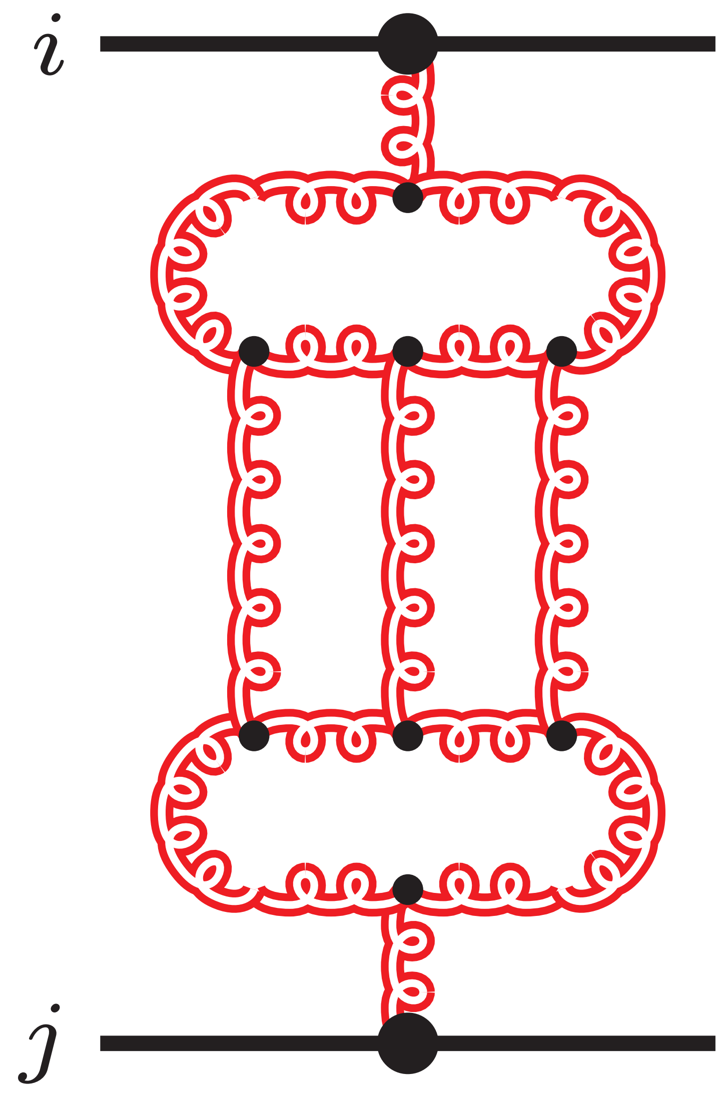
High energy limit of $2 \to 2$ scattering amplitudes at NNLL

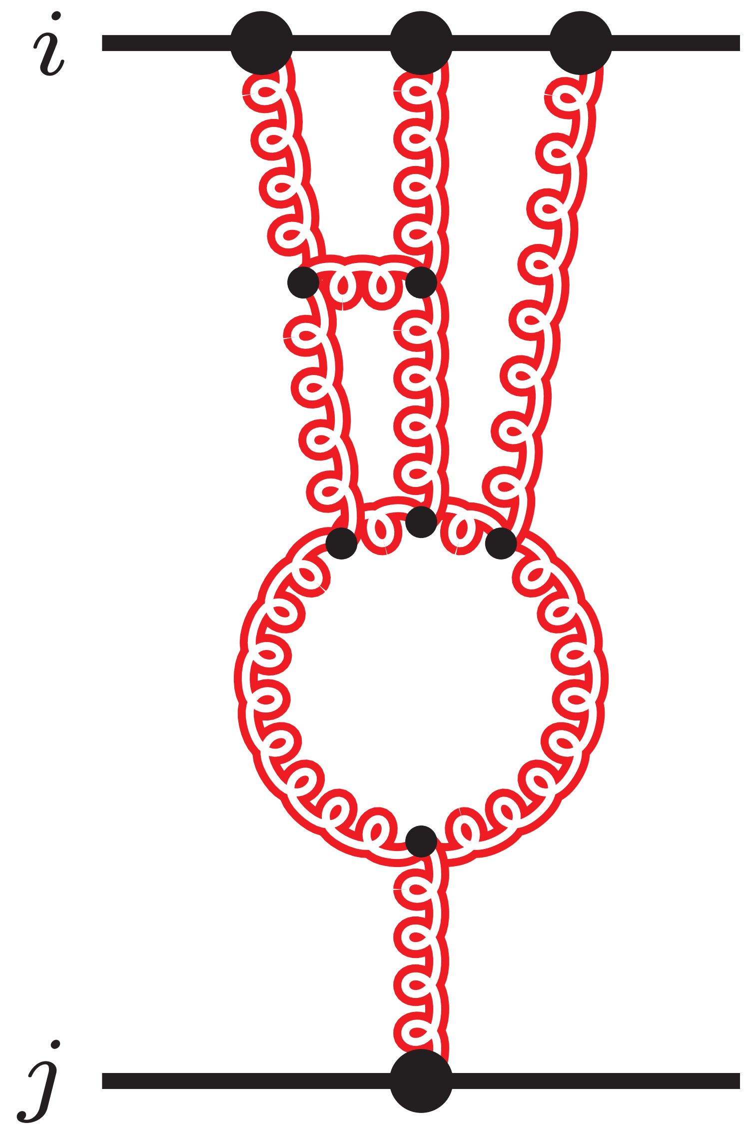


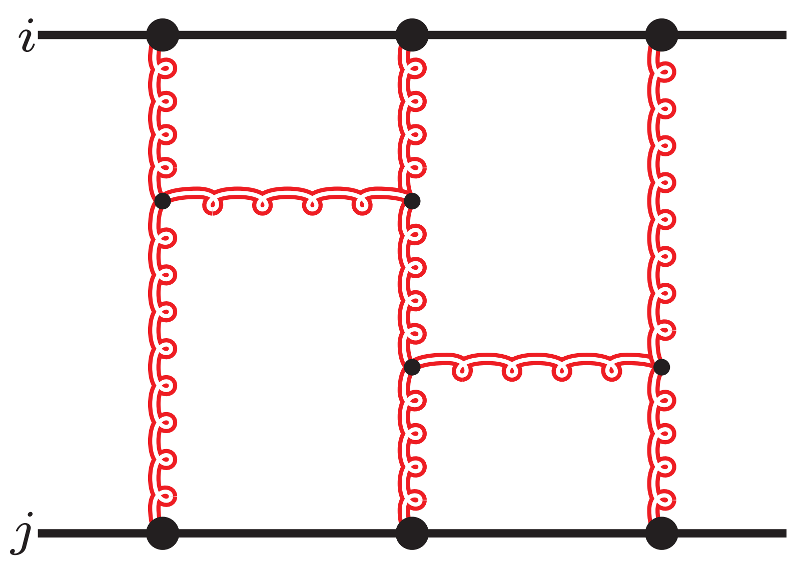
Calum Milloy

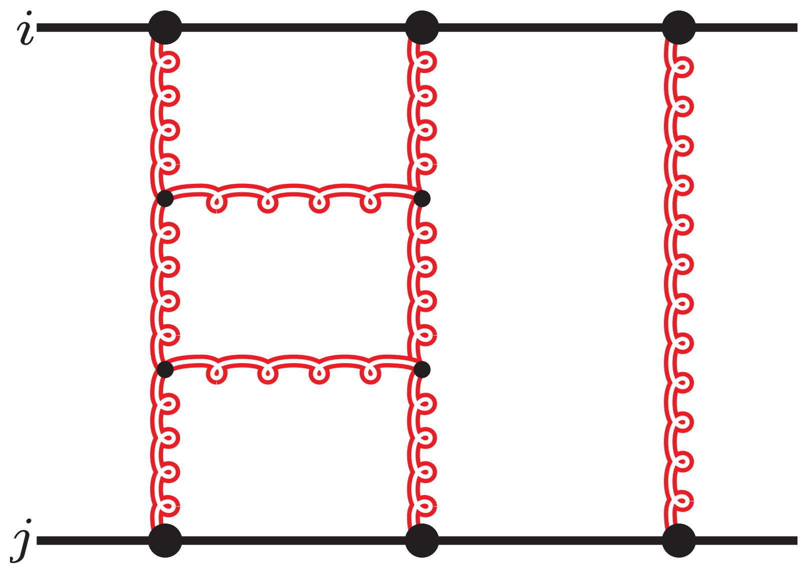
[PRD 103 (2021), 2012.00613] w/ Giulio Falcioni, Einan Gardi and Leonardo Vernazza

[JHEP 03 (2022) 053, 2111.10664] & [PRL 128 (2022) 13, 2112.11098]
w/ Giulio Falcioni, Einan Gardi, Niamh Maher and Leonardo Vernazza

