DIS coefficient functions at four loops in QCD and beyond

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Workshop *Loops and Legs in Quantum Field Theory*, Ettal, Apr 27, 2022

Based on work done in collaboration with:

- *Resummation of small-x double logarithms in QCD: inclusive deep-inelastic scattering*
  J. Davies, C.-H. Kom, S. M., and A. Vogt `arXiv:2202.10362`
- *Low moments of the four-loop splitting functions in QCD*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:2111.15561`
- *Approximate four-loop QCD corrections to the Higgs-boson production cross section*
  G. Das, S. M., and A. Vogt `arXiv:2004.00563`
- *Soft corrections to inclusive deep-inelastic scattering at four loops and beyond*
  G. Das, S. M., and A. Vogt `arXiv:1912.12920`
- *Five-loop contributions to low-N non-singlet anomalous dimensions in QCD*
  F. Herzog, S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt
  `arXiv:1812.11818`
- *On quartic colour factors in splitting functions and the gluon cusp anomalous dimension*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:1805.09638`
- *Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:1707.08315`
- Many more papers of MVV and friends . . . `2001 - ...`

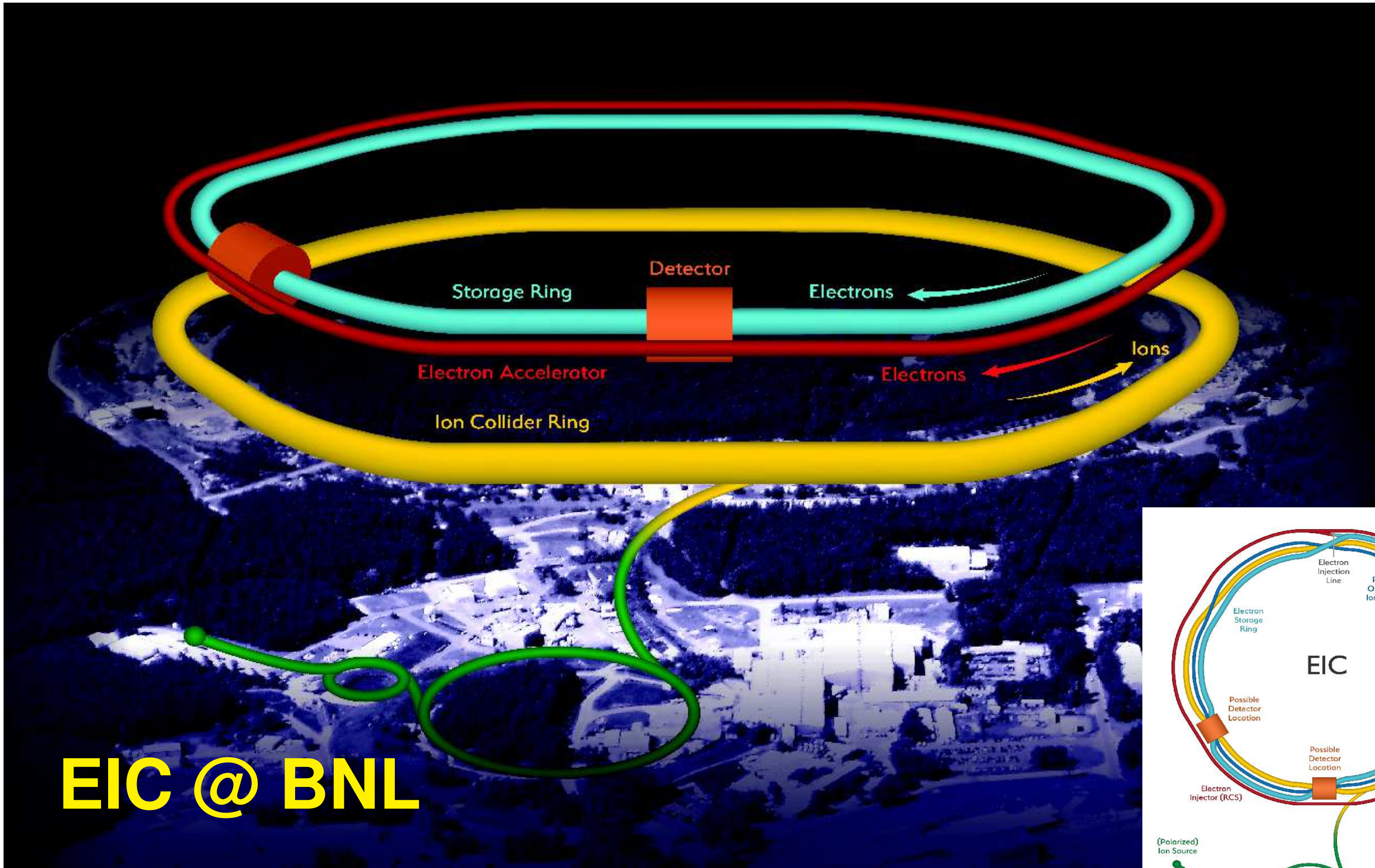
Deep-inelastic scattering

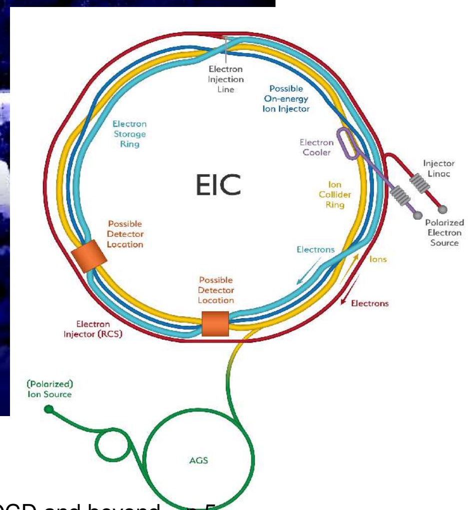
Once upon a time . . .

- HERA: deep structure of proton at highest Q^2 and smallest x

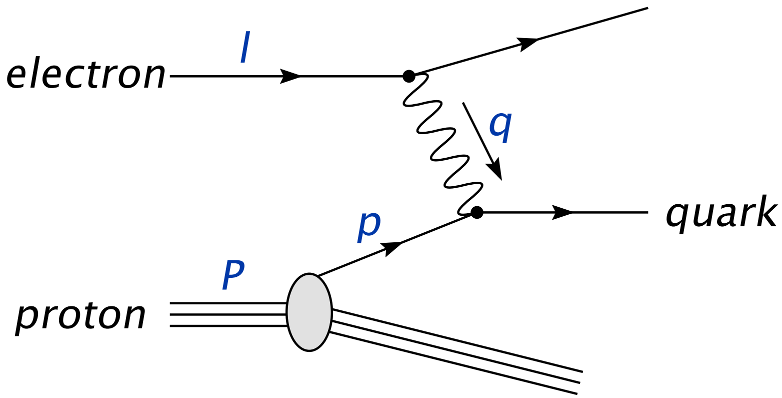
Bright future for precision hadron physics

- Electron-Ion Collider
  A machine that will unlock the secrets of the strongest force in Nature





Deep-inelastic scattering



Kinematic variables

- momentum transfer $Q^2 = -q^2$
- Bjorken variable $x = Q^2/(2p \cdot q)$

- Structure functions (up to order $\mathcal{O}(1/Q^2)$)

$$F_a\left(x, Q^2\right) = \sum_i \left[C_{a,i}\left(\alpha_s(\mu^2), \mu^2/Q^2\right) \otimes PDF\left(\mu^2\right)\right](x)$$

- Coefficient functions up to N^{4}LO (work in progress)

$$C_{a,i} = \alpha_s^n \left(c_{a,i}^{(0)} + \alpha_s c_{a,i}^{(1)} + \alpha_s^2 c_{a,i}^{(2)} + \alpha_s^3 c_{a,i}^{(3)} + \alpha_s^3 c_{a,i}^{(4)} + \dots\right)$$

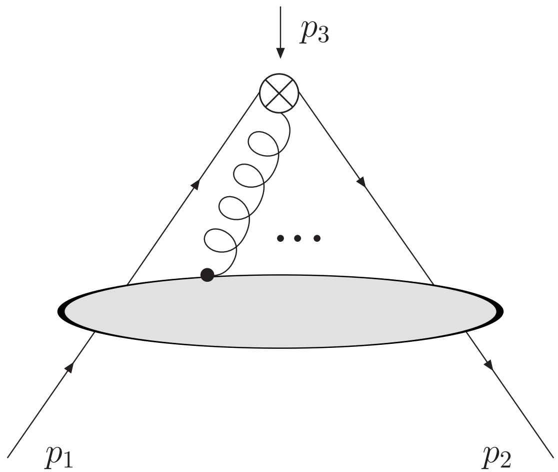
- Evolution equations up to N^{3}LO (work in progress)
  - non-singlet ($2n_f - 1$ scalar) and singlet (2×2 matrix) equations

$$\frac{d}{d \ln \mu^2} PDF(x, \mu^2) = \left[P(\alpha_s(\mu^2)) \otimes PDF(\mu^2)\right](x)$$

  - splitting functions $P_{ij} = \alpha_s P_{ij}^{(0)} + \alpha_s^2 P_{ij}^{(1)} + \alpha_s^3 P_{ij}^{(2)} + \alpha_s^4 P_{ij}^{(3)} + \dots$

QCD evolution at 1% precision

Operator matrix elements



- Quark operator of spin-N and twist two
  $O^{\psi}_{\{\mu_1,\dots,\mu_N\}} = \overline{\psi}\,\gamma_{\{\mu_1} D_{\mu_2} \dots D_{\mu_N\}}\,\psi$
- N covariant derivatives
  $D_{\mu,ij} = \partial_\mu \delta_{ij} + \mathrm{i} g_s \left(t_a\right)_{ij} A^a_\mu$
  sandwiched between quark fields ψ, $\overline{\psi}$
- Evaluation of operators in matrix elements $A^{\psi\psi}$ with external quark states

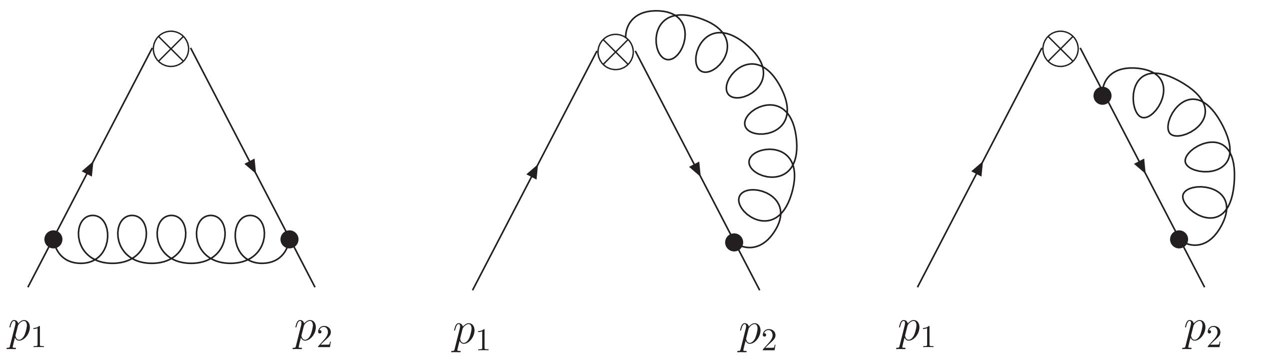
$$A^{\psi\psi}_{\{\mu_1,\dots,\mu_N\}} = \langle \psi(p_1) | O^{\psi}_{\{\mu_1,\dots,\mu_N\}}(p_3) | \overline{\psi}(p_2) \rangle$$

- Anomalous dimensions $\gamma(\alpha_s, N)$ govern scale dependence of renormalized operators

$$\frac{d}{d\ln\mu^2} O^{\rm ren} = -\gamma\, O^{\rm ren} \qquad\qquad \gamma(N) = -\int\limits_0^1 dx\, x^{N-1} P(x)$$

- Zero-momentum transfer through operator reduces problem to computation of propagator-type diagrams

