



Decomposition of Triple Collinear Splitting Functions

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Introduction

Splitting Functions

New Basis

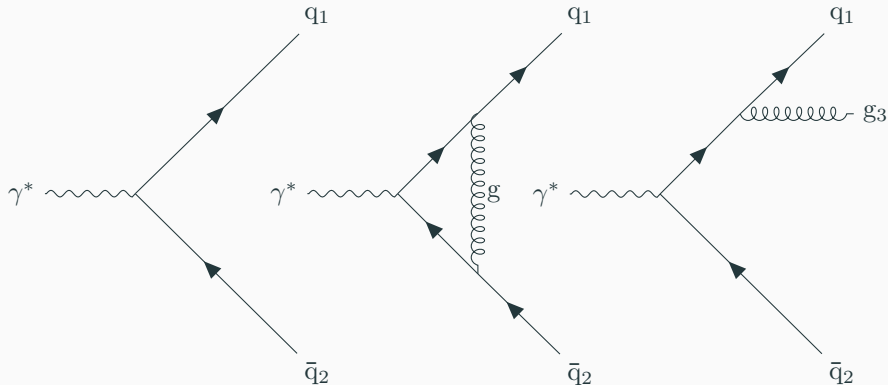
Results

Conclusions

Introduction

- Theoretical predictions of QCD observables need to match experimental precision.

$$d\hat{\sigma} = \left(\frac{\alpha_s}{2\pi}\right)^m d\hat{\sigma}^{\text{LO}} + \left(\frac{\alpha_s}{2\pi}\right)^{m+1} d\hat{\sigma}^{\text{NLO}} + \left(\frac{\alpha_s}{2\pi}\right)^{m+2} d\hat{\sigma}^{\text{NNLO}} + \mathcal{O}(\alpha_s^{m+3}) \quad (1)$$



- Infrared divergences must cancel out at each order correction by KLN theorem.
 - $d\hat{\sigma}^{\text{NLO}} = R + V$
 - $d\hat{\sigma}^{\text{NNLO}} = RR + RV + VV$
 - $d\hat{\sigma}^{\text{N3LO}} = RRR + RRV + RVV + VVV$
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 - $d\hat{\sigma}^{\text{N3LO}} = RRR + RRV + RVV + VVV$
- Subtraction or slicing scheme needed for higher order QCD calculations
- NNLOJET group uses antenna functions
- Innate structure of antennae needs attention to extend antenna subtraction to N3LO and beyond.

Splitting Functions

- Triple collinear splitting functions extracted from $|\mathcal{A}_4^0(i, j, k, l)|^2$:
- $|\mathcal{A}_4^0(i, j, k, l)|^2 \rightarrow P_{abc}(i, j, k)|\mathcal{A}_2^0(I, l)|^2$, where $p_I = p_i + p_j + p_k$
 - In the limit where i, j, k are collinear, the amplitude becomes the relevant splitting function multiplied by the amplitude the combined particle (ijk) and particle l .

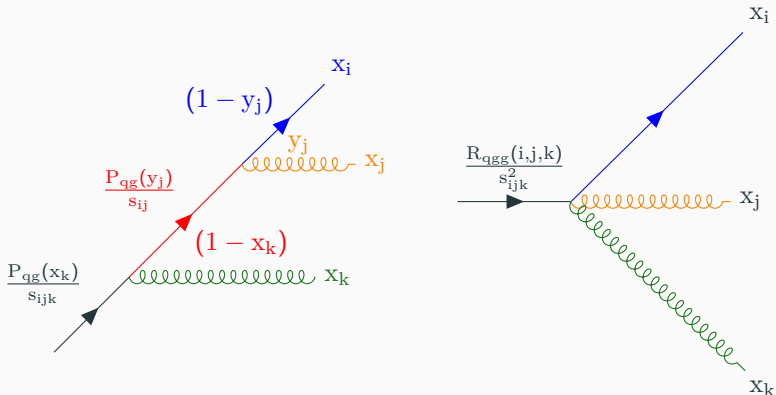
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 - In the limit where i, j, k are collinear, the amplitude becomes the relevant splitting function multiplied by the amplitude the combined particle (ijk) and particle l .
- $P_{abc}(i, j, k) \equiv P_{abc}(x_i, x_j, x_k; s_{ij}, s_{ik}, s_{jk}, s_{ijk})$
- $s_{i_1 \dots i_n} \equiv (p_{i_1} + \dots + p_{i_n})^2$
- $s_{ij} = 2p_i \cdot p_j = 2E_i E_j (1 - \cos \theta_{ij})$
 - s_{ij} is small when i, j are collinear or at least one is soft

Decomposition of Triple Collinear Splitting Functions

Idea: Can you decompose triple collinear splitting functions into a strongly-ordered iterated collinear splitting and a remainder which is finite when any two of $\{i, j, k\}$ are collinear?

