: Rescale $(n+1)$ variables $z_i = 1 \;\to\; N - (n+1)$ left

- Rescale $z_1 = \cdots z_5 = 1$. Interpret $z_6 = t/s =: z$. $\partial := \frac{\partial}{\partial z}$

Box: Bases

- **Integral basis** : canonical [Henn '13]

$$e_1 = (-s)^{\epsilon+1}\, z\, \mathcal{I}^{(4)}(0,1,0,2)$$

$$e_2 = (-s)^{\epsilon+1}\, \mathcal{I}^{(4)}(1,0,2,0)$$

$$e_3 = \epsilon\,(-s)^{\epsilon+2}\, z\, \mathcal{I}^{(4)}(1,1,1,1)$$

- **Operator basis** : $e_i = \Lambda_i\, D_i \bullet \mathcal{I}^{(0)}(0)$ with prefactors Λ_i and

$$D_1 = \frac{\epsilon\delta - 1}{(\epsilon-1)\epsilon}\partial - \frac{z}{(\epsilon-1)\epsilon}\partial^2$$

$$D_2 = \frac{(1-3\delta)(4\delta-1)\epsilon}{\epsilon-1}1 + \frac{z(7\epsilon\delta - 2\epsilon - 1)}{(\epsilon-1)\epsilon}\partial + \frac{z^2}{(\epsilon-1)\epsilon}\partial^2$$

$$D_3 = \frac{4\epsilon\delta - \epsilon - 1}{(\epsilon-1)\epsilon}\partial - \frac{z}{(\epsilon-1)\epsilon}\partial^2$$

- **Strategy for Pfaffian** : Use Macaulay matrix method in the basis

$$\texttt{Std} = (\partial^2 \;\; \partial \;\; 1)^T$$

↳ gauge transform to e via $e = G \cdot \texttt{Std}$, $G \in \mathbb{Q}^{3\times 3}(z,\epsilon,\delta)$

Box: Pfaffian system

- **Pfaffian in** $\mathtt{Std} = (\partial^2 \;\; \partial \;\; 1)^T$ **basis**:

$$\partial\,\mathtt{Std} = (\partial^3 \;\; \partial^2 \;\; \partial)^T = P^{(\mathtt{Std})} \cdot \mathtt{Std}$$

- **Macaulay matrix method**: Solve for $C = \begin{pmatrix} c_{11} \; c_{12} \; c_{13} \end{pmatrix}$ in

$$C_{\mathtt{Ext}} - C \cdot M_{\mathtt{Ext}} = 0 \quad , \quad M_{\mathtt{Ext}} = \left(z^2(z+1)\right)$$

- **Matrix** $C_{\mathtt{Ext}}$ **from**:

$$\partial\,\mathtt{Std} = C_{\mathtt{Ext}} \cdot \mathtt{Ext} \quad + \quad C_{\mathtt{Std}} \cdot \mathtt{Std}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \left(\partial^3\right) + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \partial^2 \\ \partial \\ 1 \end{pmatrix}$$

- **Solution**: $C = \left(\frac{1}{z^2(z+1)} \;\; 0 \;\; 0 \right)$ $\to$ plug into $C_{\mathtt{Std}} - C \cdot M_{\mathtt{Std}} = P^{(\mathtt{Std})}$

- Gauge transform $\partial\,\mathtt{Std} = P^{(\mathtt{Std})} \cdot \mathtt{Std}$ to $\partial\, e = P \cdot e$:

$$P = \epsilon \begin{pmatrix} -\frac{1}{z} & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{2}{z(z+1)} & \frac{2}{z+1} & -\frac{1}{z(z+1)} \end{pmatrix}$$

Linear relations

Pfaffian system induces linear relations

- A-hypergeometric function

$$f(\beta) = \frac{1}{\Gamma(\beta_0+1)} \int_{\mathcal{C}} g(z,x)^{\beta_0} x_1^{-\beta_1} \cdots x_n^{-\beta_n} \, \frac{\mathrm{d}x}{x}$$

- Can show $\partial_i f(\beta) = f(\beta - a_i)$

- Pick monomial basis $D_i = \partial^{k_i}$. We have $\partial_i D_j = D_j \partial_i$.