One-loop computation



- Computation of loop integral in $D = 4 - 2\epsilon$ dimensions and expansion in ϵ
  - anomalous dimension $\gamma(N)$ from ultraviolet divergence

$$\Delta^{\mu_1} \dots \Delta^{\mu_N} \langle \psi(p_1) | O^{\psi}_{\{\mu_1, \dots, \mu_N\}}(0) | \overline{\psi}(-p_1) \rangle =$$

$$= \quad 1 + \frac{\alpha_s}{4\pi}\frac{1}{\epsilon}\gamma^{(0)}(N) + \mathcal{O}(\alpha_s \epsilon^0) + \mathcal{O}(\alpha_s^2)$$

$$= \quad 1 + \frac{\alpha_s}{4\pi}\frac{1}{\epsilon}\left\{ C_F \left(4S_1(N) + \frac{2}{N+1} - \frac{2}{N} - 3 \right) \right\} + \mathcal{O}(\alpha_s \epsilon^0) + \mathcal{O}(\alpha_s^2)$$

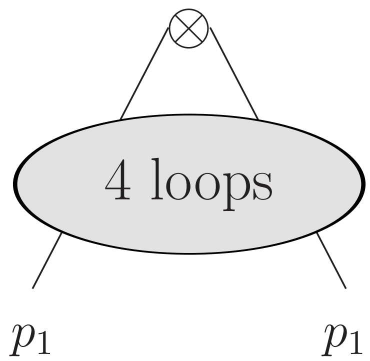
- One-loop result with harmonic sum $S_1(N) = \sum_{i=1}^{N} \frac{1}{i}$
- Details in *The Theory of Quark and Gluon Interactions*
  F.J. Yndurain

Four-loop computation

- Feynman diagrams for operator matrix elements generated up to four loops with **Qgraf** Nogueira ‘91
- Parametric reduction of four-loop massless propagator diagrams with integration-by-parts identities encoded in **Forcer** Ruijl, Ueda, Vermaseren ‘17
- Symbolic manipulations with **Form** Vermaseren ‘00; Kuipers, Ueda, Vermaseren, Vollinga ‘12 and multi-threaded version **TForm** Tentyukov, Vermaseren ‘07
- Diagrams of same topology and color factor combined to meta diagrams
- Non-singlet anomalous dimension
  - 1 one-, 7 two-, 53 three- and 650 four-loop meta diagrams for $\gamma_{\rm ns}^{\pm}$
  - 1 three- and 29 four-loop meta diagrams for $\gamma_{\rm ns}^{\rm s}$

Fixed Mellin moments

- Computation of anomalous dimensions $\gamma(N)$ for Mellin moments mostly up to $N = 18$
  - sometimes higher for complicated topologies ($N = 19$, $N = 20$, . . .)
  - much higher for “easy” topologies, e.g., n_f-dependent ($N \simeq 80$, . . .)



Analytic reconstruction

- Sufficiently many Mellin moments allow for reconstruction of analytic all-N expressions through solution of Diophantine equations
- Anomalous dimensions $\gamma(N)$ given by harmonic sums up to weight 7

$$S_{\pm m_1,\ldots,m_k}(N) = \sum_{i=1}^{N} \frac{(\pm 1)^i}{i^{m_1}} S_{m_2,\ldots,m_k}(i)$$

  - $2 \cdot 3^{w-1}$ sums at weight w
- Reciprocity relation (RR) $\gamma(N) = \gamma_{\mathrm{u}}\left(N + \gamma(N) - \beta(\alpha_s)\right)$ reduces number of 2^{w-1} sums at weight w for γ_{u}
  - additional denominators with powers $1/(N+1)$ give $2^{w+1} - 1$ objects (255 at weight 7)
- Large-n_c limit only needs harmonic sums with positive index
  - weight w RR sums given by Fibonacci number $F(w)$
  - total number of unknowns (including powers $1/(N+1)$) amount to $F(w+4) - 2$ (87 at $w = 7$)
- Additional 46 constraints from large-x/small-x ($N \to \infty / N \to 0$) limit
- Solution becomes feasible with 18 Mellin moments for $\gamma_{\mathrm{ns}}^{\pm}$

Small-x behavior (I)

The small x-limit: $x \to 0$

- Structure of non-singlet splitting functions $P_{\rm ns}^{\pm}$ at small x
  - double-logarithmic contributions with very large coefficients
  - resummation for $P_{\rm ns}^{+}$ to leading logarithmic (LL) accuracy in Mellin-N space
    Kirschner, Lipatov '83

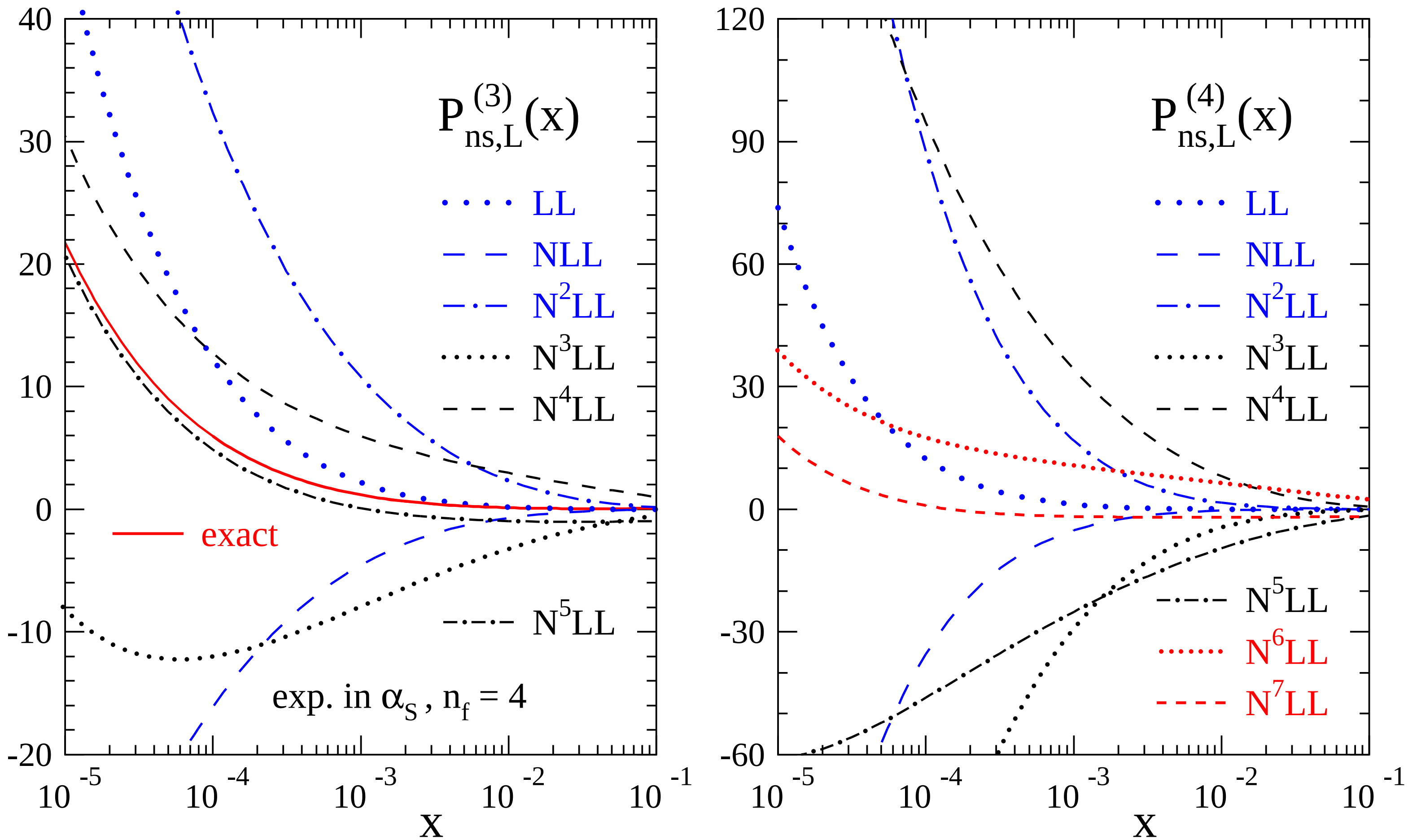
$$P_{\rm ns,LL}^{+}(N,\alpha_s) \;=\; \frac{N}{2}\left\{1 - \left(1 - \frac{2\alpha_s C_F}{\pi N^2}\right)^{1/2}\right\}$$

- Large-n_c limit with intriguing structure   Velizhanin '14

$$P_{\rm ns}^{+}(N,\alpha_s)\,\left(P_{\rm ns}^{+}(N,\alpha_s) - N + \beta(\alpha_s)/\alpha_s\right) \;=\; O(1)$$

  - Laurent expansion about $N = 0$
- Exploit structure of the (unfactorized) structure functions in dimensional regularization
- Resummation in terms of modified Bessel functions to N^{7}LL accuracy
  Davies, Kom, S.M., Vogt '22

Small-x behavior (II)

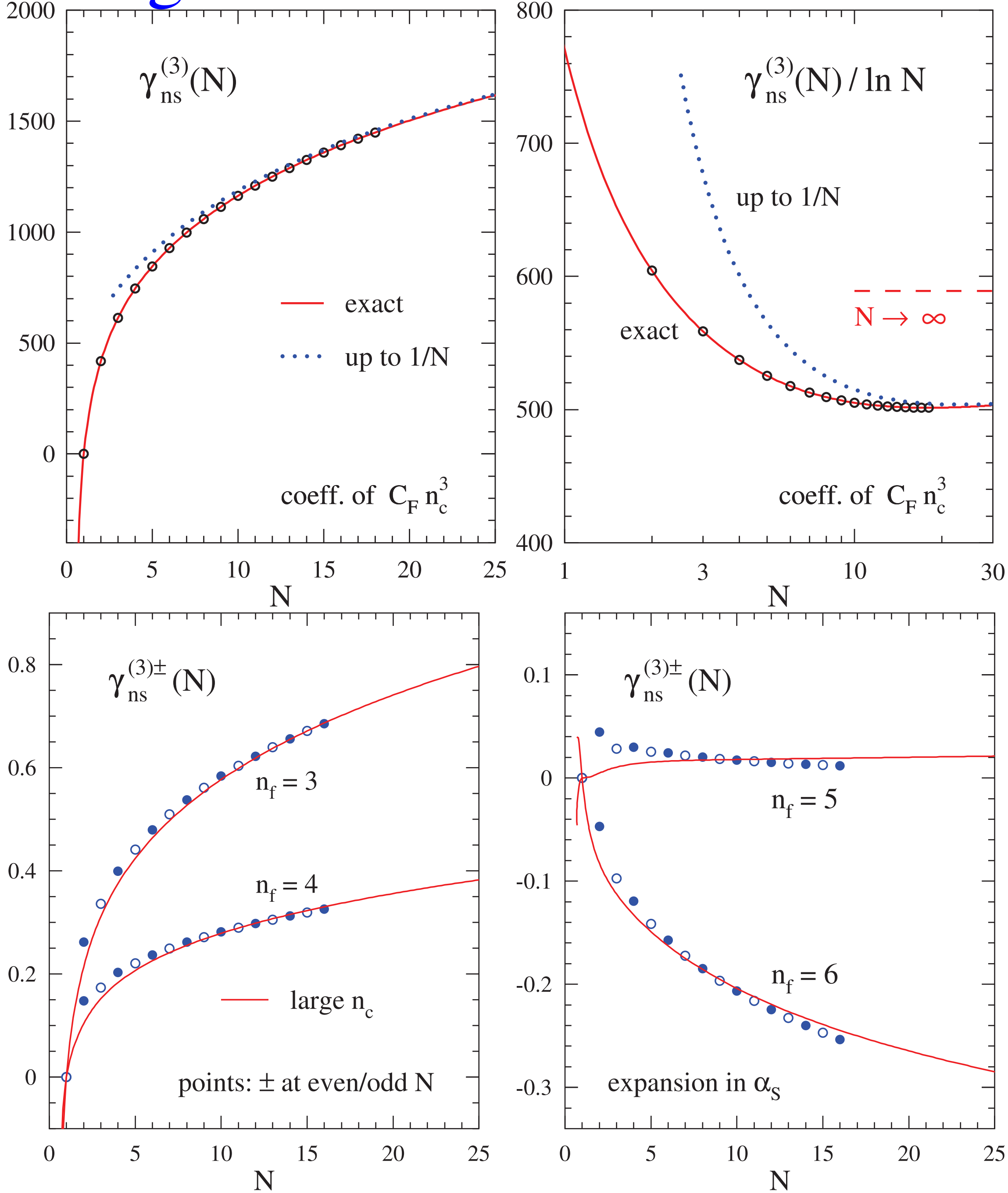


- Splitting functions $P_{\rm ns}^{(3),+}$ (left) and $P_{\rm ns}^{(4),+}$ (right) Davies, Kom, S.M., Vogt '22
  - small-x approximations to the four-flavour splitting functions $P^{(n)}_{\rm ns,\,L}(x)$ in the large-n_c limit
  - predictions up to N ^{7}LL