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$$p_i + p_j + p_k = (x_i + x_j + x_k)p = p$$

$$x_j = (1 - x_k)y_j \text{ by momentum conservation.}$$

$$P_{abc}(i, j, k) = \sum \frac{P_{(ab)c}}{s_{ijk}} \frac{P_{ab}}{s_{ij}} + \frac{R_{abc}(i, j, k)}{s_{ijk}^2} \quad (2)$$

How to decide what terms in P_{abc} go in R_{abc} ?

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How to decide what terms in P_{abc} go in R_{abc} ?

A basis change!

New Basis

$$\begin{aligned}
& \frac{\beta_1}{s_{jk}s_{ijk}} + \frac{\beta_2}{s_{ij}s_{ijk}} + \frac{\beta_3}{s_{ik}s_{ijk}} + \frac{\beta_4}{s_{jk}^2} + \frac{\beta_5}{s_{ij}^2} + \frac{\beta_6}{s_{ik}^2} \\
& + \frac{\beta_7}{s_{jk}s_{ij}} + \frac{\beta_8}{s_{jk}s_{ik}} + \frac{\beta_9}{s_{ij}s_{ik}} + \frac{\beta_{10}s_{ij}}{s_{jk}^2s_{ijk}} + \frac{\beta_{11}s_{ij}}{s_{ik}^2s_{ijk}} + \frac{\beta_{12}s_{jk}}{s_{ij}^2s_{ijk}} \\
& + \frac{\beta_{13}}{s_{ijk}^2} + \frac{\beta_{14}s_{ij}}{s_{jk}s_{ijk}^2} + \frac{\beta_{15}s_{ij}}{s_{ik}s_{ijk}^2} + \frac{\beta_{16}s_{jk}}{s_{ij}s_{ijk}^2} + \frac{\beta_{17}s_{ij}^2}{s_{jk}^2s_{ijk}^2} + \frac{\beta_{18}s_{ij}^2}{s_{ik}^2s_{ijk}^2} + \frac{\beta_{19}s_{jk}^2}{s_{ij}^2s_{ijk}^2} \\
& + \left[\frac{\beta_{20}s_{jk}}{s_{ik}^2s_{ijk}} + \frac{\beta_{21}s_{jk}}{s_{ik}s_{ijk}^2} + \frac{\beta_{22}s_{jk}^2}{s_{ik}^2s_{ijk}^2} + \frac{\beta_{23}s_{ij}^2}{s_{ik}s_{jk}s_{ijk}^2} + \frac{\beta_{24}s_{ij}}{s_{ik}s_{jk}s_{ijk}} + \frac{\beta_{25}s_{jk}}{s_{ik}s_{jk}s_{ijk}} \right. \\
& + \frac{\beta_{26}s_{ik}}{s_{ij}^2s_{ijk}} + \frac{\beta_{27}s_{ik}}{s_{jk}^2s_{ijk}} + \frac{\beta_{28}s_{ik}}{s_{ij}s_{ijk}^2} + \frac{\beta_{29}s_{ik}}{s_{jk}s_{ijk}^2} + \frac{\beta_{30}s_{ik}^2}{s_{ij}^2s_{ijk}^2} + \frac{\beta_{31}s_{ik}^2}{s_{jk}^2s_{ijk}^2} + \frac{\beta_{32}s_{ik}}{s_{ij}s_{jk}s_{ijk}} \\
& \left. + \frac{\beta_{33}s_{ik}^2}{s_{ij}s_{jk}s_{ijk}^2} + \frac{\beta_{34}s_{jk}^2}{s_{ij}s_{ik}s_{ijk}^2} + \frac{\beta_{35}s_{ij}s_{ik}}{s_{jk}^2s_{ijk}^2} + \frac{\beta_{36}s_{ij}s_{jk}}{s_{ik}^2s_{ijk}^2} + \frac{\beta_{37}s_{ik}s_{jk}}{s_{ij}^2s_{ijk}^2} \right]
\end{aligned} \tag{3}$$

- Initially $P_{abc}(i, j, k)$ expressed as β_i
- Linear basis change to ' α ' basis which organises how terms contribute in the $\{i, j, k\}$ single collinear limits.