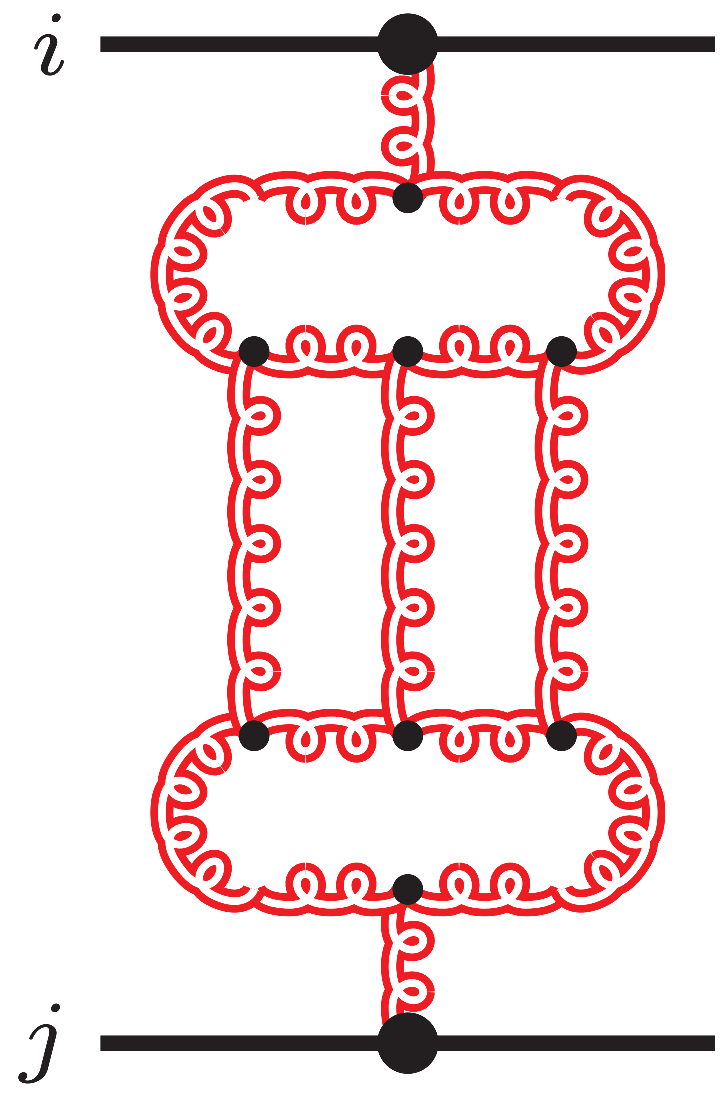
Arnold-Regge Center, Università di Torino, and INFN

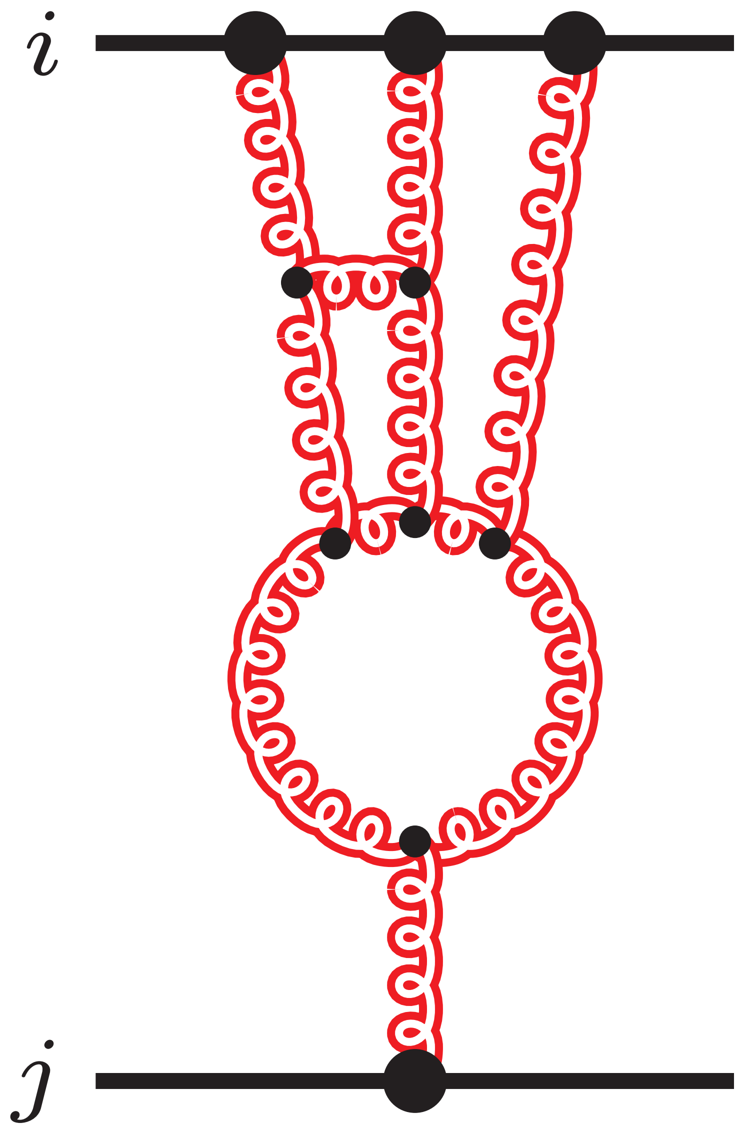






Disentangling the Regge pole and Regge cut in perturbative QCD





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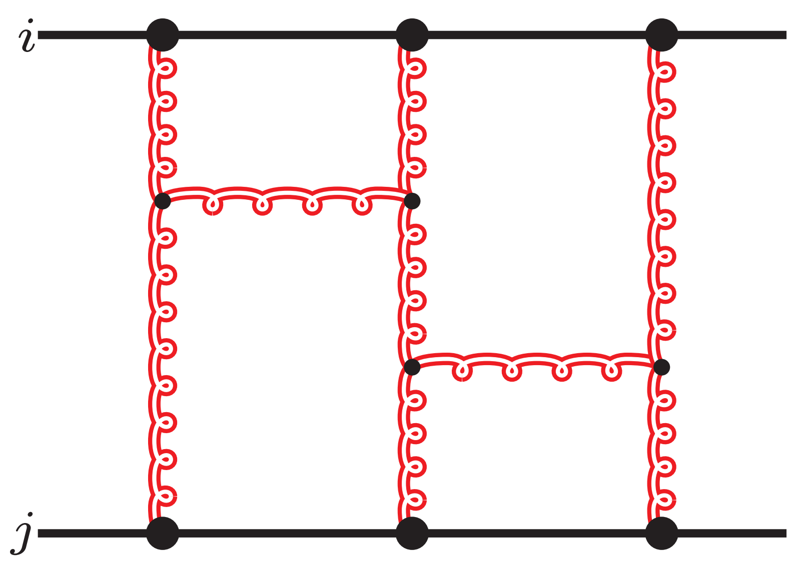
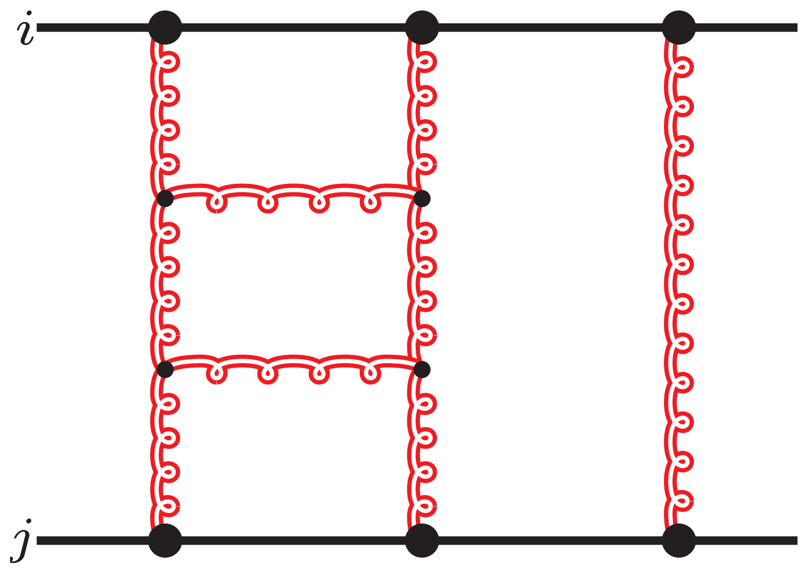