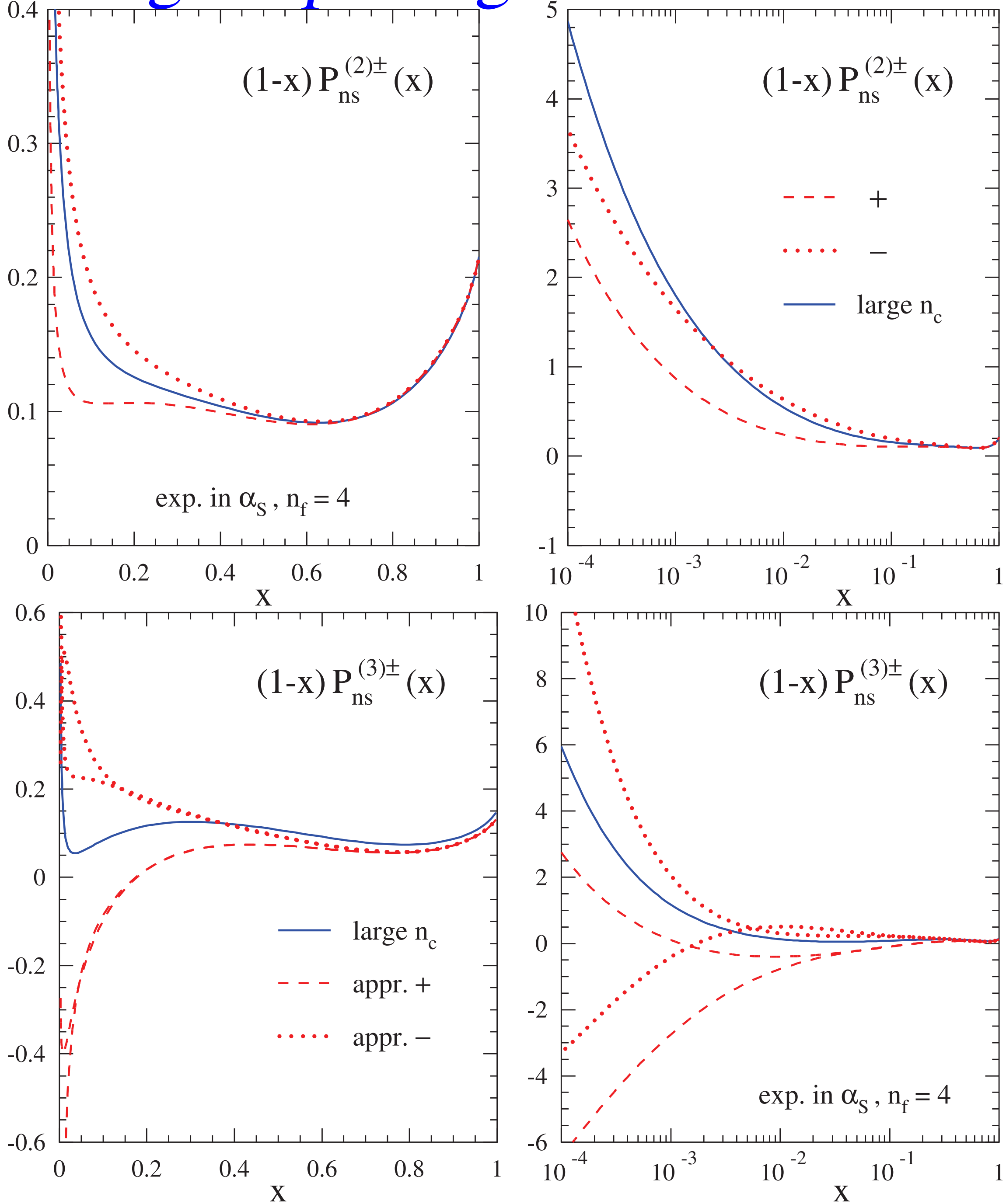
Four-loop non-singlet Mellin moments

- Top:
  n_f^0 part of anomalous dimensions $\gamma_{\rm ns}^{(3)\pm}(N)$
  in large-n_c limit
  and large-N expansion

- Bottom: results for even-N ($\gamma_{\rm ns}^{(3)+}(N)$)
  and odd-N ($\gamma_{\rm ns}^{(3)-}(N)$)
  in large-n_c limit
  for $n_f = 3, \ldots 6$



Four-loop non-singlet splitting functions



- Top:
  three-loop $P_{\rm ns}^{(2)\pm}(x)$
  and large-n_c limit
  with $n_f = 4$

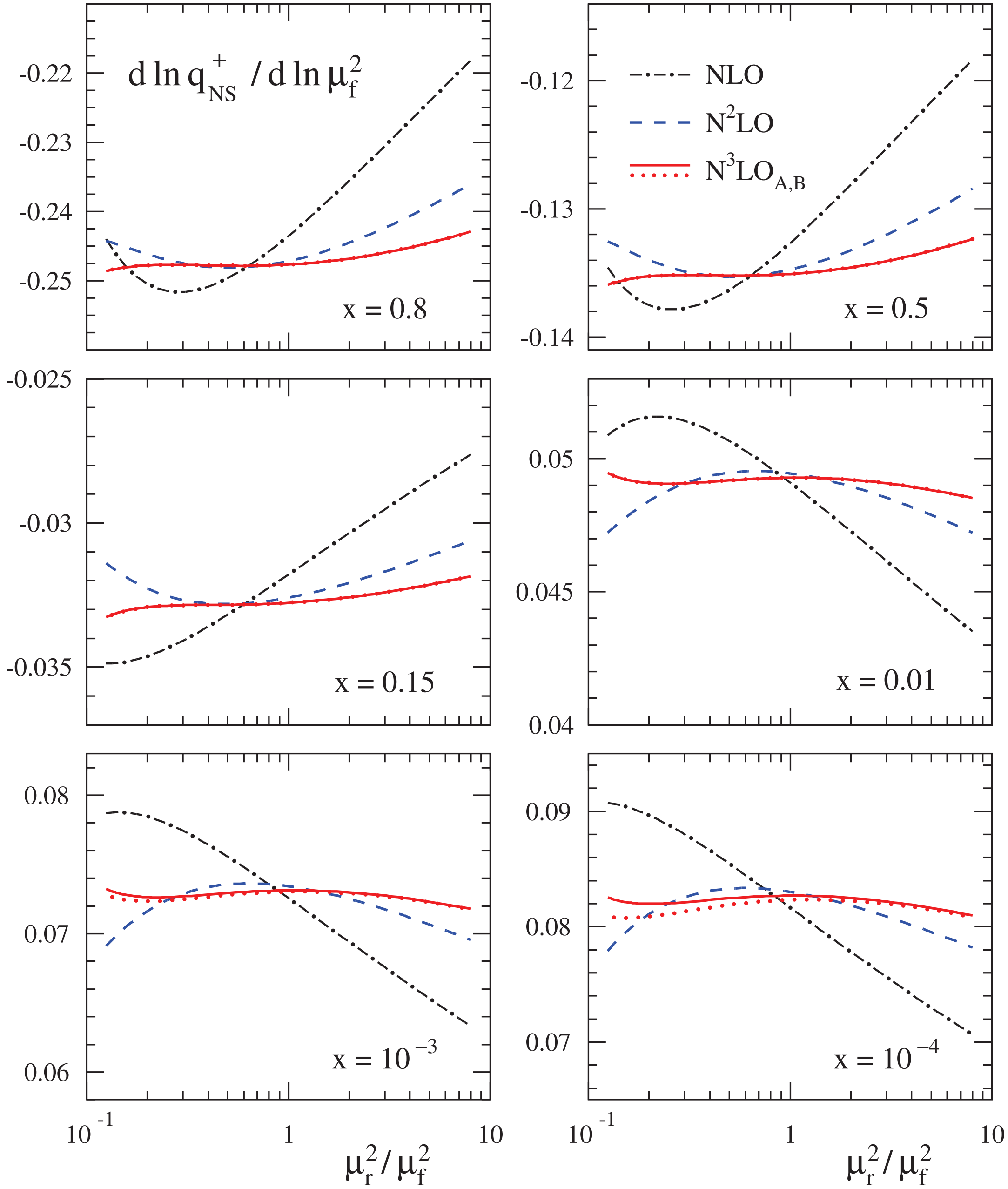
- Bottom:
  four-loop $P_{\rm ns}^{(3)\pm}(x)$
  and uncertainty bands
  beyond large-n_c limit
  with $n_f = 4$

Scale stability of evolution

- Renormalization scale dependence of evolution kernel $d \ln q_{\rm ns}^{+} / d \ln \mu_f^2$
  - non-singlet shape

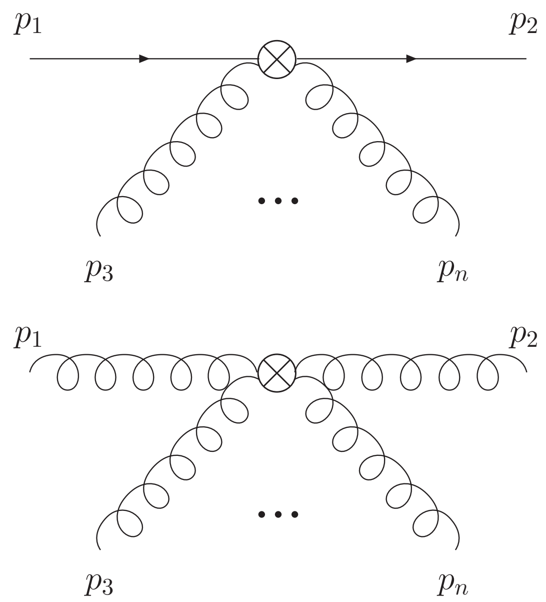
$$xq_{\rm ns}^{+}(x, \mu_0^2) \;=\; x^{\,0.5}(1-x)^3$$

- NLO, NNLO and N^{3}LO predictions
  - remaining uncertainty of four-loop splitting function $P_{\rm ns}^{(3)+}$ almost invisible



Singlet

Operator matrix elements



- Singlet operators of spin-N and twist two

$$O^{\rm q}_{\{\mu_1,\dots,\mu_N\}} = \overline{\psi}\,\gamma_{\{\mu_1} D_{\mu_2} \dots D_{\mu_N\}}\,\psi\;,$$

$$O^{\rm g}_{\{\mu_1,\dots,\mu_N\}} = F_{\nu\{\mu_1} D_{\mu_2} \cdots D_{\mu_{N-1}} F_{\mu_N\}}{}^{\nu}$$

- Quartic Casimir terms at four loops are effectively 'leading-order'
  - $d_{xy}^{(4)} \equiv d_x^{abcd} d_y^{abcd}$ for representations labels x, y with generators T_r^a

$$d_r^{abcd} = \frac{1}{6}\,{\rm Tr}\,(\,T_r^a T_r^b T_r^c T_r^d + \text{ five } bcd \text{ permutations}\,)$$