- Then $e(\beta) = \big(D_1 \bullet f(\beta), \ldots, D_r \bullet f(\beta)\big)$ satisfies

$$\partial_i \, e(\beta) = P_i(\beta) \cdot e(\beta) = e(\beta - a_i)$$

- Opposite shift by $Q_i(\beta) := P_i(\beta + a_i)^{-1}$

$$\partial_i^{-1} \, e(\beta) = Q_i(\beta) \cdot e(\beta) = e(\beta + a_i)$$

- **Recurrence relations** from *matrix multiplication* (fast with rational reconstruction)

- **For GFIs**: IBP-like relations from DEQs. Ordinarily: DEQs from IBPs.

Conclusions

Conclusions

GKZ:

- Feynman integrals are special cases of A-hypergeometric functions
- Can represent GFIs as differential operators

Macaulay matrix:

- Can derive Pfaffian systems (DEQs for MIs) without IBPs
- *External* variables
- Alternative to Gröbner bases (usually adopted by mathematicians)

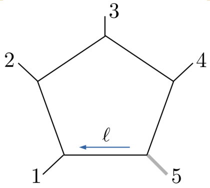
Pfaffian systems for GFIs:

- Linear relations
- Evaluation of intersection numbers [See talk by Seva Chestnov on Friday]

Thank you for listening

Extra slides

Pentagon: Setup



- **Kinematics**: One off-shell leg. $p_{1,2,3,4}^2 = 0$, $p_5^2 := p^2$, $s_{ij} := 2p_i \cdot p_j$
- **Integrand**: $n = 5$ integration variables, $N = 11$ monomials

$$\mathcal{G}(z,x) = \sum_{i=1}^{5} z_i x_i + z_6 x_1 x_3 + z_7 x_1 x_4 + z_8 x_1 x_5 + z_9 x_2 x_4 + z_{10} x_2 x_5 + z_{11} x_3 x_5$$

- **Homogeneity**: Rescale $z_1 = \ldots = z_6 = 1$. Remaining 5 GKZ variables

$$z_{7,8,9,10,11} = \sum_i \pm y_i$$

in terms of 5 kinematic variables

$$y_1 = \frac{p^2}{s_{12}}\ ,\ y_2 = \frac{s_{13}}{s_{12}}\ ,\ y_3 = \frac{s_{14}}{s_{12}}\ ,\ y_4 = \frac{s_{23}}{s_{12}}\ ,\ y_5 = \frac{s_{24}}{s_{12}}$$

Pentagon: Bases

Integral basis:

$$
\begin{aligned}
e_1 &= (-s_{12})^{\epsilon} I(2,0,0,1,0,1) & e_2 &= (-s_{12})^{\epsilon} I(2,0,1,0,0,1) \\
e_3 &= (-s_{12})^{\epsilon} I(2,0,1,0,1,0) & e_4 &= (-s_{12})^{\epsilon} I(2,1,0,0,0,1) \\
e_5 &= (-s_{12})^{\epsilon} I(2,1,0,0,1,0) & e_6 &= (-s_{12})^{\epsilon} I(2,1,0,1,0,0) \\
e_7 &= \epsilon(-s_{12})^{\epsilon} I(4,1,0,1,0,1) & e_8 &= \epsilon(-s_{12})^{\epsilon+1} I(4,0,1,1,1,1) \\
e_9 &= \epsilon(2\epsilon-1)(-s_{12})^{\epsilon} I(6,1,0,1,1,1) & e_{10} &= \epsilon(-s_{12})^{\epsilon+1} I(4,1,1,0,1,1) \\
e_{11} &= \epsilon(-s_{12})^{\epsilon+1} I(4,1,1,1,0,1) & e_{12} &= \epsilon(-s_{12})^{\epsilon+1} I(4,1,1,1,1,0) \\
e_{13} &= \epsilon^2(-s_{12})^{\epsilon+1} I(6,1,1,1,1,1)\,.
\end{aligned}
\tag{5.35}
$$