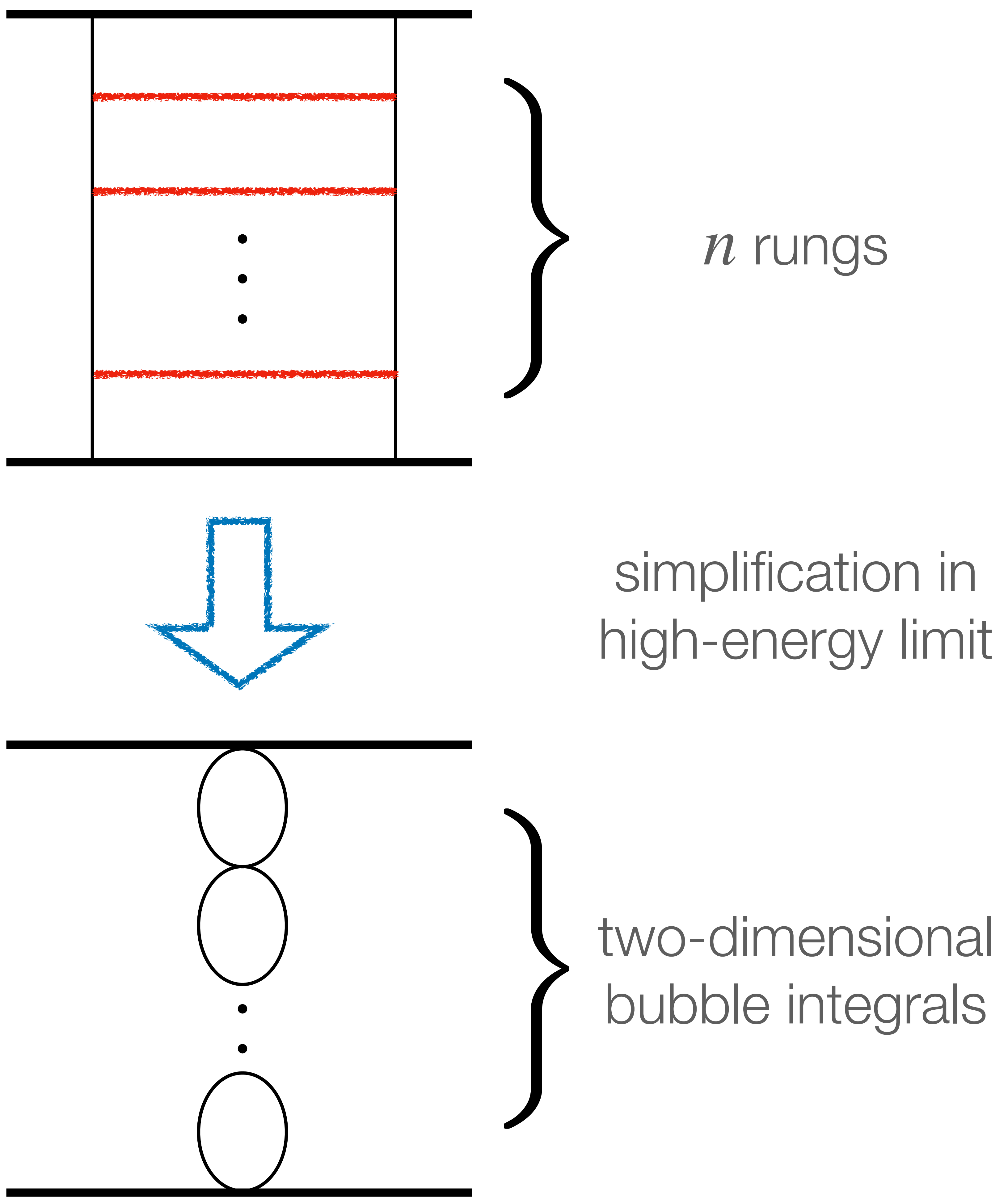
Contents

- Introduction to the high-energy limit of $2 \to 2$ scattering amplitudes
  - What are Regge poles and cuts?
  - Leading Logarithm (LL) Approximation
  - Next-to-Leading Logarithms (NLL)
- NNLL
  - Calculation of multi-Reggeon exchanges
  - How to distinguish pole and cut contributions
- Applications
  - The three-loop Regge trajectory in QCD
  - Infrared constraints

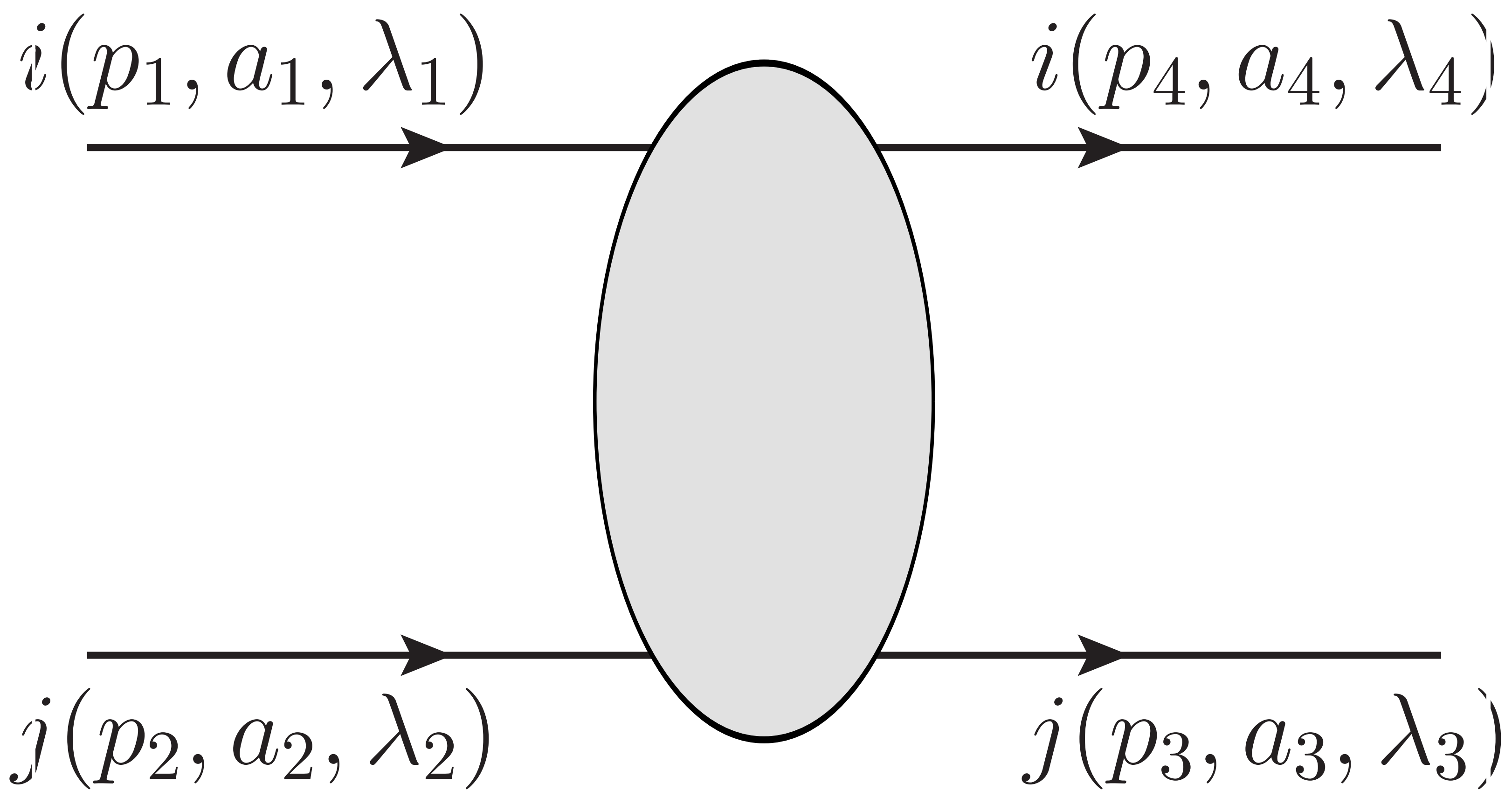
Motivation

Why care about the high-energy limit?

- Interesting theoretical question
- Use it as a toy model for scattering amplitudes
- Has still rich structure in both kinematics and colour
- Understand all-order behaviour of amplitudes
- Regge poles and Regge cuts in perturbative QCD
- Provide constraints for universal infrared divergences



$2 \to 2$ Scattering Amplitudes



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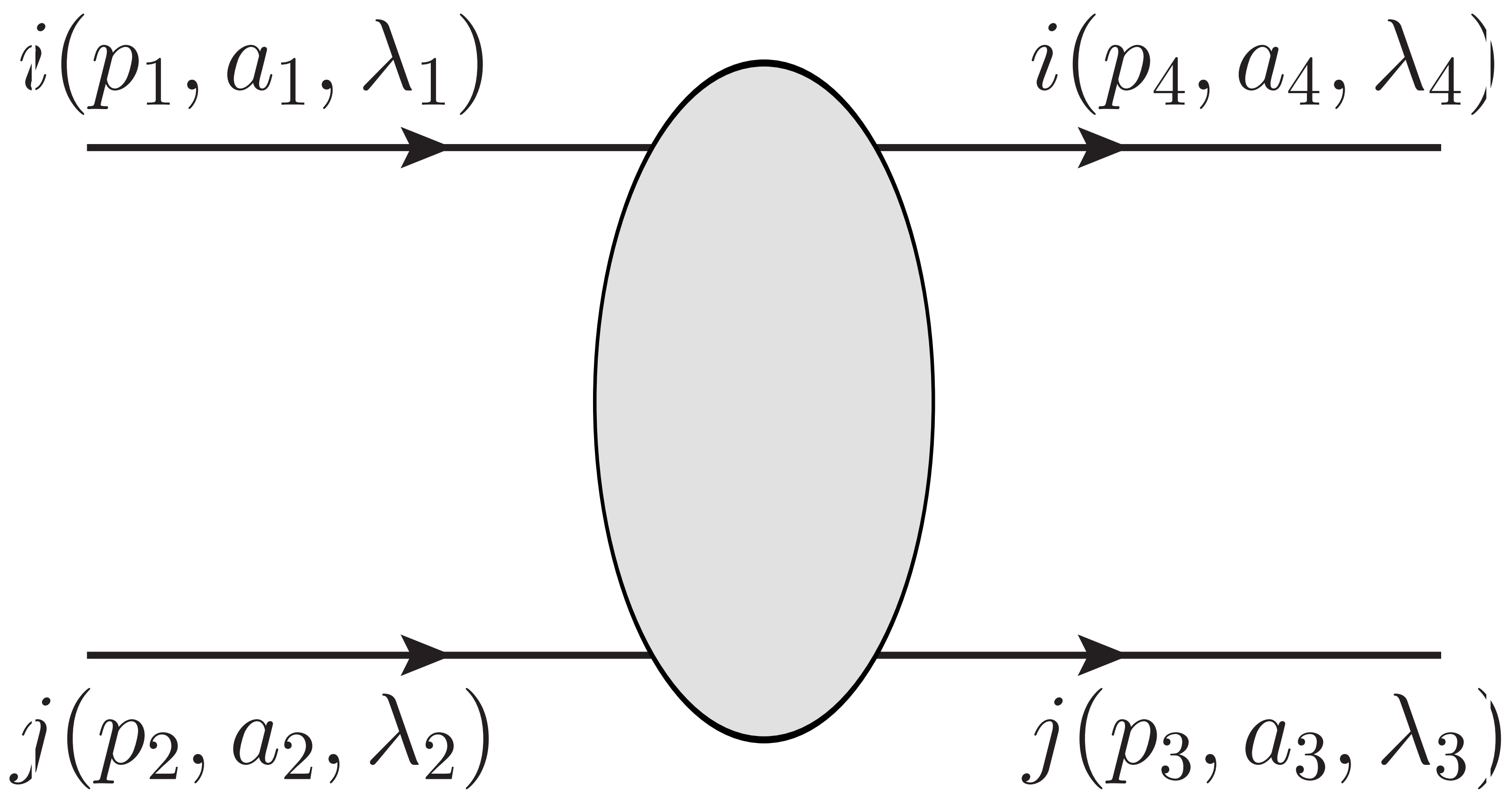