  - anomalous dimensions fulfil relation for $\mathcal{N} = 1$ supersymmetry

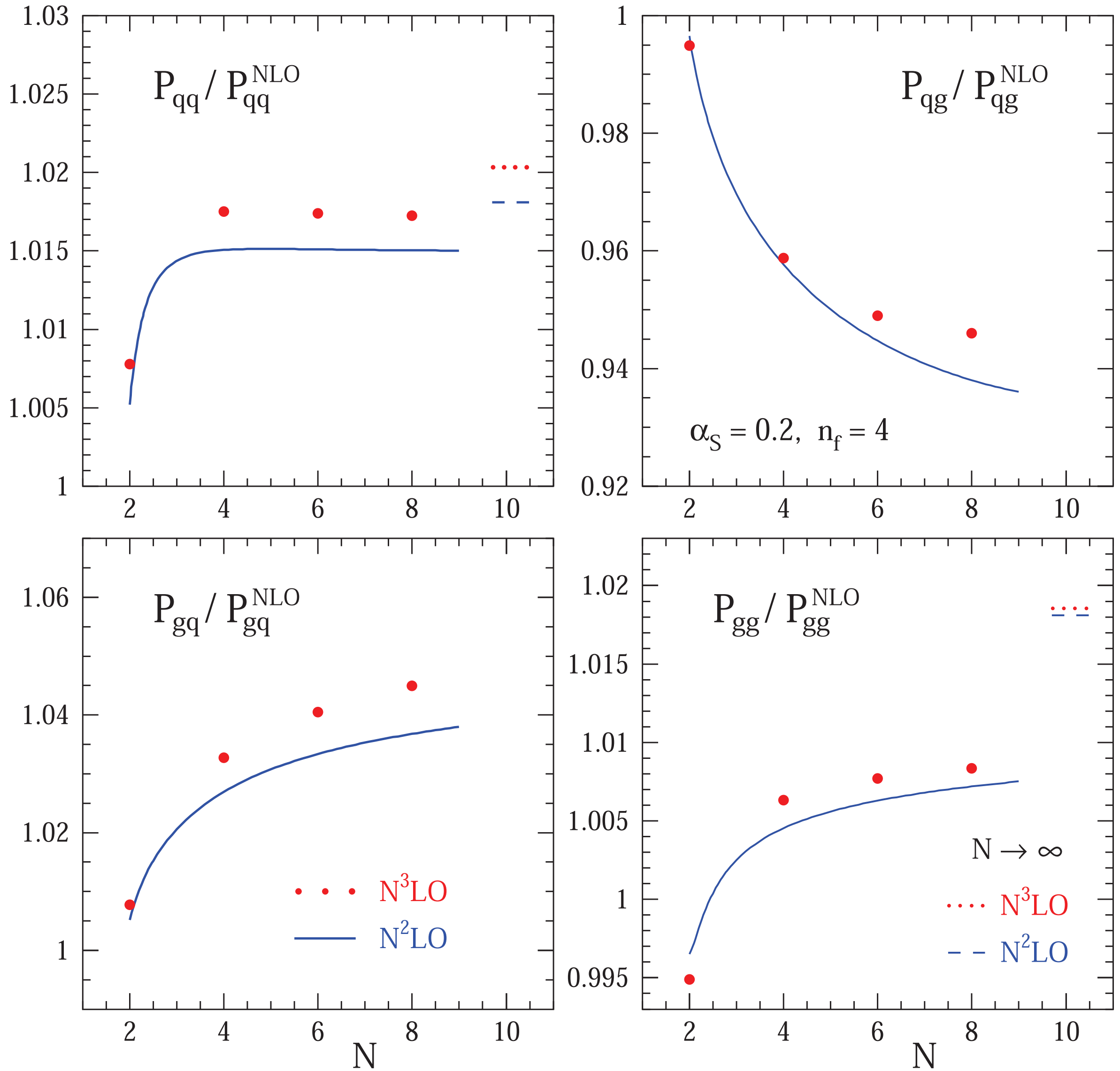
$$\gamma^{(3)}_{\rm qq}(N) + \gamma^{(3)}_{\rm gq}(N) - \gamma^{(3)}_{\rm qg}(N) - \gamma^{(3)}_{\rm gg}(N) \overset{Q}{=} 0$$

- Eigenvalues of singlet splitting functions (conjectured to be) composed of reciprocity-respecting sums
  - quartic Casimir terms fulfil stronger condition Belitsky, Müller, Schäfer '99

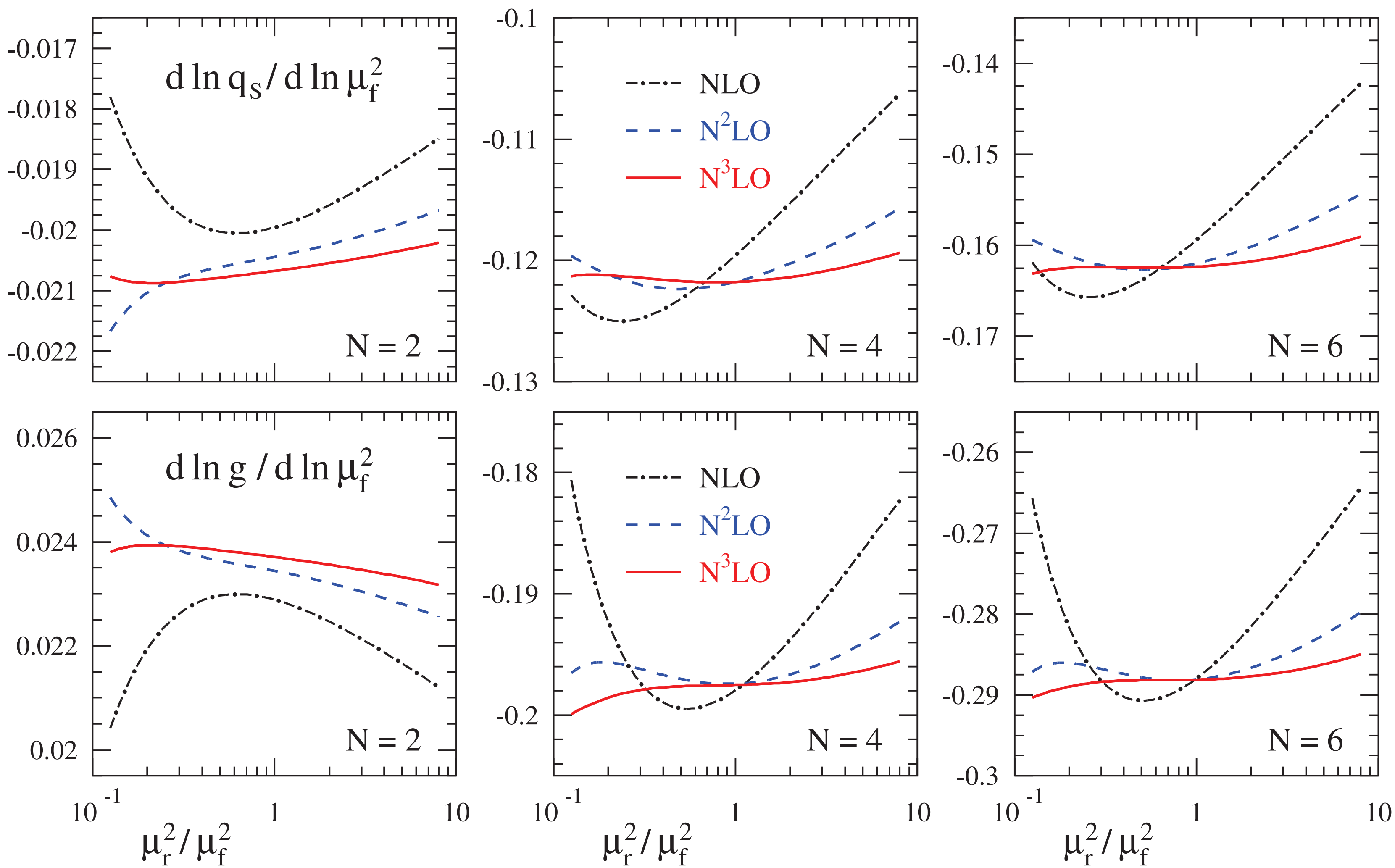
$$\gamma^{(0)}_{\rm qg}(N)\,\gamma^{(3)}_{\rm gq}(N) \overset{Q}{=} \gamma^{(0)}_{\rm gq}(N)\,\gamma^{(3)}_{\rm qg}(N)$$

Four-loop singlet Mellin moments

- Singlet moments at NNLO (lines) and N^3LO (even-N points) normalized to NLO results
  - $\alpha_s(\mu_f) = 0.2$ and $n_f = 4$
- Large-N limits in qq- and gg-channel



Scale stability of singlet evolution



- Renormalization-scale dependence of singlet PDFs $d \ln q_s^{\pm} / d \ln \mu_f^2$ and $d \ln g^{\pm} / d \ln \mu_f^2$ at $N = 2,\ 4,$ and 6 using NLO, NNLO and N^{3}LO predictions with $\alpha_s(\mu_f) = 0.2$ and $n_f = 4$

Five-loop Mellin moments

Five-loop Mellin moments

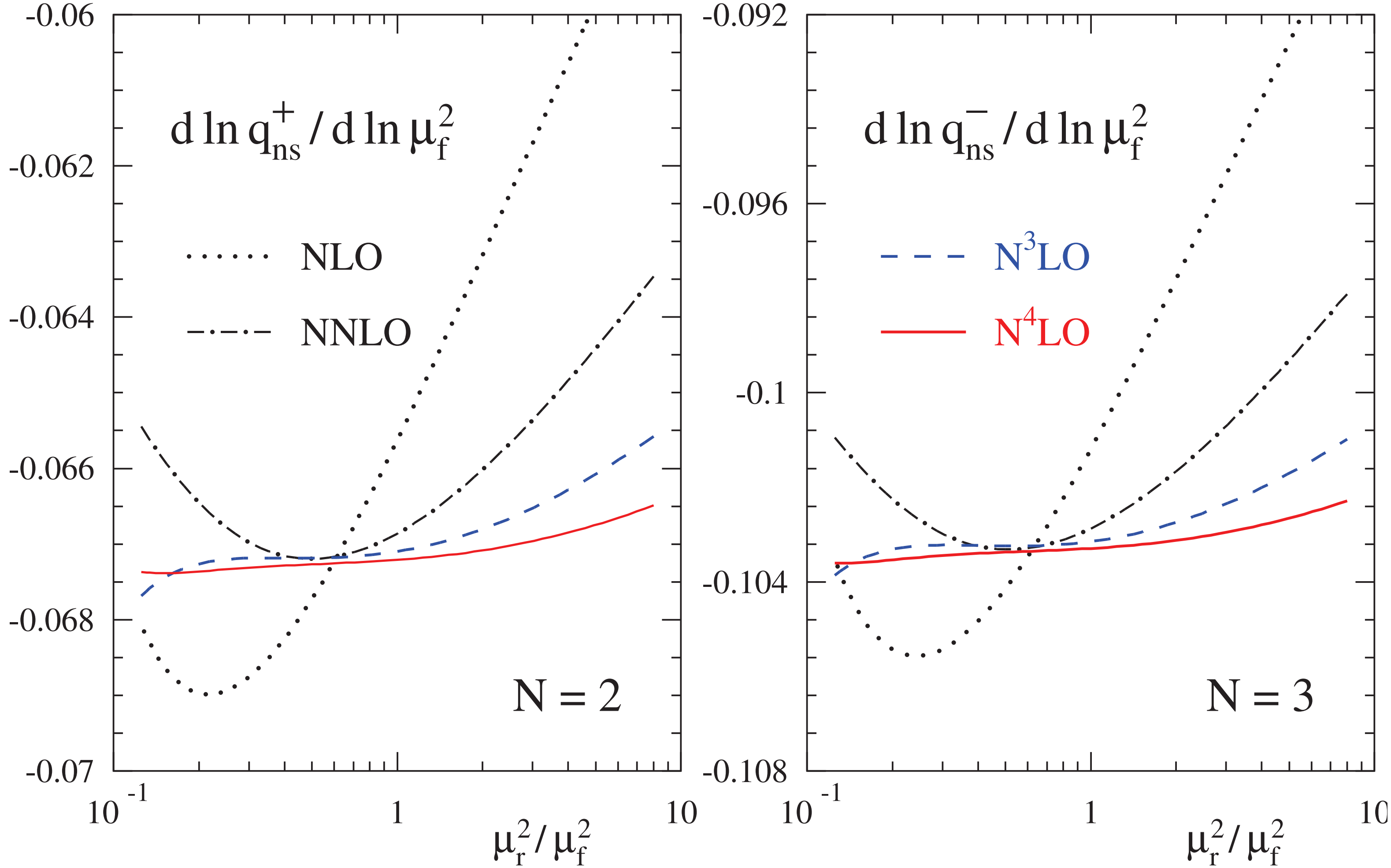
- Moments $N = 2$ and $N = 3$ for nonsinglet anomalous dimensions $\gamma_{\rm ns}^{\pm}$
- Implementation by Herzog, Ruijl '17 of local R^* operation Chetyrkin, Tkachov '82; Chetyrkin, Smirnov '84 for reduction of five-loop self-energy diagrams to four-loop ones computed with **Forcer** Ruijl, Ueda, Vermaseren '17