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- $\text{Tr}(\not{i}\not{j}\not{k}\not{l}) = (s_{kl}s_{ij} + s_{il}s_{jk} - s_{jl}s_{ik})$

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 - Special properties that $\text{Tr}(\not{i}\not{j}\not{k}\not{l})/s_{ij}$ is not singular in the i, j collinear limit.
 - When in the i, j, k collinear limit, we allow the momentum of particle l to be normalised so that $\text{Tr}(\not{i}\not{j}\not{k}\not{l}) = (x_k s_{ij} + x_i s_{jk} - x_j s_{ik})$

$$\frac{\alpha_{12}}{s_{jk}s_{ijk}} + \frac{\alpha_{13}}{s_{ij}s_{ijk}} + \frac{\alpha_{14}}{s_{ik}s_{ijk}} \quad (4a)$$

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$$+ \frac{\alpha_1}{s_{ijk}^2} + \frac{\alpha_2 \text{Tr}(i\bar{k}j\bar{l})}{s_{jk}s_{ijk}^2} + \frac{\alpha_3 \text{Tr}(i\bar{j}k\bar{l})}{s_{ij}s_{ijk}^2} + \frac{\alpha_4 \text{Tr}(j\bar{i}k\bar{l})}{s_{ik}s_{ijk}^2} \quad (4b)$$

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$$+ \frac{\alpha_{23} \text{Tr}(i\bar{j}k\bar{l})}{s_{ijk}s_{jk}s_{ij}} + \frac{\alpha_{24} \text{Tr}(i\bar{k}j\bar{l})}{s_{jk}s_{ik}s_{ijk}} + \frac{\alpha_{34} \text{Tr}(j\bar{i}k\bar{l})}{s_{ij}s_{ik}s_{ijk}} \quad (4c)$$

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$$+ \frac{\alpha_{22} W_{jk}}{s_{jk}^2 s_{ijk}^2} + \frac{\alpha_{33} W_{ij}}{s_{ij}^2 s_{ijk}^2} + \frac{\alpha_{44} W_{ik}}{s_{ik}^2 s_{ijk}^2} \quad (4d)$$

$\frac{(\text{Tr}(ijk\bar{l}))^2}{s_{jk}^2}$ contains some terms in the j, k collinear limit. This must be treated carefully.

$$\text{Tr}(ijk\bar{l}) = s_{il}s_{jk} + Y_{jk}$$

Expanding $Y_{jk} = s_{ij}s_{kl} - s_{ik}s_{jl}$ in the j, k collinear limit, we can show that $Y_{jk} = \mathcal{O}(\sqrt{s_{jk}})$.

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This is why we introduce W_{jk} rather than the above structure:

$$\frac{W_{jk}}{s_{jk}^2 s_{ijk}^2} = \frac{Y_{jk}^2}{s_{jk}^2 s_{ijk}^2} - \frac{2x_i x_j x_k}{(1 - \epsilon)(1 - x_i)} \frac{1}{s_{jk} s_{ijk}} \quad (5)$$

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Y_{jk} is a truncated trace and the Y_{jk}^2 term has the same single collinear structure as the second term.

Integrable singularity which vanishes upon azimuthal integration (in d dimensions).

Results

- All terms with two of $\{i, j, k\}$ collinear, $(\alpha_{12}, \alpha_{13}, \alpha_{14})$, appear as
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- In practice, due to colour ordering, many ' α ' coefficients are zero.
- All splitting functions describable with one Trace if $\{\gamma_\mu, \gamma_\nu\}$ used

Internal singularities involve only small invariants in $\{s_{ij}, s_{jk}, s_{ik}, s_{ijk}\}$.