- We will consider $2 \to 2$ scattering amplitudes in any massless theory
  - $i(p_1, a_1, \lambda_1) + j(p_2, a_2, \lambda_2) \to j(p_3, a_3, \lambda_3) + i(p_4, a_4, \lambda_4)$
  - Described in terms of Mandelstam invariants
  - $s = (p_1 + p_2)^2 > 0$ $\quad t = (p_1 - p_4)^2 < 0$ $\quad u = (p_1 - p_3)^2 < 0$

momentum conservation implies $s + t + u = 0$

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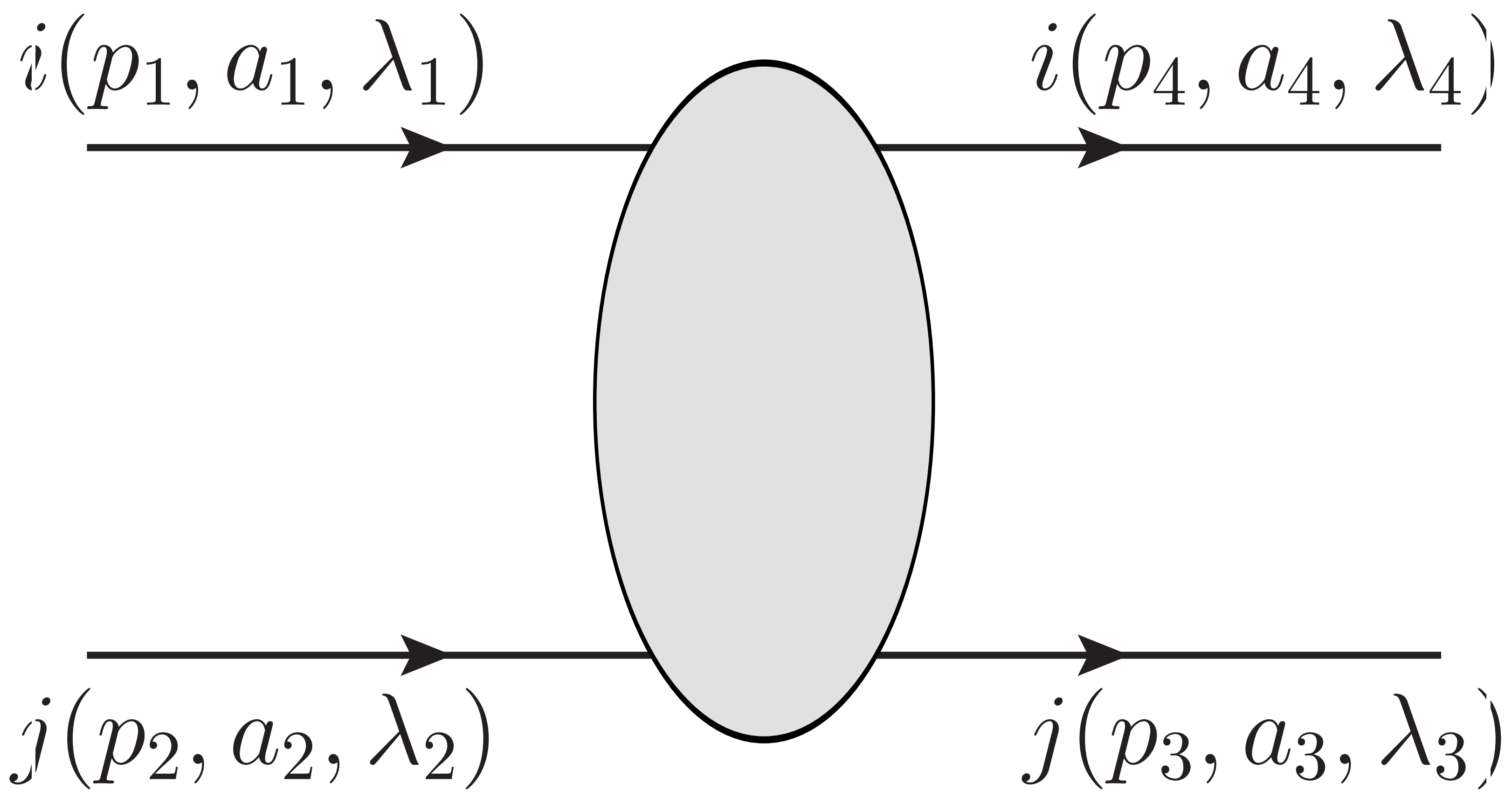

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momentum conservation implies $s + t + u = 0$

- The high-energy limit is defined to be

centre-of-mass energy much larger than momentum transfer

$$s \gg -t$$

$$s \approx -u$$

amplitudes have crossing symmetry properties under

$$s \leftrightarrow u$$

What are Regge poles and cuts?

Let us travel back to the 1960s, prior to QCD

[Regge ’59, ’60; Eden, Landshoff, Olive, Polkinghorne ’66; Collins ‘77]

Start with partial wave expansion of the scattering amplitude

angular momentum in t-channel

dependence of a state of angular momentum ℓ are the Legendre polynomials

$$\mathcal{M} \sim \sum_{\ell=0}^{\infty} (2\ell + 1) A_\ell(t) P_\ell(s)$$

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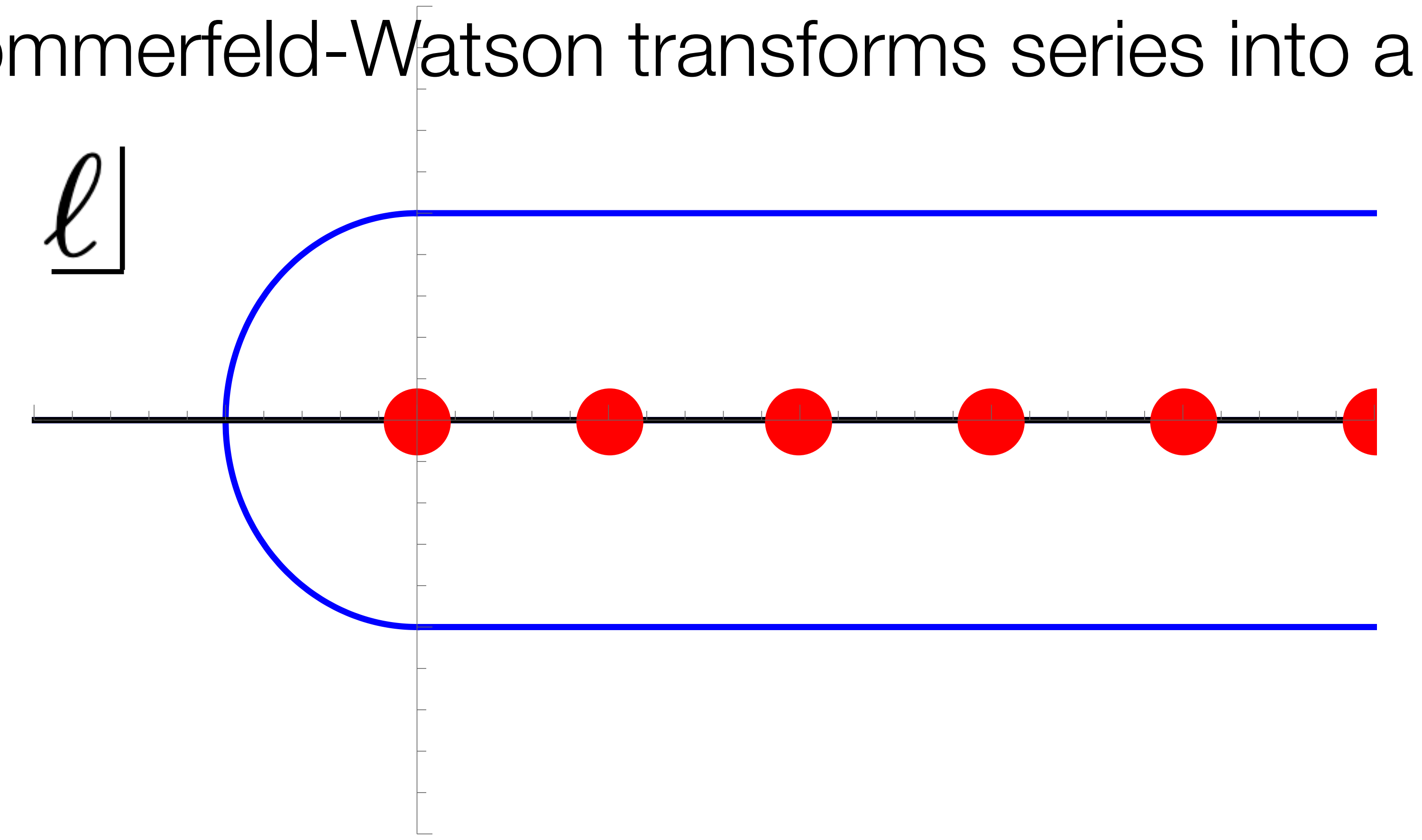