$$
\begin{aligned}
\gamma_{\rm ns}^{(4)+}(N=2) \;=\;& \\
& C_F^5\left[\frac{9306376}{19683}-\frac{802784}{729}\zeta_3-\frac{557440}{81}\zeta_5+\frac{12544}{9}\zeta_3^2+8512\zeta_7\right]\\
&-C_AC_F^4\left[\frac{81862744}{19683}-\frac{1600592}{243}\zeta_3+\frac{59840}{81}\zeta_4-\frac{142240}{27}\zeta_5+3072\zeta_3^2-\frac{35200}{9}\zeta_6+19936\zeta_7\right]\\
&+C_A^2C_F^3\left[\frac{63340406}{6561}-\frac{1003192}{243}\zeta_3-\frac{229472}{81}\zeta_4+\frac{61696}{27}\zeta_5+\frac{30976}{9}\zeta_3^2-\frac{35200}{9}\zeta_6+15680\zeta_7\right]\\
&-C_A^3C_F^2\left[\frac{220224724}{19683}+\frac{4115536}{729}\zeta_3-\frac{170968}{27}\zeta_4-\frac{3640624}{243}\zeta_5+\frac{70400}{27}\zeta_3^2+\frac{123200}{27}\zeta_6+\frac{331856}{27}\zeta_7\right]\\
&+C_A^4C_F\left[\frac{266532611}{39366}+\frac{2588144}{729}\zeta_3-\frac{221920}{81}\zeta_4-\frac{3102208}{243}\zeta_5+\frac{74912}{81}\zeta_3^2+\frac{334400}{81}\zeta_6+\frac{178976}{27}\zeta_7\right]\\
&-\frac{d_{AA}^{(4)}}{N_A}C_F\left[\frac{15344}{81}-\frac{12064}{27}\zeta_3-\frac{704}{3}\zeta_4+\frac{58400}{27}\zeta_5-\frac{6016}{3}\zeta_3^2-\frac{19040}{9}\zeta_7\right]\\
&+\frac{d_{FA}^{(4)}}{N_F}C_F\left[\frac{23968}{81}-\frac{733504}{81}\zeta_3+\frac{176320}{81}\zeta_5+\frac{6400}{3}\zeta_3^2+\frac{77056}{9}\zeta_7\right]\\
&-\frac{d_{FA}^{(4)}}{N_F}C_A\left[\frac{82768}{81}-\frac{555520}{81}\zeta_3+\frac{10912}{9}\zeta_4-\frac{1292960}{81}\zeta_5+\frac{84352}{27}\zeta_3^2+\frac{140800}{27}\zeta_6+12768\zeta_7\right]\\
&+n_fC_F^4\left[\frac{1824964}{19683}-\frac{463520}{243}\zeta_3+\frac{21248}{81}\zeta_4-\frac{16480}{81}\zeta_5+\frac{6656}{9}\zeta_3^2-\frac{6400}{9}\zeta_6+\frac{8960}{3}\zeta_7\right]\\
&-n_fC_AC_F^3\left[\frac{3375082}{6561}-\frac{420068}{243}\zeta_3-\frac{48256}{81}\zeta_4+\frac{458032}{81}\zeta_5+\frac{3968}{3}\zeta_3^2-\frac{8000}{3}\zeta_6+\frac{4480}{3}\zeta_7\right]\\
&+n_fC_A^2C_F^2\left[\frac{15291499}{13122}+\frac{1561600}{243}\zeta_3-\frac{114536}{27}\zeta_4-\frac{252544}{243}\zeta_5+\frac{24896}{27}\zeta_3^2+\frac{13600}{27}\zeta_6+\frac{11200}{27}\zeta_7\right]\\
&-n_fC_A^3C_F\left[\frac{48846580}{19683}+\frac{4314308}{729}\zeta_3-\frac{274768}{81}\zeta_4-\frac{1389080}{243}\zeta_5+\frac{27808}{81}\zeta_3^2+\frac{184000}{81}\zeta_6+\frac{39088}{27}\zeta_7\right]\\
&+n_f\frac{d_{FA}^{(4)}}{N_F}\left[\frac{22096}{27}+\frac{43712}{81}\zeta_3-\frac{512}{9}\zeta_4-\frac{217280}{81}\zeta_5-\frac{25088}{27}\zeta_3^2+\frac{25600}{27}\zeta_6-2464\zeta_7\right]\\
&-n_fC_F\frac{d_{FF}^{(4)}}{N_F}\left[\frac{170752}{81}-\frac{328832}{81}\zeta_3+\frac{650240}{81}\zeta_5-\frac{8192}{9}\zeta_3^2-\frac{35840}{9}\zeta_7\right]\\
&+n_fC_A\frac{d_{FF}^{(4)}}{N_F}\left[\frac{207824}{81}+\frac{251392}{81}\zeta_3-\frac{5632}{9}\zeta_4-\frac{522880}{81}\zeta_5+\frac{15872}{27}\zeta_3^2+\frac{70400}{27}\zeta_6-\frac{29120}{9}\zeta_7\right]\\
&+n_f^2C_F^3\left[\frac{1082297}{6561}-\frac{145792}{243}\zeta_3+\frac{1072}{81}\zeta_4+\frac{55552}{81}\zeta_5+\frac{1792}{9}\zeta_3^2-\frac{3200}{9}\zeta_6\right]\\
&+n_f^2C_AC_F^2\left[\frac{332254}{2187}-\frac{85016}{243}\zeta_3+\frac{20752}{27}\zeta_4-\frac{28544}{81}\zeta_5-\frac{13952}{27}\zeta_3^2+\frac{1600}{27}\zeta_6\right]\\
&+n_f^2C_A^2C_F\left[\frac{631400}{6561}+\frac{214268}{243}\zeta_3-784\zeta_4-\frac{53344}{243}\zeta_5+\frac{25472}{81}\zeta_3^2+\frac{22400}{81}\zeta_6\right]\\
&-n_f^2\frac{d_{FF}^{(4)}}{N_F}\left[\frac{43744}{81}-\frac{35648}{81}\zeta_3-\frac{1792}{9}\zeta_4-\frac{52480}{81}\zeta_5+\frac{2048}{27}\zeta_3^2+\frac{12800}{27}\zeta_6\right]\\
&+n_f^3C_F^2\left[\frac{265510}{19683}+\frac{11872}{729}\zeta_3-\frac{128}{3}\zeta_4+\frac{512}{27}\zeta_5\right]\\
&+n_f^3C_AC_F\left[\frac{168677}{19683}+\frac{11872}{729}\zeta_3+\frac{2752}{81}\zeta_4-\frac{4096}{81}\zeta_5\right]-n_f^4C_F\left[\frac{5504}{19683}+\frac{1024}{729}\zeta_3-\frac{128}{81}\zeta_4\right]
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{\rm ns}^{(4)-}(N=3) \;=\;& \\
& C_F^5\left[\frac{81472935625}{80621568}+\frac{99382175}{23328}\zeta_3-\frac{3395975}{162}\zeta_5-\frac{9650}{9}\zeta_3^2+\frac{34685}{2}\zeta_7\right]\\
&-C_AC_F^4\left[\frac{286028134219}{80621568}-\frac{23916529}{7776}\zeta_3-4490\zeta_3^2+\frac{134090}{81}\zeta_4-\frac{2468075}{108}\zeta_5-\frac{55000}{9}\zeta_6+\frac{155155}{4}\zeta_7\right]\\
&+C_A^2C_F^3\left[\frac{20173099267}{3359232}-\frac{15401281}{864}\zeta_3+\frac{732787}{1296}\zeta_4+\frac{1972075}{216}\zeta_5-\frac{63830}{9}\zeta_3^2-\frac{79750}{9}\zeta_6+\frac{139895}{4}\zeta_7\right]\\
&-C_A^3C_F^2\left[\frac{166662991819}{20155392}-\frac{36397493}{2916}\zeta_3-\frac{103763}{54}\zeta_4+\frac{30994565}{3888}\zeta_5-\frac{133990}{27}\zeta_3^2-\frac{72875}{54}\zeta_6+\frac{2127335}{108}\zeta_7\right]\\
&+C_A^4C_F\left[\frac{75932079965}{10077696}-\frac{27693563}{23328}\zeta_3-\frac{1791229}{1296}\zeta_4-\frac{9417425}{1944}\zeta_5-\frac{96700}{81}\zeta_3^2+\frac{163625}{81}\zeta_6+\frac{199640}{27}\zeta_7\right]\\
&-\frac{d_{AA}^{(4)}}{N_A}C_F\left[\frac{81725}{162}-\frac{33505}{18}\zeta_3-\frac{1100}{3}\zeta_4+\frac{52025}{18}\zeta_5-\frac{7000}{3}\zeta_3^2-\frac{48125}{36}\zeta_7\right]\\
&-\frac{d_{FA}^{(4)}}{N_F}C_F\left[\frac{231575}{36}+\frac{6351445}{324}\zeta_3-\frac{2927225}{162}\zeta_5+\frac{23210}{3}\zeta_3^2-\frac{200410}{9}\zeta_7\right]\\
&+\frac{d_{FA}^{(4)}}{N_F}C_A\left[\frac{165871}{54}+\frac{1816625}{162}\zeta_3-\frac{41800}{9}\zeta_4-\frac{4456145}{162}\zeta_5+\frac{196880}{27}\zeta_3^2+\frac{200750}{27}\zeta_6-\frac{7525}{4}\zeta_7\right]\\
&+n_fC_F^4\left[\frac{1776521549}{40310784}-\frac{1332919}{486}\zeta_3+\frac{5000}{9}\zeta_3^2+\frac{33290}{81}\zeta_4-\frac{30325}{81}\zeta_5-\frac{10000}{9}\zeta_6+\frac{14000}{3}\zeta_7\right]\\
&-n_fC_AC_F^3\left[\frac{3737356319}{3359232}-\frac{2327111}{432}\zeta_3+\frac{1280}{3}\zeta_3^2+\frac{262069}{648}\zeta_4+\frac{1693715}{162}\zeta_5-\frac{14000}{3}\zeta_6+\frac{7000}{3}\zeta_7\right]\\
&+n_fC_A^2C_F^2\left[\frac{5637513931}{3359232}+\frac{2711207}{486}\zeta_3-\frac{5020}{27}\zeta_3^2-\frac{457499}{108}\zeta_4+\frac{508820}{243}\zeta_5-\frac{20375}{27}\zeta_6+\frac{50155}{108}\zeta_7\right]\\
&-n_fC_A^3C_F\left[\frac{8766012215}{2519424}+\frac{45697231}{5832}\zeta_3+\frac{1195}{81}\zeta_3^2-\frac{2848403}{648}\zeta_4-\frac{1808870}{243}\zeta_5+\frac{222250}{81}\zeta_6+\frac{250915}{108}\zeta_7\right]\\
&-n_fC_F\frac{d_{FF}^{(4)}}{N_F}\left[\frac{24385}{27}-\frac{334010}{81}\zeta_3-\frac{8480}{9}\zeta_3^2+\frac{1622600}{81}\zeta_5-\frac{135380}{9}\zeta_7\right]\\
&+n_f\frac{d_{FA}^{(4)}}{N_F}\left[\frac{297889}{162}+\frac{154970}{81}\zeta_3-\frac{62600}{27}\zeta_3^2+\frac{3700}{9}\zeta_4-\frac{122780}{81}\zeta_5-\frac{36500}{27}\zeta_6-910\zeta_7\right]\\
&+n_fC_A\frac{d_{FF}^{(4)}}{N_F}\left[\frac{241835}{162}+\frac{333487}{81}\zeta_3+\frac{30560}{27}\zeta_3^2-10780/9\,\zeta_4-\frac{316900}{81}\zeta_5+\frac{110000}{27}\zeta_6-\frac{71960}{9}\zeta_7\right]\\
&+n_f^2C_F^3\left[\frac{512848319}{1679616}-\frac{57109}{54}\zeta_3+\frac{2800}{9}\zeta_3^2+\frac{9118}{81}\zeta_4+\frac{86440}{81}\zeta_5-\frac{5000}{9}\zeta_6\right]\\
&+n_f^2C_AC_F^2\left[\frac{1080083}{5832}-\frac{296729}{972}\zeta_3-\frac{21800}{27}\zeta_3^2+\frac{56327}{54}\zeta_4-\frac{42860}{81}\zeta_5+\frac{2500}{27}\zeta_6\right]\\
&+n_f^2C_A^2C_F\left[\frac{61747877}{419904}+\frac{2496811}{1944}\zeta_3+\frac{39800}{81}\zeta_3^2-\frac{3503}{3}\zeta_4-\frac{88990}{243}\zeta_5+\frac{35000}{81}\zeta_6\right]\\
&-n_f^2\frac{d_{FF}^{(4)}}{N_F}\left[\frac{19435}{27}-\frac{53366}{81}\zeta_3+\frac{3200}{27}\zeta_3^2-\frac{3160}{9}\zeta_4-\frac{70000}{81}\zeta_5+\frac{20000}{27}\zeta_6\right]\\
&+n_f^3C_F^2\left[\frac{28758139}{1259712}+\frac{21673}{729}\zeta_3-\frac{610}{9}\zeta_4+\frac{800}{27}\zeta_5\right]\\
&+n_f^3C_AC_F\left[\frac{13729181}{1259712}+\frac{14947}{729}\zeta_3+\frac{4390}{81}\zeta_4-\frac{6400}{81}\zeta_5\right]-n_f^4C_F\left[\frac{259993}{629856}+\frac{1660}{729}\zeta_3-\frac{200}{81}\zeta_4\right]
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{\rm ns}^{(4)\,\rm v}(N=3) \;=\;& \gamma_{\rm ns}^{(4)-}(N=3)\\
&+n_f\frac{d_{abc}d^{abc}}{N_F}\Bigg\{C_F^2\left[\frac{79906955}{46656}+\frac{246955}{54}\zeta_3-\frac{504550}{81}\zeta_5\right]\\
&\quad-C_AC_F\left[\frac{9797321}{3888}-\frac{475655}{54}\zeta_3+\frac{17600}{9}\zeta_4+\frac{516950}{81}\zeta_5-\frac{500}{9}\zeta_3^2+\frac{2800}{9}\zeta_7\right]\\
&\quad+C_A^2\left[\frac{166535}{486}-\frac{1783913}{324}\zeta_3+\frac{5555}{9}\zeta_4+\frac{507515}{81}\zeta_5-\frac{2035}{27}\zeta_3^2-\frac{5500}{27}\zeta_6-\frac{2765}{18}\zeta_7\right]\\
&\quad+n_fC_A\left[\frac{285985}{3888}+\frac{41954}{81}\zeta_3+\frac{160}{27}\zeta_3^2-\frac{1010}{9}\zeta_4-\frac{56480}{81}\zeta_5+\frac{1000}{27}\zeta_6\right]\\
&\quad+n_fC_F\left[\frac{1098323}{3888}-\frac{49720}{81}\zeta_3+\frac{3200}{9}\zeta_4\right]-n_f^2\left[\frac{21823}{1944}\right]\Bigg\}
\end{aligned}
$$