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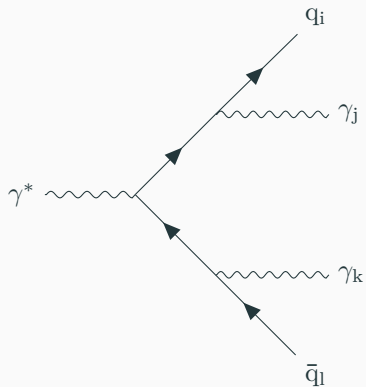
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- External single soft singularities like $x_i/x_j/s_{ij}$ appear only in the iterated $P \times P$ terms.

Double soft singularities are defined as external.

Example diagram:



$$\begin{aligned}
 P_{q\gamma\gamma}(i, j, k) &= \frac{P_{qg}(x_k)}{s_{ijk}} \frac{P_{qg}\left(\frac{x_j}{1-x_k}\right)}{s_{ij}} \\
 &+ \frac{1}{s_{ijk}^2} \left[\tilde{f}_0^{\text{sub}}(x_i, x_j, x_k) - (1-\epsilon)^2 \frac{\text{Tr}(j\tilde{i}k\tilde{l})}{(1-x_k)s_{ij}} \right. \\
 &\left. + \tilde{f}^{\text{sub}}(x_i, x_j, x_k) \frac{s_{ijk} \text{Tr}(j\tilde{i}k\tilde{l})}{s_{ij}s_{ik}} \right] + (j \leftrightarrow k),
 \end{aligned} \tag{6a}$$

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 &+ \frac{1}{s_{ijk}^2} \left[\tilde{f}_0^{\text{sub}}(x_i, x_j, x_k) - (1-\epsilon)^2 \frac{\text{Tr}(j\tilde{i}k\tilde{l})}{(1-x_k)s_{ij}} \right. \\
 &\left. + \tilde{f}^{\text{sub}}(x_i, x_j, x_k) \frac{s_{ijk} \text{Tr}(j\tilde{i}k\tilde{l})}{s_{ij}s_{ik}} \right] + (j \leftrightarrow k), \tag{6a}
 \end{aligned}$$

where $\tilde{f}_0(x_i, x_j, x_k)$ and $\tilde{f}(x_i, x_j, x_k)$ are given by

$$\tilde{f}_0^{\text{sub}}(x_i, x_j, x_k) = (1-\epsilon)(1-(1-\epsilon)A_0(x_j, x_k)), \tag{6b}$$

$$\tilde{f}^{\text{sub}}(x_i, x_j, x_k) = -\frac{x_k P_{qg}(x_k)}{x_j(1-x_i)} + \frac{2}{(1-x_i)} - 2(1-\epsilon) + \frac{1}{2}(1-\epsilon)^2. \tag{6c}$$

$$A_0(x, y) = 1 - \frac{(1-x)}{(1-y)}, \quad (7)$$

$$B_0(x, y) = 1 + \frac{2x(x-2)}{(1-y)^2} + \frac{4x}{(1-y)}. \quad (8)$$

A feature of basis change.