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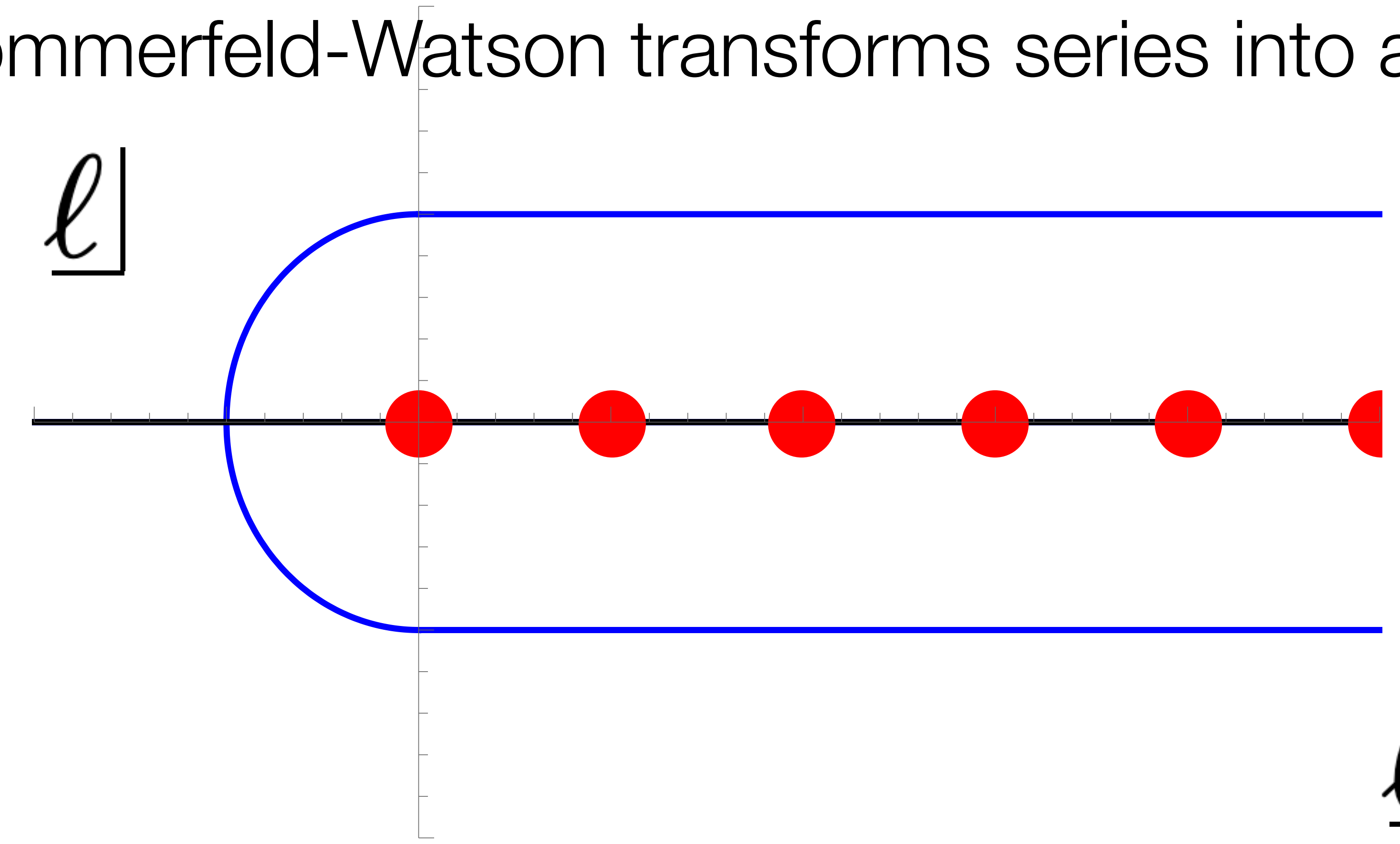

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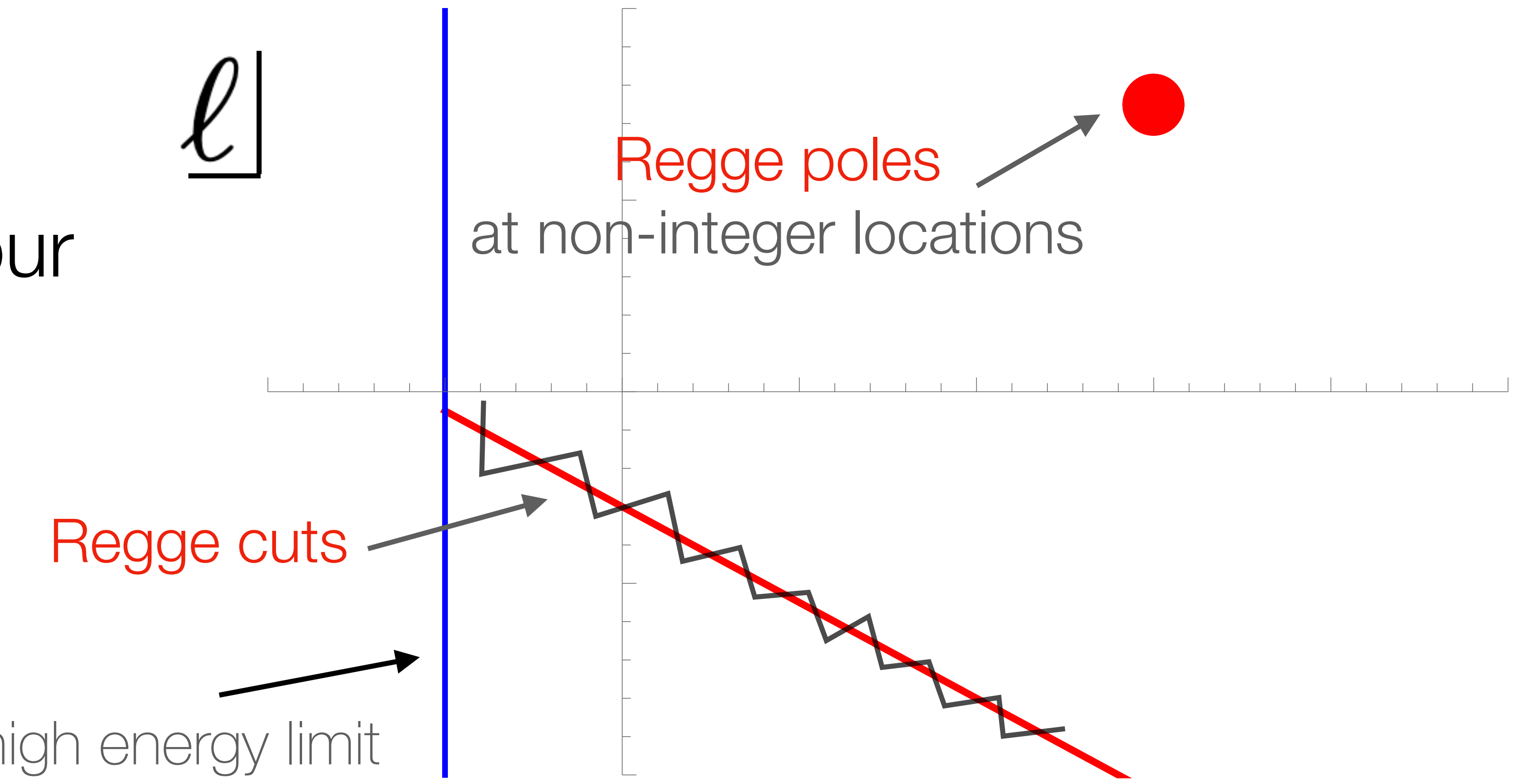
Regge's idea was to open up the contour

Doing so we will pick up
other analytic behaviour



This contour is subleading in high energy limit

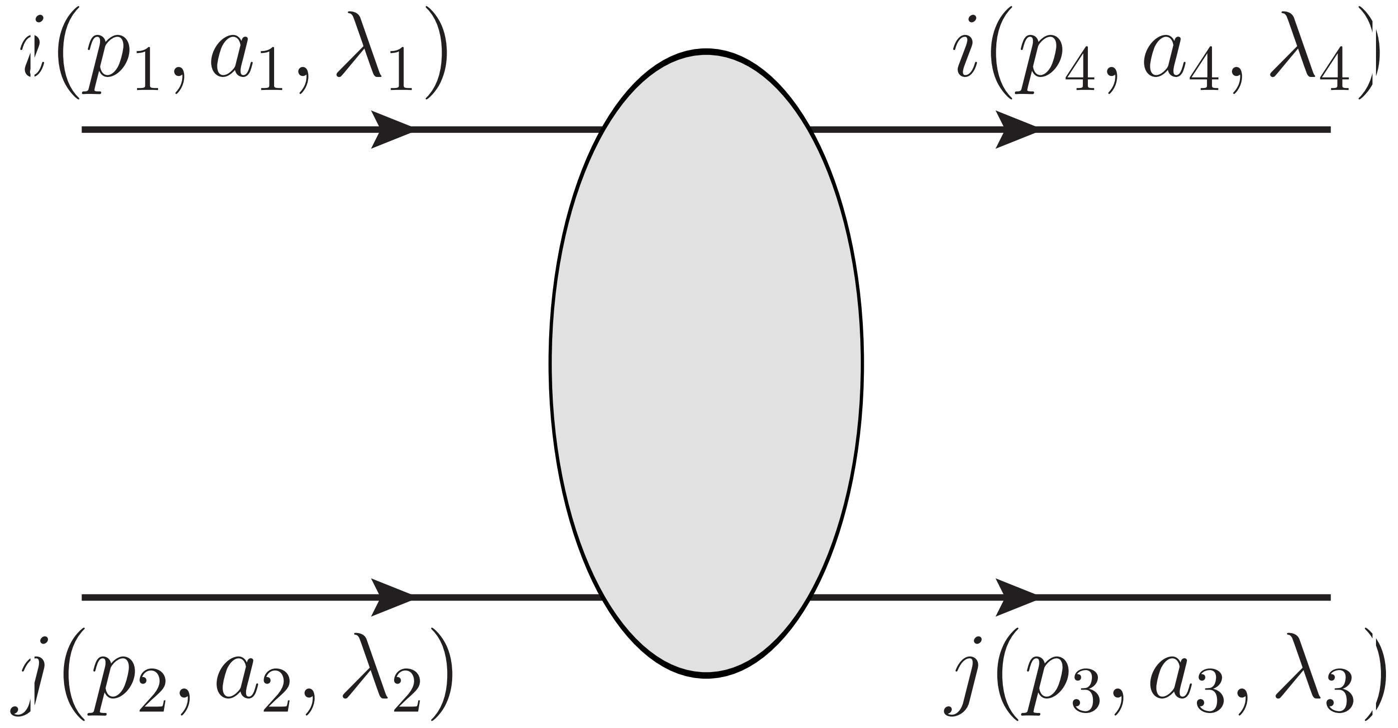
Regge cuts arise from only nonplanar diagrams [Mandelstam '63]