Scale stability of evolution



- Renormalization-scale dependence of $d \ln q_{\rm ns}^{\pm} / d \ln \mu_f^2$ at $N = 2$ and $N = 3$ using NLO, NNLO, N^{3}LO and N^{4}LO predictions with $\alpha_s(\mu_f) = 0.2$ and $n_f = 4$

Coefficient functions at four loops

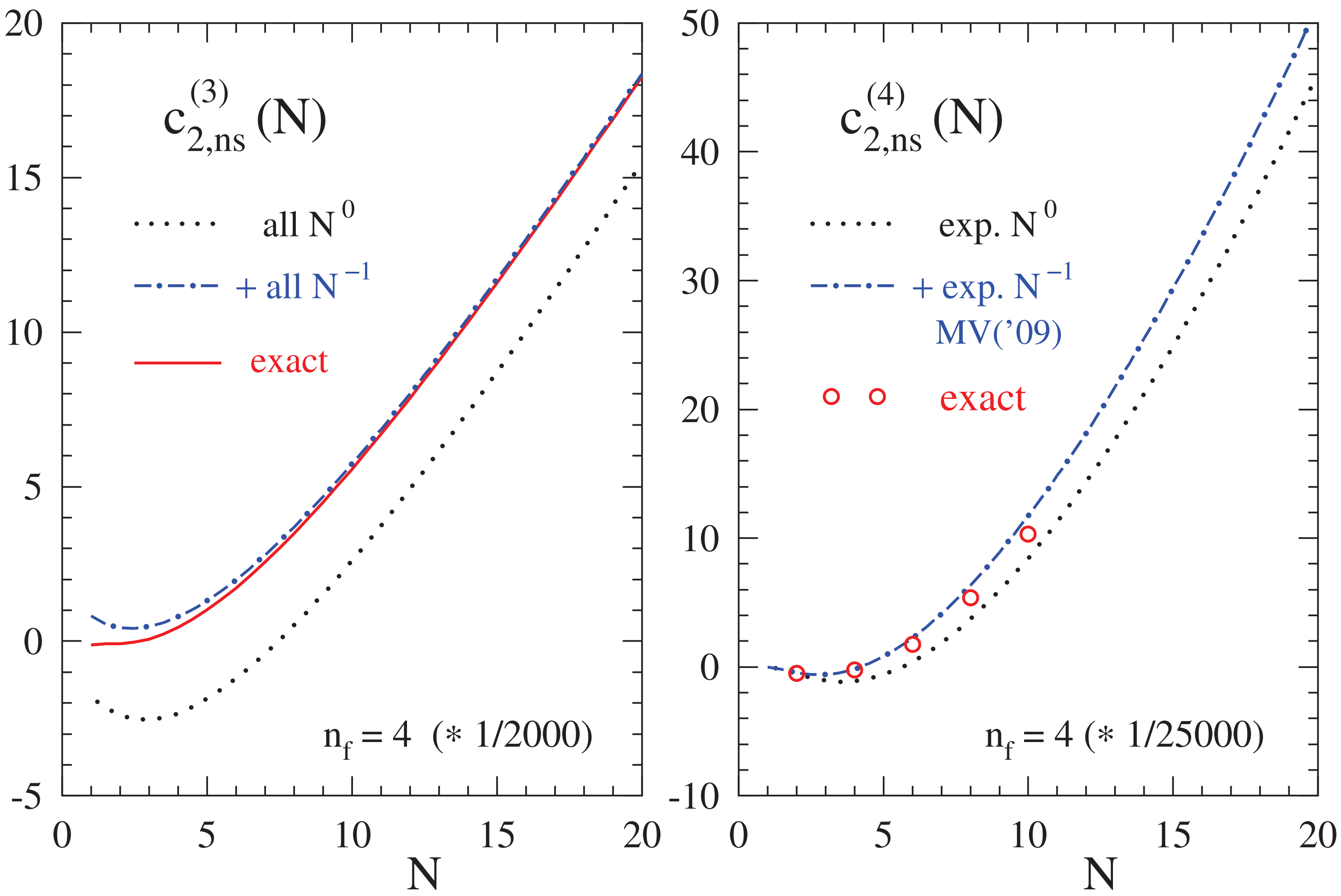
Four-loop non-singlet Mellin moments

- Perturbative expansion of non-singlet coefficient functions
  - Mellin moments $N = 2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 14$ of $C_{2,\mathrm{ns}}$ and $C_{L,\mathrm{ns}}$ (moments $N = 12,\ 14$ in limit of large n_c)
  - Mellin moments $N = 1,\ 3,\ 5,\ 7,\ 9,\ 11,\ 13,\ 15$ of $C_{3,-}$ (moments $N = 11,\ 13,\ 15$ in limit of large n_c)
- Numerical results for $C_{2,\mathrm{ns}}(N, n_f)$

S.M., Ruijl, Ueda, Vermaseren, Vogt *to appear*

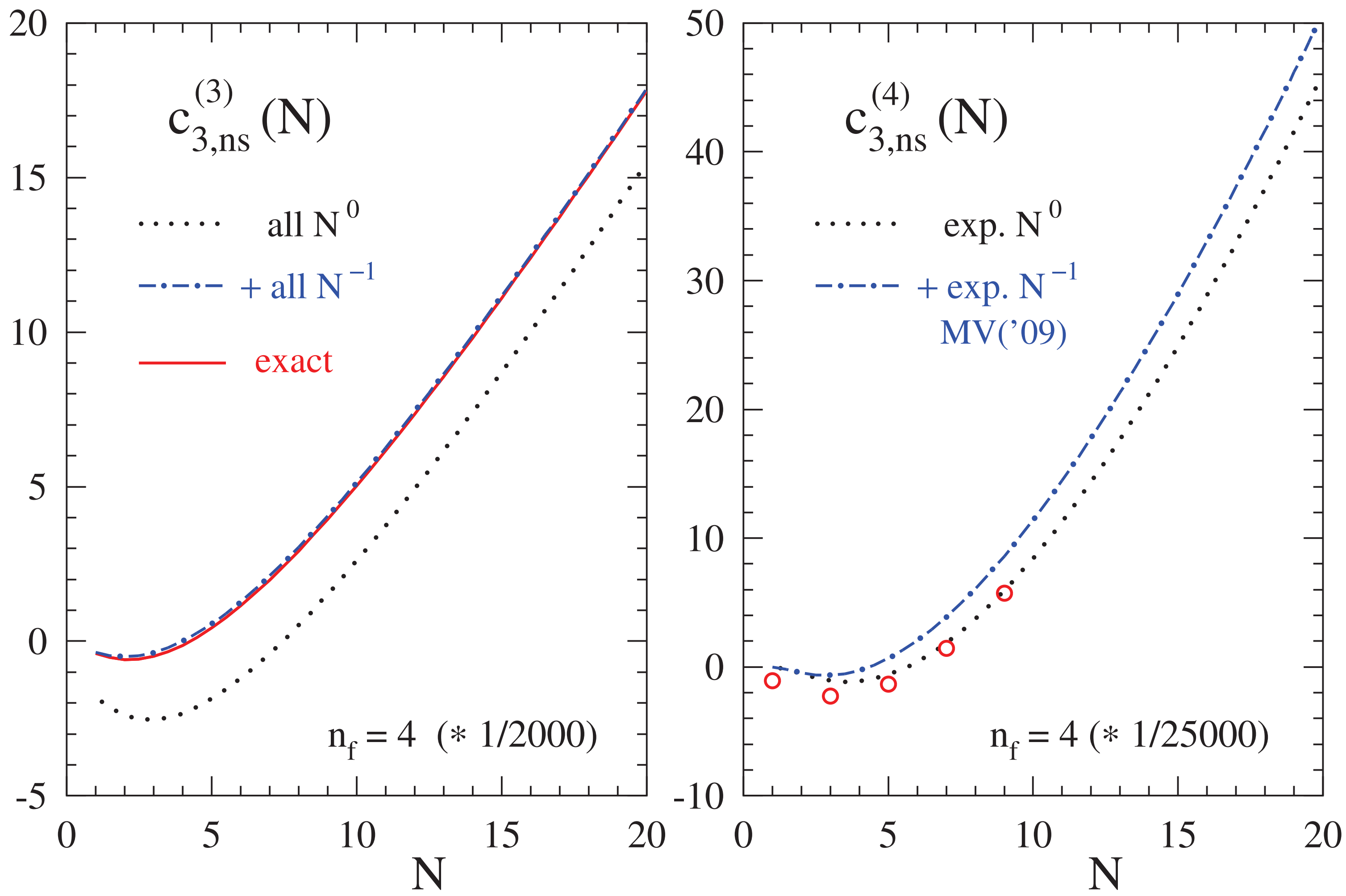
$$
\begin{aligned}
C_{2,\mathrm{ns}}(2,4) &= 1 + 0.0354\ \alpha_s - 0.0231\ \alpha_s^2 - 0.0613\ \alpha_s^3 - 0.4746\ \alpha_s^4\ , \\
C_{2,\mathrm{ns}}(4,4) &= 1 + 0.4828\ \alpha_s + 0.4711\ \alpha_s^2 + 0.4727\ \alpha_s^3 - 0.2458\ \alpha_s^4\ , \\
C_{2,\mathrm{ns}}(6,4) &= 1 + 0.8894\ \alpha_s + 1.2054\ \alpha_s^2 + 1.7572\ \alpha_s^3 + 1.7748\ \alpha_s^4\ , \\
C_{2,\mathrm{ns}}(8,4) &= 1 + 1.2358\ \alpha_s + 2.0208\ \alpha_s^2 + 3.5294\ \alpha_s^3 + 5.3921\ \alpha_s^4\ , \\
C_{2,\mathrm{ns}}(10,4) &= 1 + 1.5359\ \alpha_s + 2.8608\ \alpha_s^2 + 5.6244\ \alpha_s^3 + 10.324\ \alpha_s^4\ .
\end{aligned}
$$