$x_I \rightarrow 0$ exposing external singularities

$q\gamma\gamma \rightarrow q$	$\frac{P_{qg}(x_k)}{s_{ijk}} \frac{P_{qg}\left(\frac{x_j}{1-x_k}\right)}{s_{ij}}$ + ($j \leftrightarrow k$)	$\frac{1}{s_{ijk}} R_{q\gamma\gamma \rightarrow q}(i, j, k)$	$\frac{1}{s_{ijk}} P_{q\gamma\gamma \rightarrow q}(i, j, k)$
$x_i \rightarrow 0$	0	0	0
$x_j \rightarrow 0$	$+\frac{1}{s_{ij}s_{ijk}} \frac{x_i}{x_j} \left[2P_{qg}(x_k) \right]$ $+\frac{1}{s_{ik}s_{ijk}} \frac{1}{x_j} \left[2P_{qg}(x_k) \right]$	$+\frac{1}{s_{ij}s_{ik}} \frac{x_i}{x_j} \left[P_{qg}(x_k) \right]$ $+\frac{1}{s_{ij}s_{ijk}} \frac{x_i}{x_j} \left[-P_{qg}(x_k) \right]$ $+\frac{1}{s_{ik}s_{ijk}} \frac{1}{x_j} \left[-P_{qg}(x_k) \right]$	$+\frac{1}{s_{ij}s_{ik}} \frac{x_i}{x_j} \left[P_{qg}(x_k) \right]$ $+\frac{1}{s_{ij}s_{ijk}} \frac{x_i}{x_j} \left[P_{qg}(x_k) \right]$ $+\frac{1}{s_{ik}s_{ijk}} \frac{1}{x_j} \left[P_{qg}(x_k) \right]$
$x_k \rightarrow 0$	$+\frac{1}{s_{ij}s_{ijk}} \frac{1}{x_k} \left[2P_{qg}(x_j) \right]$ $+\frac{1}{s_{ik}s_{ijk}} \frac{x_i}{x_k} \left[2P_{qg}(x_j) \right]$	$+\frac{1}{s_{ij}s_{ik}} \frac{x_i}{x_k} \left[P_{qg}(x_j) \right]$ $+\frac{1}{s_{ij}s_{ijk}} \frac{1}{x_k} \left[-P_{qg}(x_j) \right]$ $+\frac{1}{s_{ik}s_{ijk}} \frac{x_i}{x_k} \left[-P_{qg}(x_j) \right]$	$+\frac{1}{s_{ij}s_{ik}} \frac{x_i}{x_k} \left[P_{qg}(x_j) \right]$ $+\frac{1}{s_{ij}s_{ijk}} \frac{1}{x_k} \left[P_{qg}(x_j) \right]$ $+\frac{1}{s_{ik}s_{ijk}} \frac{x_i}{x_k} \left[P_{qg}(x_j) \right]$

Table 1: Singular behaviour of the $P_{q\gamma\gamma \rightarrow q}$ triple collinear splitting function in the limit where individual momentum fractions are small.

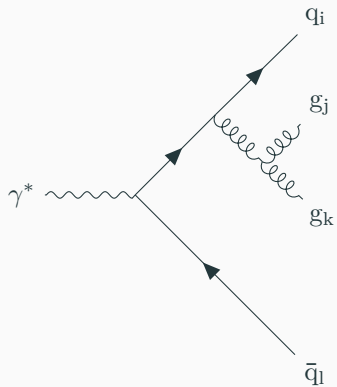
$$P_{q\gamma\gamma\rightarrow q}(i, j, k) \xrightarrow{j \text{ soft}} \frac{2x_i}{s_{ij}x_j} \frac{1}{s_{ik}} P_{qg}(x_k), \quad (9)$$

External soft singularity.

All comes from $P \times P$ terms. $R_{q\gamma\gamma\rightarrow q}$ terms cancel out in soft j limit.

Soft k limit is the same with $j \leftrightarrow k$.

Example diagram:



$$\begin{aligned}
 P_{\text{qgg}}(i, j, k) &= \frac{P_{\text{qg}}(x_k)}{s_{ijk}} \frac{P_{\text{qg}}\left(\frac{x_j}{1-x_k}\right)}{s_{ij}} + \frac{P_{\text{qg}}(1-x_i)}{s_{ijk}} \frac{P_{\text{gg}}\left(\frac{x_j}{1-x_i}\right)}{s_{jk}} \\
 &+ \frac{1}{s_{ijk}^2} \left[\frac{W_{jk}}{s_{jk}^2} \frac{2(1-\epsilon)}{(1-x_i)^2} + \frac{\text{Tr}(ijk\bar{l})}{s_{jk}} \frac{4(1-\epsilon)x_k}{(1-x_i)^2} + \frac{\text{Tr}(ijk\bar{l})}{s_{ij}} \frac{(1-\epsilon)^2}{1-x_k} \right. \\
 &\left. + f_0(x_i, x_j, x_k) + f(x_i, x_j, x_k) \frac{s_{ijk} \text{Tr}(ijk\bar{l})}{s_{ij}s_{jk}} \right],
 \end{aligned}
 \tag{10}$$