Only from Feynman integral analysis, there is no colour (yet)

We will show how to disentangle cuts and poles in perturbative QCD

The High-Energy Limit of QCD

At Leading Logarithmic Accuracy: The Regge trajectory



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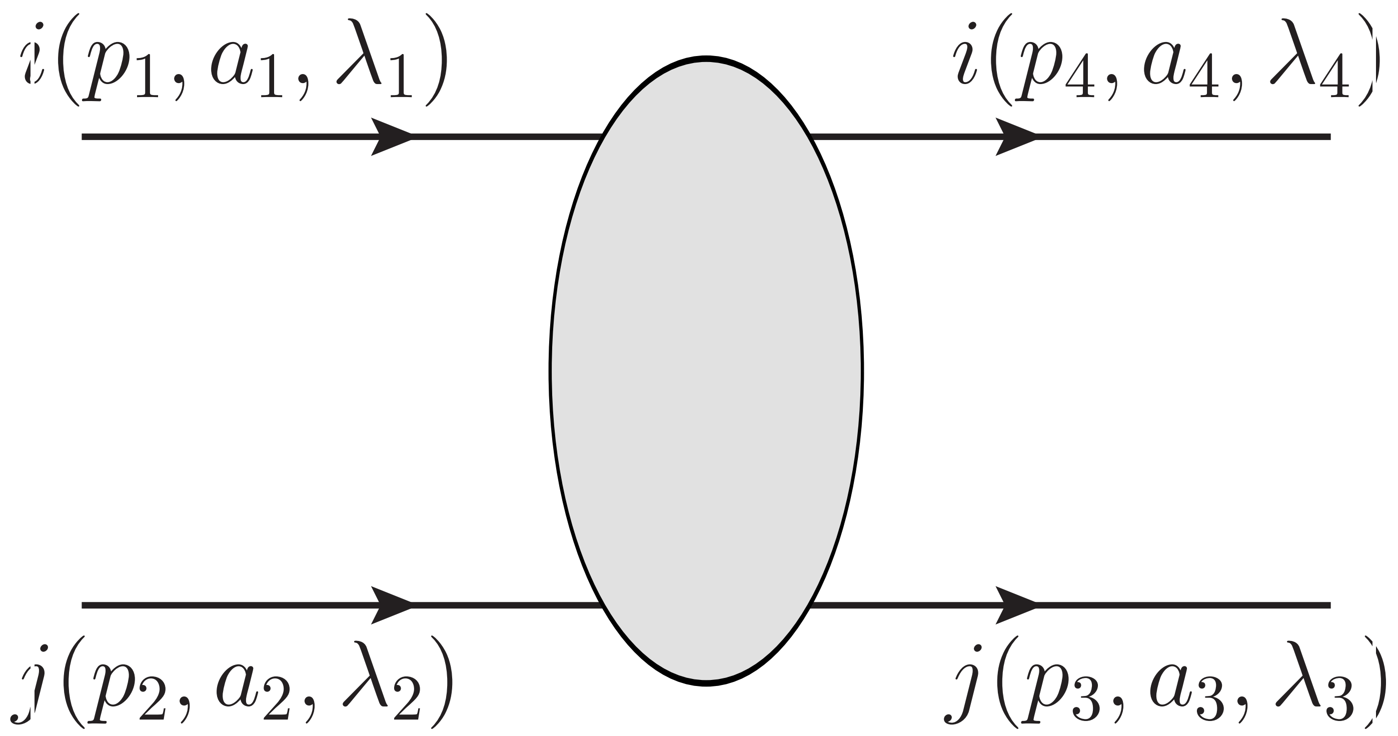

- At leading power the amplitude is

helicity conservation

$$\mathcal{M}^{\rm tree}_{ij\to ij} = g_s^2 \, \frac{2s}{t} \, (T_i^b)_{a_1a_4} (T_j^b)_{a_2a_3} \, \delta_{\lambda_1\lambda_4}\delta_{\lambda_2\lambda_3}$$

colour operator notation valid for any representation

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convenient to define the colour flow operators corresponding to Mandelstam invariants

$$\mathbf{T}_s = \mathbf{T}_1 + \mathbf{T}_2$$
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colour charge in t-channel is the adjoint Casimir (gluon)

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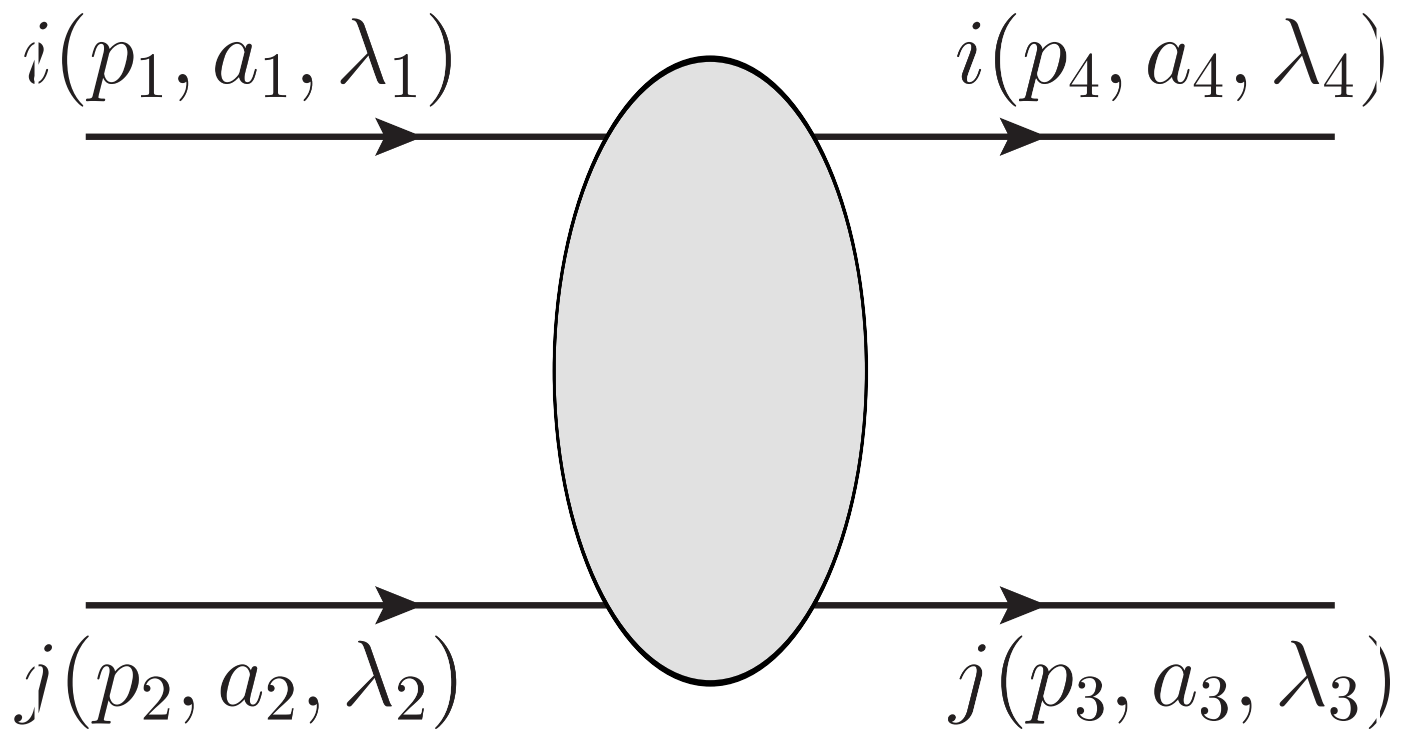