Four-loop non-singlet Mellin moments



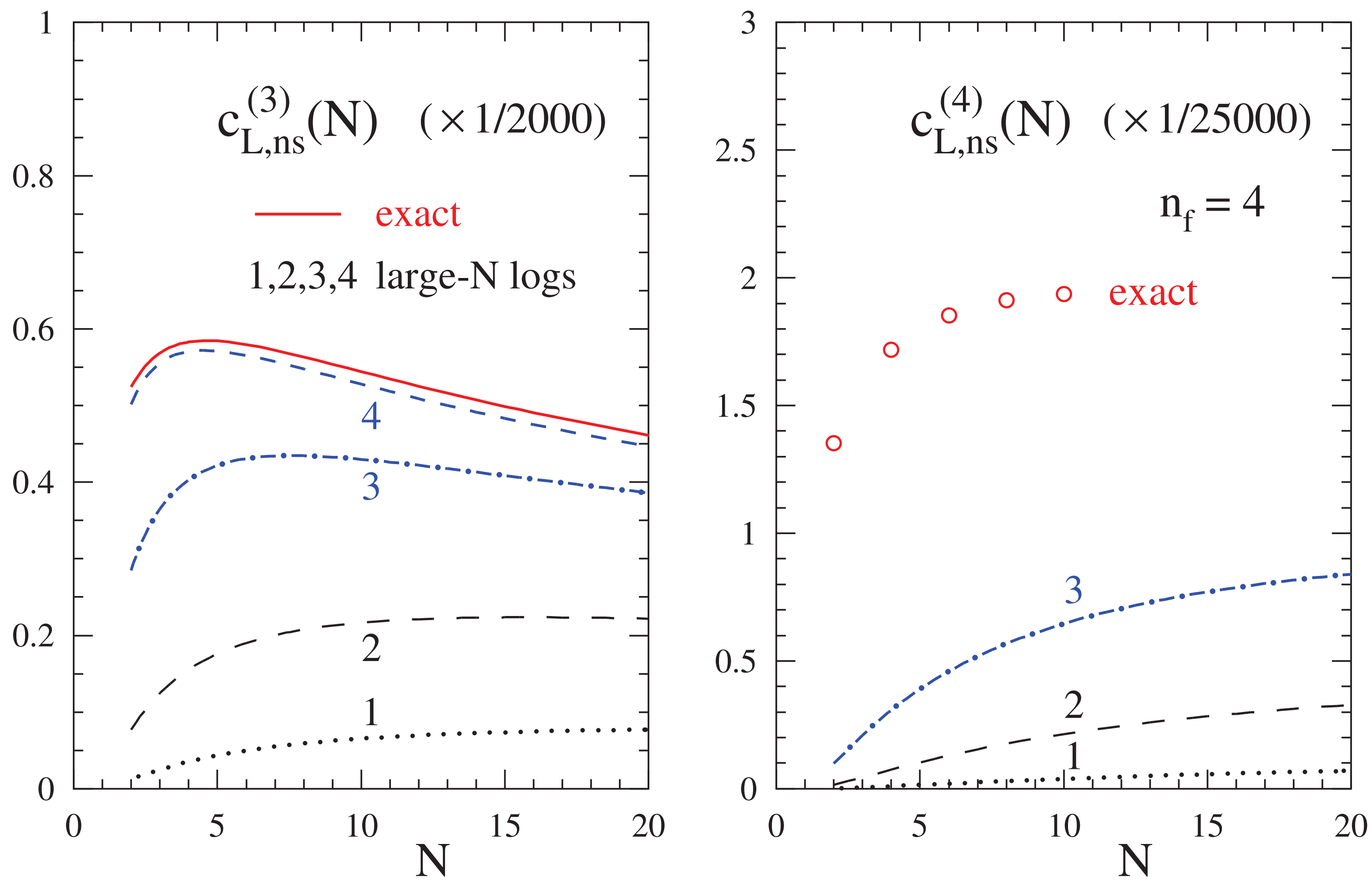
- Exact results for $c^{(3)}_{2,\mathrm{ns}}$ (N^3LO) at $n_f = 4$ (rescaled by $2000 \simeq (4\pi)^3$)
- Moments for $c^{(4)}_{2,\mathrm{ns}}$ (N^4LO) at $n_f = 4$ (rescaled by $25000 \simeq (4\pi)^4$)
- Comparison with contributions provided by large-N resummations

Four-loop non-singlet Mellin moments



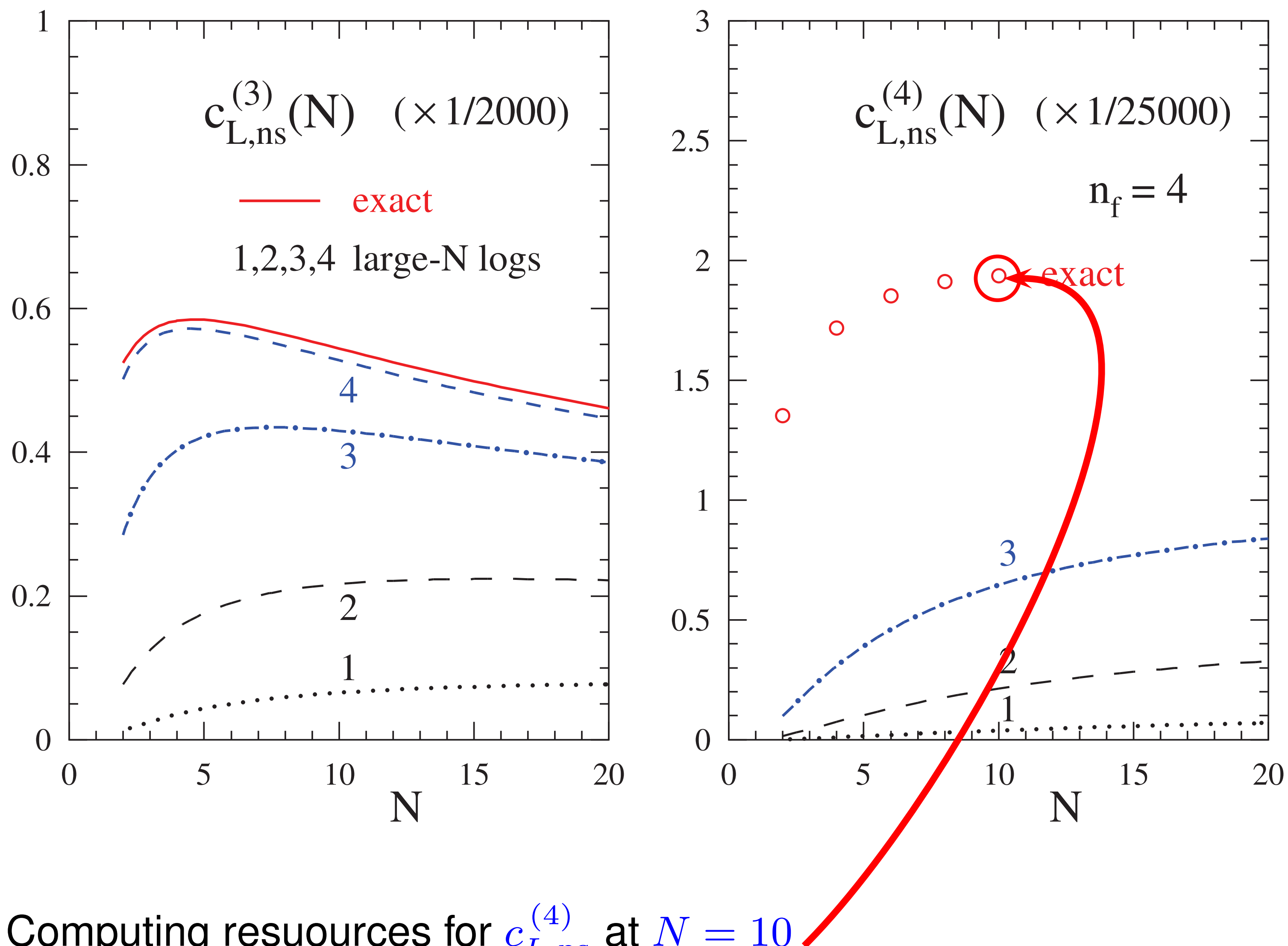
- Exact results for $c^{(3)}_{3,ns}$ (N^3LO) at $n_f = 4$ (rescaled by $2000 \simeq (4\pi)^3$)
- Moments for $c^{(4)}_{3,ns}$ (N^4LO) at $n_f = 4$ (rescaled by $25000 \simeq (4\pi)^4$)
- Comparison with contributions provided by large-N resummations

Four-loop non-singlet Mellin moments



- Exact results for $c^{(3)}_{L,\mathrm{ns}}$ ($\mathrm{N^3LO}$) and moments for $c^{(4)}_{3,\mathrm{ns}}$ ($\mathrm{N^4LO}$) at $n_f = 4$
- Tower of logarithms $\ln^4(N)/N$, . . . , $\ln(N)/N$ at $\mathrm{N^3LO}$
- Tower of logarithms $\ln^6(N)/N$, . . . , $\ln^4(N)/N$ at $\mathrm{N^4LO}$

Four-loop non-singlet Mellin moments



- Computing resuources for $c^{(4)}_{L,ns}$ at $N = 10$
  - single core CPU time $\mathcal{O}(800.000)$h (**TForm** speed-up is $\mathcal{O}(10)$h)
  - $\mathcal{O}(20)$ TByte of disk space at intermediate stages of computation

Threshold resummation

- Coefficient function in large x-limit have large logarithms at n$^{\rm th}$-order

$$\alpha_s^n \, \frac{\ln^{2n-1}(1-x)}{(1-x)_+} \longleftrightarrow \alpha_s^n \, \ln^{2n}(N)$$

- Threshold resummation in Mellin space

$$C^N = (1 + \alpha_s \, g_{01} + \alpha_s^2 \, g_{02} + \ldots) \cdot \exp(G^N) + \mathcal{O}(N^{-1} \ln^n N)$$

- Control over logarithms $\ln(N)$ with $\lambda = \beta_0 \alpha_s \ln(N)$ to N^{k}LL accuracy

$$G^N = \ln(N) g_1(\lambda) + g_2(\lambda) + \alpha_s g_3(\lambda) + \alpha_s^2 g_4(\lambda) + \alpha_s^3 g_5(\lambda) + \ldots$$

  - $g_1(\lambda)$: LL Sterman '87; Appell, Mackenzie, Sterman '88
  - $g_2(\lambda)$: NLL Catani Trenatdue '89
  - $g_3(\lambda)$: NNLL or N^{2}LL Vogt '00; Catani, Grazzini, de Florian, Nason '03
  - $g_4(\lambda)$: N^{3}LL S.M., Vermaseren, Vogt '05
  - $g_5(\lambda)$: N^{4}LL Das, S.M., Vogt '19

- Resummed G^N predicts fixed orders in perturbation theory
  - generating functional for towers of large logarithms