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 P_{\text{qgg}}(i, j, k) &= \frac{P_{\text{qg}}(x_k)}{s_{ijk}} \frac{P_{\text{qg}}\left(\frac{x_j}{1-x_k}\right)}{s_{ij}} + \frac{P_{\text{qg}}(1-x_i)}{s_{ijk}} \frac{P_{\text{gg}}\left(\frac{x_j}{1-x_i}\right)}{s_{jk}} \\
 &+ \frac{1}{s_{ijk}^2} \left[\frac{W_{jk}}{s_{jk}^2} \frac{2(1-\epsilon)}{(1-x_i)^2} + \frac{\text{Tr}(ijk\bar{l})}{s_{jk}} \frac{4(1-\epsilon)x_k}{(1-x_i)^2} + \frac{\text{Tr}(ijk\bar{l})}{s_{ij}} \frac{(1-\epsilon)^2}{1-x_k} \right. \\
 &\left. + f_0(x_i, x_j, x_k) + f(x_i, x_j, x_k) \frac{s_{ijk} \text{Tr}(ijk\bar{l})}{s_{ij}s_{jk}} \right],
 \end{aligned} \tag{10}$$

$$f_0(x_i, x_j, x_k) = (1-\epsilon)(B_0(x_k, x_i) - 1 + (1-\epsilon)A_0(x_i, x_k)),$$

$$f(x_i, x_j, x_k) = -\frac{x_j P_{\text{qg}}(x_j)}{x_k(1-x_i)} - \frac{2x_k P_{\text{qg}}(x_k)}{x_j(1-x_i)} + \frac{4}{(1-x_i)} - 3(1-\epsilon).$$

$qgg \rightarrow q$	$\frac{P_{qg}(x_k)}{s_{ijk}} \frac{P_{qg}\left(\frac{x_j}{1-x_k}\right)}{s_{ij}}$ $+ \frac{P_{qg}(1-x_i)}{s_{ijk}} \frac{P_{qg}\left(\frac{x_j}{1-x_i}\right)}{s_{jk}}$	$\frac{1}{s_{ijk}^2} R_{qgg \rightarrow q}(i, j, k)$	$\frac{1}{s_{ijk}^2} P_{qgg \rightarrow q}(i, j, k)$
$x_i \rightarrow 0$	0	0	0
$x_j \rightarrow 0$	$+ \frac{1}{s_{ij}s_{ijk}} \frac{x_i}{x_j} \left[2P_{qg}(x_k) \right]$ $+ \frac{1}{s_{jk}s_{ijk}} \frac{x_k}{x_j} \left[2P_{qg}(x_k) \right]$	$+ \frac{1}{s_{ij}s_{ijk}} \frac{x_i}{x_j} \left[-2P_{qg}(x_k) \right]$ $+ \frac{1}{s_{jk}s_{ijk}} \frac{x_k}{x_j} \left[-2P_{qg}(x_k) \right]$	0
$x_k \rightarrow 0$	$+ \frac{1}{s_{ij}s_{ijk}} \frac{1}{x_k} \left[2P_{qg}(x_j) \right]$ $+ \frac{1}{s_{jk}s_{ijk}} \frac{x_j}{x_k} \left[2P_{qg}(x_j) \right]$	$+ \frac{1}{s_{ij}s_{jk}} \frac{x_j}{x_k} \left[P_{qg}(x_j) \right]$ $+ \frac{1}{s_{ij}s_{ijk}} \frac{1}{x_k} \left[-P_{qg}(x_j) \right]$ $+ \frac{1}{s_{jk}s_{ijk}} \frac{x_j}{x_k} \left[-P_{qg}(x_j) \right]$	$+ \frac{1}{s_{ij}s_{jk}} \frac{x_j}{x_k} \left[P_{qg}(x_j) \right]$ $+ \frac{1}{s_{ij}s_{ijk}} \frac{1}{x_k} \left[P_{qg}(x_j) \right]$ $+ \frac{1}{s_{jk}s_{ijk}} \frac{x_j}{x_k} \left[P_{qg}(x_j) \right]$

Table 2: Singular behaviour of the $P_{qgg \rightarrow q}$ triple collinear splitting function in the limit where individual momentum fractions are small.

$$P_{qgg \rightarrow q}(i, j, k) \xrightarrow{k \text{ soft}} \frac{2x_j}{s_{jk}x_k} \frac{1}{s_{ij}} P_{qg}(x_j). \quad (11)$$

External soft singularity. All from $P \times P$. Like in $P_{q\gamma\gamma \rightarrow q}$ case.