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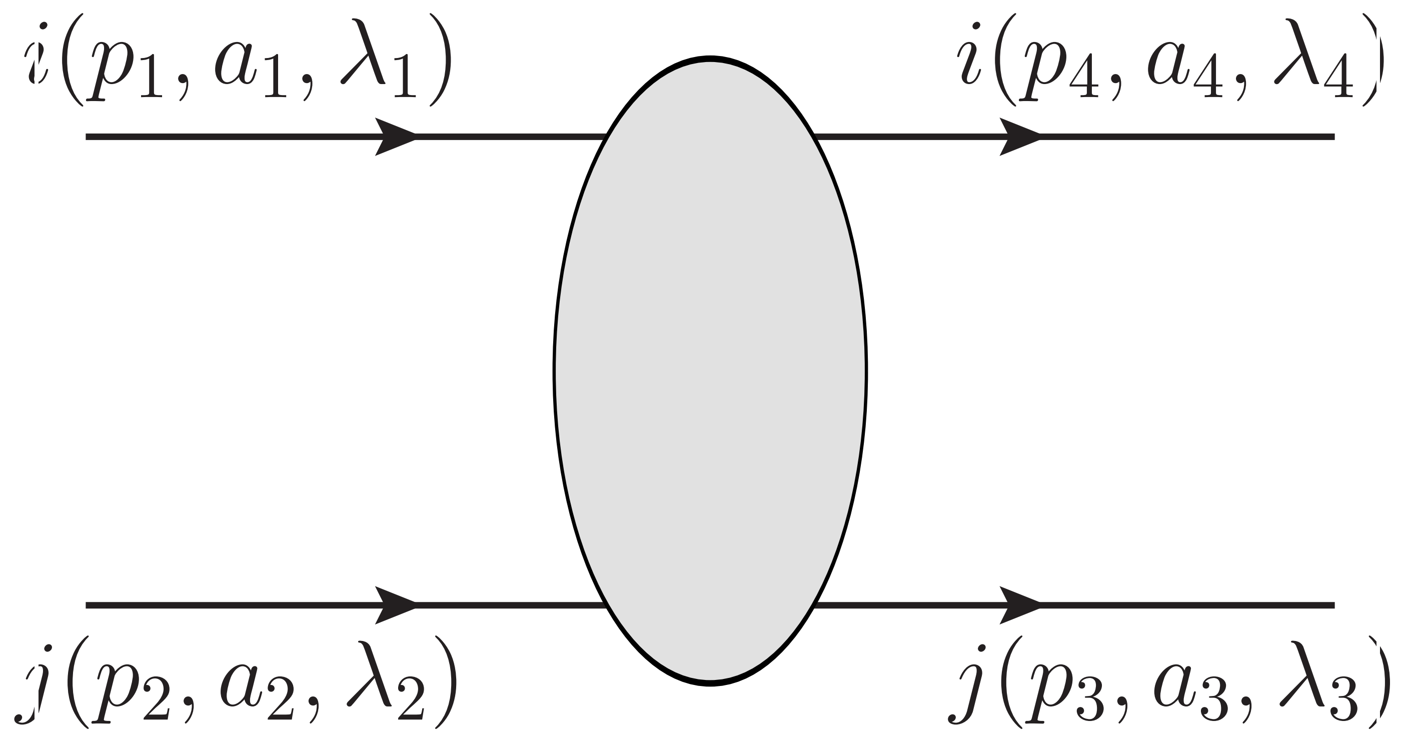

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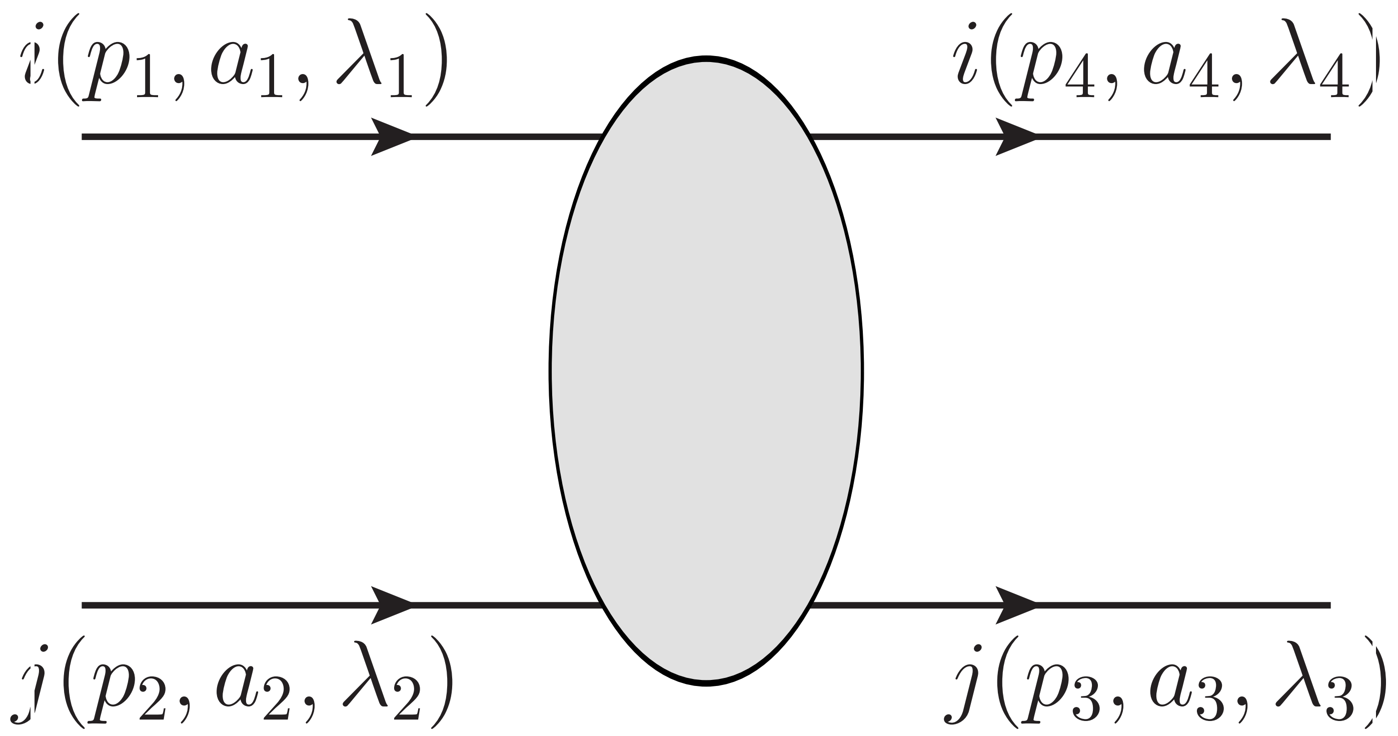

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$$\alpha_g^{(1)}(t,\mu^2) = \frac{r_\Gamma}{2\epsilon}\left(\frac{\mu^2}{-t}\right)^{\epsilon} \qquad r_\Gamma = e^{\epsilon\gamma_E}\frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}$$

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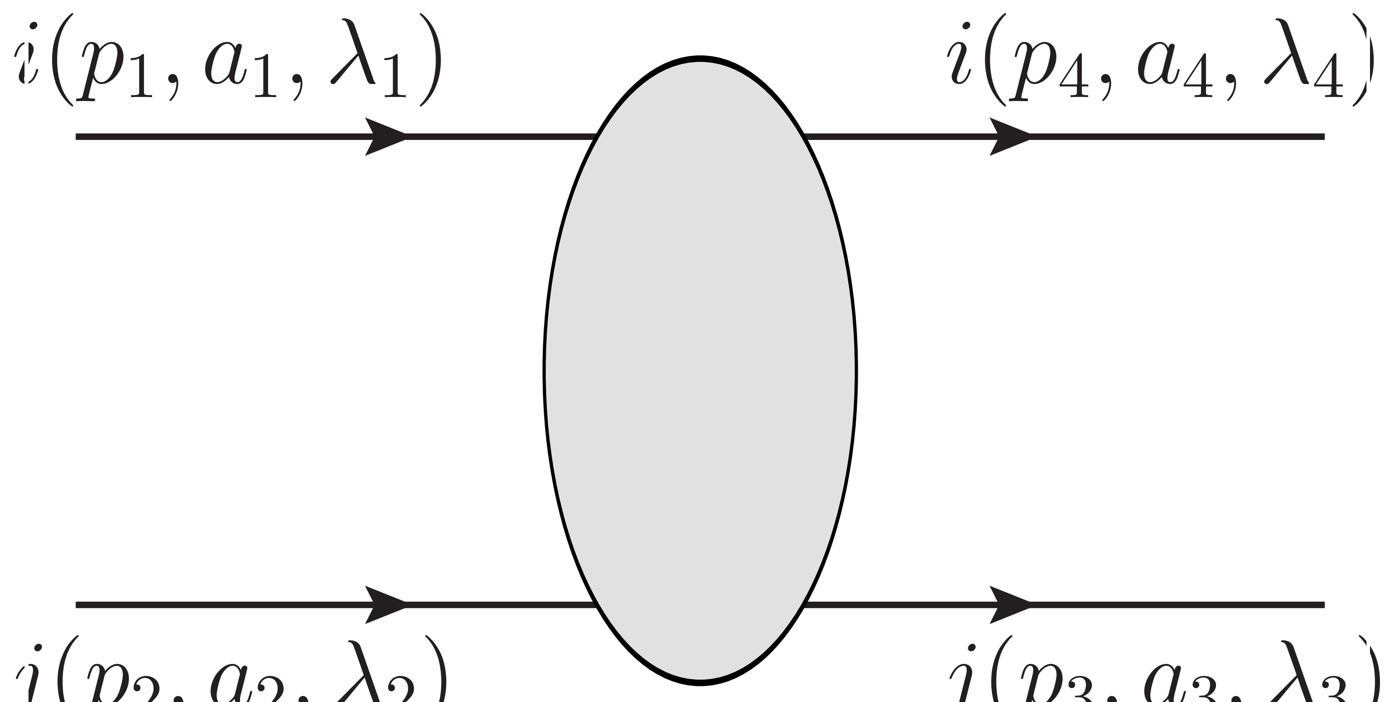

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- This captures logarithms of the form $\alpha_s^n \log^n\left(\dfrac{s}{-t}\right)$, what about subleading logarithms?