DIS coefficient functions at four loops

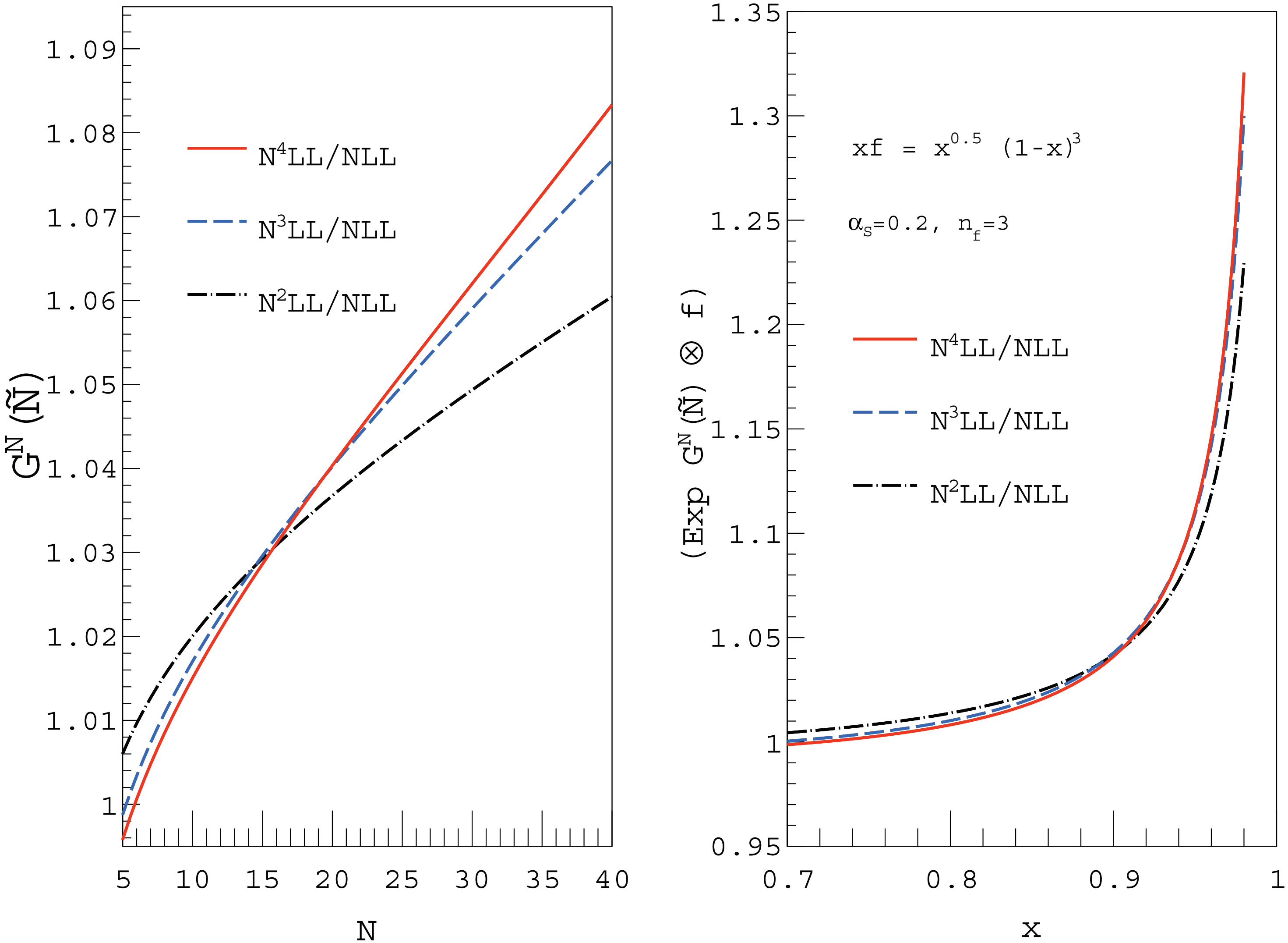
Result

- Four-loop coefficient function $c^{(4)}_{2,\mathrm{q}}$ known $\dfrac{\ln^7(1-x)}{(1-x)_+}, \ldots, \dfrac{1}{(1-x)_+}$
- New result for $\dfrac{1}{(1-x)_+}$ term
  - best estimate (using partial large-n_c information)

$$
\begin{aligned}
c^{(4)}_{2,\mathrm{q}}\Big|_{\frac{1}{(1-x)_+}} \quad &= \quad (3.874 \pm 0.010) \cdot 10^4 + (-3.496490 \pm 0.000003) \cdot 10^4\, n_f \\
&\quad +2062.715\, n_f^2 - 12.08488\, n_f^3 + 47.55183\, n_f\, fl_{11}
\end{aligned}
$$

- Based on results for
  - Quark and gluon form factors at four loops in QCD
    Lee, von Manteuffel, Schabinger, Smirnov, Smirnov, Steinhauser '22
  - eikonal anomalous dimensions Dixon, Magnea, Sterman '08
  - Mellin moments of DIS structure functions at four loops

Numerical results for DIS

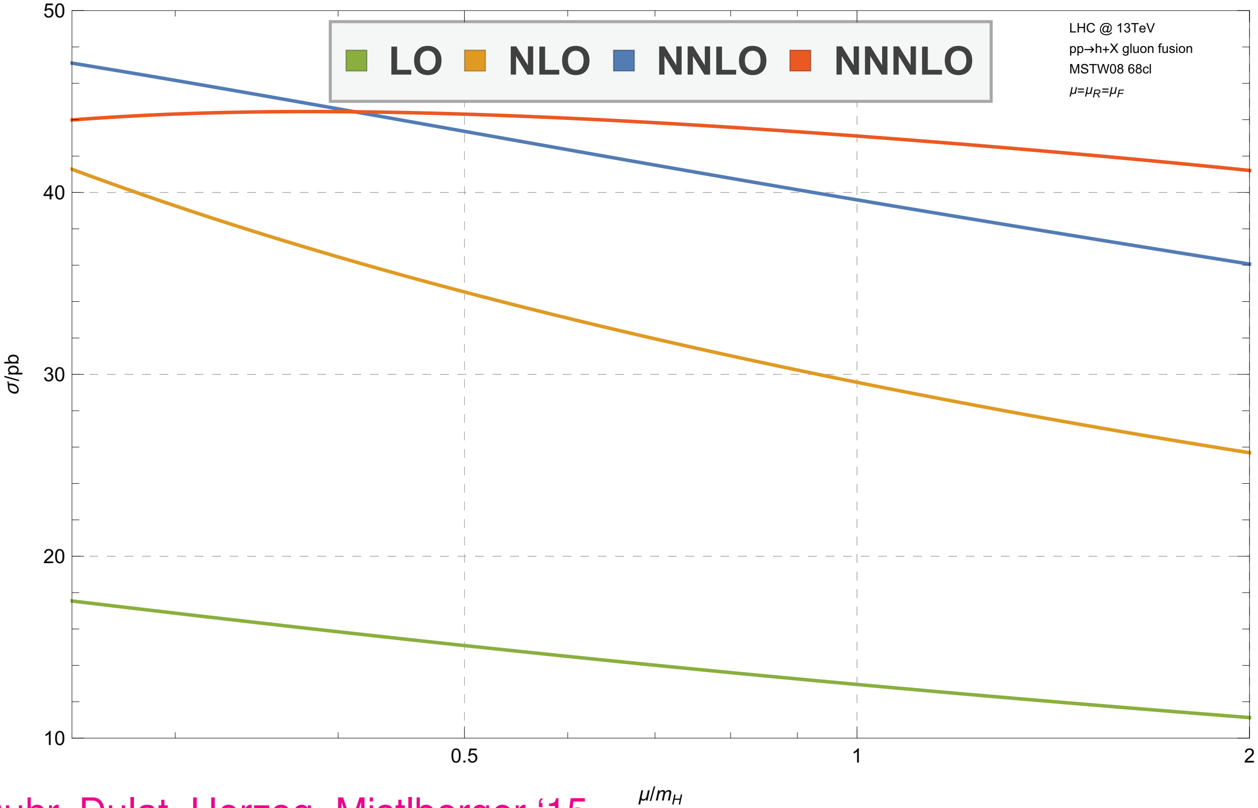


- Left: Resummed exponent G^N normalized to NLL for DIS plotted successively up to N^{4}LL for $\alpha_s = 0.2$ and $n_f = 3$
- Right: Resummed series convoluted with typical shape for a quark distribution $xf = x^{0.5}(1-x)^3$ up to N^{4}LL

Higgs boson production

Higgs cross section in gluon-gluon fusion

Exact N^3LO QCD corrections

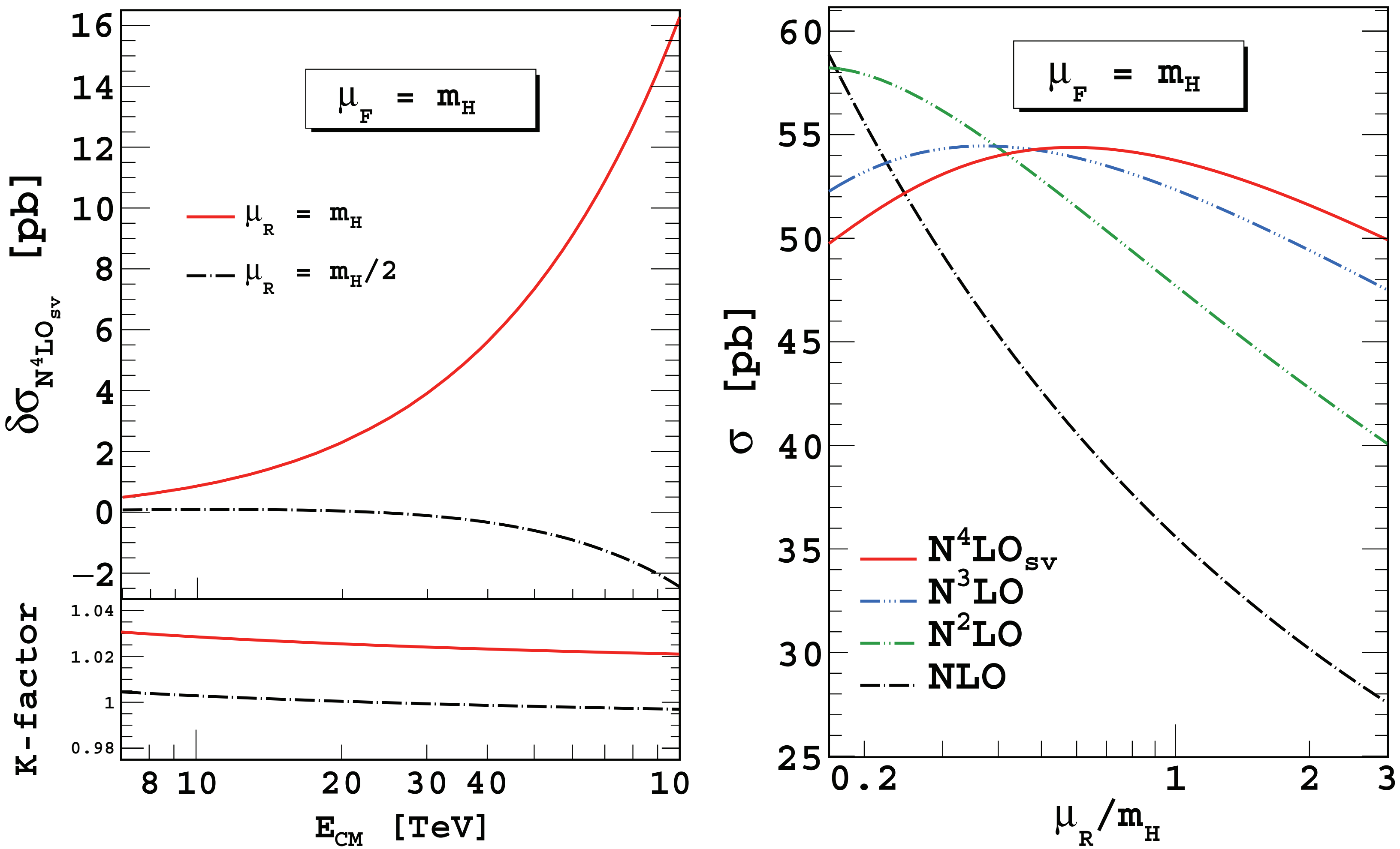


Anastasiou, Duhr, Dulat, Herzog, Mistlberger '15

- Apparent convergence of perturbative expansion
- Scale dependence of exact N^3LO prediction with residual uncertainty 3%
- Minimal sensitivity at scale $\mu = m_H/2$

Approximate N^4LO QCD corrections

Das, S.M., Vogt '20



- Left: Consistency check with approximate N^4LO corrections at two scales $\mu = m_H$ and $\mu = m_H/2$ as function of $\sqrt{s}$
- Right: μ_R scale dependence at approximate N^4LO for $\sqrt{s} = 14$ TeV

Summary

- Experimental precision of $\lesssim 1\%$ motivates computations at higher order in perturbative QCD
  - theoretical predictions at NNLO in QCD nowadays standard
- Push for theory results at N^{3}LO and N^{4}LO
  - evolutions equations and inclusive cross sections
  - massive use of computer algebra
- Precision studies of hadron structure
  - great prospects for DIS at future colliders