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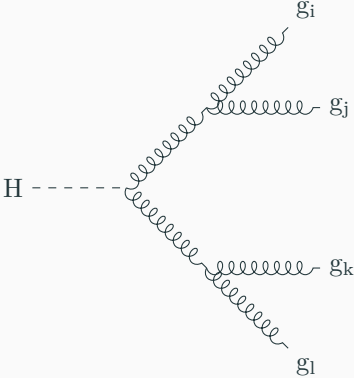
External soft singularity. All from $P \times P$. Like in $P_{q\gamma\gamma \rightarrow q}$ case.

However, in the soft j limit the $1/x_j/s_{ij}$ and $1/x_j/s_{jk}$ terms cancel between the $P \times P$ and $R_{qgg \rightarrow q}$ contributions.

$$P_{qgg \rightarrow q}(i, j, k) \xrightarrow{j \text{ soft}} \frac{2s_{ik}}{s_{ij}s_{jk}} \frac{1}{s_{ik}} P_{qg}(x_k), \quad (12)$$

Internal soft singularity. All from $R_{qgg \rightarrow q}$. Inherently un-iterated.

Example diagram:



$$\begin{aligned}
 P_{ggg}(i, j, k) &= \frac{P_{gg}(1 - x_i)}{s_{ijk}} \frac{P_{gg}\left(\frac{x_j}{1 - x_i}\right)}{s_{jk}} + \frac{1}{s_{ijk}^2} \left[\frac{2(1 - \epsilon)W_{jk}}{(1 - x_i)^2 s_{jk}^2} \right. \\
 &+ g_0(x_i, x_j, x_k) + \frac{4(1 - \epsilon)x_k}{(1 - x_i)^2} \frac{\text{Tr}(ijk\ell)}{s_{jk}} + g(x_i, x_j, x_k) \frac{s_{ijk} \text{Tr}(ijk\ell)}{s_{ij}s_{jk}} \left. \right] \\
 &+ (i \leftrightarrow k),
 \end{aligned} \tag{13}$$

$$\begin{aligned}
 P_{ggg}(i, j, k) &= \frac{P_{gg}(1-x_i)}{s_{ijk}} \frac{P_{gg}\left(\frac{x_j}{1-x_i}\right)}{s_{jk}} + \frac{1}{s_{ijk}^2} \left[\frac{2(1-\epsilon)W_{jk}}{(1-x_i)^2 s_{jk}^2} \right. \\
 &+ g_0(x_i, x_j, x_k) + \frac{4(1-\epsilon)x_k}{(1-x_i)^2} \frac{\text{Tr}(ijk\ell)}{s_{jk}} + g(x_i, x_j, x_k) \frac{s_{ijk} \text{Tr}(ijk\ell)}{s_{ij}s_{jk}} \left. \right] \\
 &+ (i \leftrightarrow k),
 \end{aligned} \tag{13}$$

$$g_0(x_i, x_j, x_k) = (1-\epsilon)B_0(x_k, x_i),$$

$$g(x_i, x_j, x_k) = -\frac{x_k P_{gg}(x_k)}{x_j(1-x_i)} - \frac{P_{gg}(x_j)}{x_k} + \frac{2}{x_j(1-x_k)} - 1 - \frac{1}{(1-x_i)(1-x_k)}.$$

Conclusions

- We have decomposed all triple collinear splitting functions into strongly-ordered terms and a remainder in one trace
- All internal and external singularities exposed in paper
- The strongly-ordered terms take a form of $\sum P_{(ab)c} P_{ab}$ as predicted

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Thank you very much! Questions?

More detail in arXiv:2204.10755

1 \rightarrow 2 Splitting Functions

$$P_{qg}(x) = P_{gq}(1-x) = \frac{2(1-x)}{x} + (1-\epsilon)x = \frac{1+(1-x)^2}{x} - \epsilon x, \quad (14)$$

$$P_{gg}(x) = \frac{2(1-x)}{x} + \frac{2x}{1-x} + 2x(1-x) = \frac{1+x^4+(1-x)^4}{x(1-x)}, \quad (15)$$

$$P_{q\bar{q}}(x) = P_{\bar{q}q}(x) = 1 - \frac{2x(1-x)}{1-\epsilon} = \frac{x^2+(1-x)^2-\epsilon}{1-\epsilon} \quad (16)$$

The soft gluon contribution is the $2(1-x)/x$ term in Eqs. (14,15) and the azimuthal contribution has the $2x(1-x)$ structure in Eqs. (15,16).