Operator basis:

$$
\begin{aligned}
e_1'^{(\mathcal{D})} &= \partial_{11}\,,\ e_2'^{(\mathcal{D})} = \partial_{10}\,,\ e_3'^{(\mathcal{D})} = \partial_9\,,\ e_4'^{(\mathcal{D})} = \partial_8\,,\ e_5'^{(\mathcal{D})} = \partial_7 \\
e_6'^{(\mathcal{D})} &= \epsilon(5\delta+1) + z_7\partial_7 + z_8\partial_8 + z_9\partial_9 + z_{10}\partial_{10} + z_{11}\partial_{11} \\
e_7'^{(\mathcal{D})} &= (4\delta\epsilon+\epsilon+1)\partial_{11} + z_{11}\partial_{11}^2 + z_9\partial_9\partial_{11} + z_{10}\partial_{10}\partial_{11} \\
e_8'^{(\mathcal{D})} &= \partial_9\partial_{11} \\
e_9'^{(\mathcal{D})} &= \delta\epsilon\,(4\delta\epsilon+1)\,\partial_{11} + \delta z_{11}\epsilon\partial_{11}^2 + z_7\,(4\delta\epsilon+\epsilon+1)\,\partial_7\partial_{11} + z_9\,(5\delta\epsilon+\epsilon+2)\,\partial_9\partial_{11} + \delta z_{10}\epsilon\partial_{10}\partial_{11} \\
&\quad + z_7 z_{11}\partial_7\partial_{11}^2 + z_9 z_{11}\partial_9\partial_{11}^2 + z_9^2\partial_9^2\partial_{11} + z_7 z_9\partial_7\partial_9\partial_{11} + z_7 z_{10}\partial_7\partial_{10}\partial_{11} + z_9 z_{10}\partial_9\partial_{10}\partial_{11} \\
e_{10}'^{(\mathcal{D})} &= \partial_7\partial_{10} \\
e_{11}'^{(\mathcal{D})} &= (5\delta\epsilon+\epsilon+1)\partial_{10} + z_{10}\partial_{10}^2 + z_7\partial_7\partial_{10} + z_8\partial_8\partial_{10} + z_9\partial_9\partial_{10} + z_{11}\partial_{11}\partial_{10} \\
e_{12}'^{(\mathcal{D})} &= (5\delta\epsilon+\epsilon+1)\partial_9 + z_9\partial_9^2 + z_7\partial_7\partial_9 + z_8\partial_8\partial_9 + z_{10}\partial_{10}\partial_9 + z_{11}\partial_{11}\partial_9 \\
e_{13}'^{(\mathcal{D})} &= (4\delta\epsilon+\epsilon+2)\partial_{11}\partial_9 + z_9\partial_{11}\partial_9^2 + z_{10}\partial_{10}\partial_{11}\partial_9 + z_{11}\partial_{11}^2\partial_9\,.
\end{aligned}
\tag{5.38}
$$

Pentagon: Pfaffian system

- **Strategy**: Obtain $\partial_i\,\texttt{Std} = P_i^{(\texttt{Std})} \cdot \texttt{Std} \to$ gauge to basis e. Here,

$$\texttt{Std} = \left(\partial_9\partial_{11}^2, \partial_9^2, \partial_{10}^2, \partial_8\partial_{11}, \partial_9\partial_{11}, \partial_{10}\partial_{11}, \partial_{11}^2, \partial_7, \partial_8, \partial_9, \partial_{10}, \partial_{11}\right)^T$$

- **Macaulay matrix**: Solve for $C \in \mathbb{Q}^{13\times 133}(z, \epsilon, \delta)$ in

$$C_{\texttt{Ext}} - C \cdot M_{\texttt{Ext}} = 0 \quad , \quad M_{\texttt{Ext}} \in \mathbb{Q}^{133\times 133}(z, \epsilon, \delta) \text{ is sparse}$$

- **Solution**: Few minutes on a laptop with `FiniteFlow` [Peraro '19]

- **Pfaffian for Std**: $P_i^{(\texttt{Std})} = C_{\texttt{Std}} - C \cdot M_{\texttt{Std}}$

- **Gauge to $\partial_i\, e = P_i \cdot e$**: Can expand e as

$$e = \sum_{k\in K} G_k^{(1)} \left(\partial^k\,\texttt{Std}\right) = \sum_{k\in K} G_k^{(1)} \cdot G_k^{(2)} \cdot \texttt{Std} \quad , \quad K \subset \mathbb{N}_0^5$$

because $\partial^k\,\texttt{Std} = G_k^{(2)} \cdot \texttt{Std}$ known from $\partial_i\,\texttt{Std} = P_i^{(\texttt{Std})} \cdot \texttt{Std}$