Odd/even amplitudes

Odd/even amplitudes

- It is convenient to exploit $s \leftrightarrow u$ symmetry in the high-energy limit. We define even and odd amplitudes such that they have definite sign under crossing

$$\mathcal{M}^{(\pm)}(s,t) = \frac{1}{2}\Big(\mathcal{M}(s,t) \pm \mathcal{M}(-s-t,t)\Big)$$

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$$L \equiv \log\left(\frac{s}{-t}\right) - \frac{i\pi}{2} = \frac{1}{2}\left[\log\left(\frac{-s-i0}{-t}\right) + \log\left(\frac{-u-i0}{-t}\right)\right]$$

[Caron-Huot, Gardi, Vernazza '17]

Odd amplitudes $\longleftrightarrow$ Real part

Even amplitudes $\longleftrightarrow$ Imaginary part

We will focus on the odd amplitudes

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As the kinematic term $\dfrac{s}{t}$ is odd, the odd amplitude will have an even number of $\mathbf{T}^2_{s-u}$

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- The amplitude, written as an expansion $\mathcal{M}^{(\pm)}_{ij\to ij} = \sum_{n=0}^{\infty}\left(\dfrac{\alpha_s}{\pi}\right)^n \sum_{m=0}^{n} L^m \mathcal{M}^{(\pm,n,m)}_{ij\to ij}$

Next-to-Leading-Logarithms

Two-loop Regge trajectory and impact factors

Next-to-Leading-Logarithms

Two-loop Regge trajectory and impact factors

- At the next logarithmic accuracy we need to generalise the Regge pole - Regge factorisation

Odd amplitude $$\mathcal{M}^{(-)}_{ij\to ij} = e^{C_A \alpha_g(t) L} C_i(t) C_j(t) \mathcal{M}^{\text{tree}}_{ij\to ij}$$

Regge trajectory

impact factors

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Regge trajectory

impact factors

- Expanding to two loops we have

$$\mathcal{M}^{(-,2)}_{ij\to ij} = \left\{ \frac{1}{2}\left(C_A \alpha_g^{(1)}\right)^2 L^2 + C_A \left[\alpha_g^{(2)} + \alpha_g^{(1)}\left(C_i^{(1)} + C_j^{(1)}\right)\right] L + C_i^{(1)} C_j^{(1)} + C_i^{(2)} + C_j^{(2)} \right\} \mathcal{M}^{\text{tree}}_{ij\to ij}$$

Leading logs

NLL pole parameters are
the two-loop Regge trajectory
and one-loop impact factors

Infrared Divergences
The cusp and collinear anomalous dimensions

A similar exponentiation occurs for infrared divergences of scattering amplitudes

Long distance singularities factorise

e.g. [Catani '98; Aybat, Dixon, Sterman '06; Gardi, Magnea '09; Becher, Neubert '09]

$$\mathcal{M} = \mathbf{Z} \cdot \underbrace{\mathcal{H}}_{\text{finite hard function}}$$

$$\mathbf{Z} = \mathbb{P} \exp \left\{ -\frac{1}{2} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \overbrace{\mathbf{\Gamma}}^{\text{soft anomalous dimension}} \right\}$$

integrating over the running coupling generates infrared poles

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integrating over the running coupling generates infrared poles

Soft anomalous dimension in the high-energy limit is

[Del Duca, Duhr, Gardi, Magnea, White '11]

collinear anomalous dimensions for the external legs

$$\mathbf{\Gamma} = \frac{1}{2}\gamma_K \left[L\mathbf{T}_t^2 + i\pi \mathbf{T}_{s-u}^2 \right] + \Gamma_i + \Gamma_j + \mathbf{\Delta}$$

cusp anomalous dimension captures the LL divergences

starts at NLL for even amplitudes and NNLL for odd amplitudes

Comparing Regge and infrared factorisation

Regge factorisation valid to NLL

$$\mathcal{M}^{(-)}_{ij\to ij} = e^{C_A \alpha_g(t) L}\, C_i(t)\, C_j(t)\, \mathcal{M}^{\text{tree}}_{ij\to ij}$$

Infrared factorisation valid to NLL

$$\begin{aligned}\mathcal{M}^{(-)}_{ij\to ij}|_{\text{poles}} &= \exp\left[-\frac{1}{4}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\gamma_K(\alpha_s(\lambda^2))L\mathbf{T}_t^2\right] Z_i(t)Z_j(t)\mathcal{M}^{\text{tree}}_{ij\to ij}\\ &= e^{\mathbf{T}_t^2 K(\alpha_s)L} Z_i(t)Z_j(t)\mathcal{M}^{\text{tree}}_{ij\to ij}\end{aligned}$$

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$$= e^{\mathbf{T}_t^2 K(\alpha_s) L} Z_i(t) Z_j(t) \mathcal{M}^{\text{tree}}_{ij\to ij}$$

Comparing the two exponentiations gives two obvious equalities at NLL

$$K(\alpha_s(\mu^2)) \equiv -\frac{1}{4}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\gamma_K(\alpha_s(\lambda^2)) \qquad \mu^2 = -t$$

$$Z_i(t) = \exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\Gamma_i\left(\alpha_s(\lambda^2), \frac{-t}{\lambda^2}\right)\right\}$$

$$\alpha_g(t) = K + \mathcal{O}(\epsilon^0)$$

holds at two loops

$$C_{i/j}(t) = Z_{i/j}(t)\, D_{i/j}(t)$$

finite at one loop

[Del Duca, Falcioni, Magnea, Vernazza '14]

This identification suggests it is eikonal

[Korchemskaya, Korchemsky '94, '96]

We will see later how these generalise at NNLL and beyond

Next-to-Next-to-Leading-Logarithms

Non-factorising term

Next-to-Next-to-Leading-Logarithms

Non-factorising term

- At the next logarithmic order the factorisation fails and we need to add a non-factorising term

$$\begin{aligned}\mathcal{M}^{(-)}_{ij\to ij} &= \mathcal{M}^{(-),\,\mathrm{F}}_{ij\to ij} + \mathcal{M}^{(-),\,\mathrm{NF}}_{ij\to ij} \\ &= e^{C_A \alpha_g(t) L}\, C_i(t)\, C_j(t)\, \mathcal{M}^{\mathrm{tree}}_{ij\to ij} + \mathcal{M}^{(-),\,\mathrm{NF}}_{ij\to ij}\end{aligned}$$

While the factorising term is associated to Regge poles the non-factorising term is associated to Regge cuts

universal - independent of fundament/adjoint scattering particles

depends only on i or j not together

The pole term can only contain Casimirs

But the amplitude no longer carries just the colour structure of the tree-level

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But the amplitude no longer carries just the colour structure of the tree-level

- To see this, let us return to colour

Use a basis that is orthonormal and one of the elements is the tree-level antisymmetric octet

$$\begin{aligned}
(8\otimes 8)_{gg\to gg} &= 1\oplus 8_s \oplus 8_a \oplus (10\oplus \overline{10}) \oplus 27 \oplus 0\\
(8\otimes 8)_{qg\to qg} &= 1\oplus 8_s \oplus 8_a\\
(3\otimes \bar{3})_{qq\to qq} &= 1\oplus 8_a
\end{aligned}$$

tree-level

contributes to the odd amplitude at NNLL

the others contribute to the even amplitude

The Scheme Dependence at NNLL

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- Expanding the above two loops at NNLL we have

loop order log order

$$\mathcal{M}^{(-,2,0)}_{ij\to ij} = \left(C_i^{(1)}C_i^{(1)} + C_i^{(2)} + C_j^{(2)}\right)\mathcal{M}^{\rm tree}_{ij\to ij} + \mathcal{M}^{(-,2,0),\,\rm NF}_{ij\to ij}$$

fixed

We can shift terms that are colour proportional to the tree-level between these
There is an ambiguity in the definition of the two-loop impact factors

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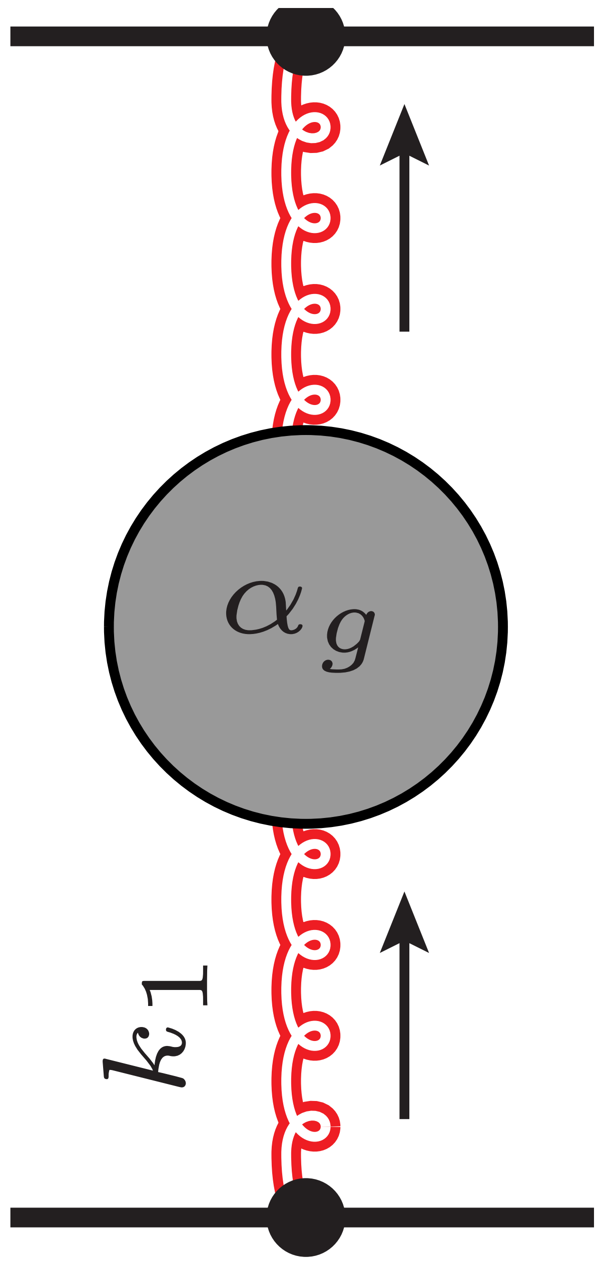

$$e^{C_A \alpha_g(t) L}\, C_i(t)\, C_j(t)\, \mathcal{M}^{\mathrm{tree}}_{ij\to ij}$$

The pole term can be thought of as a single “Reggeized” gluon (Reggeon) in the t-channel

Two Reggeon exchange appears in the even amplitude. We shall not consider it.

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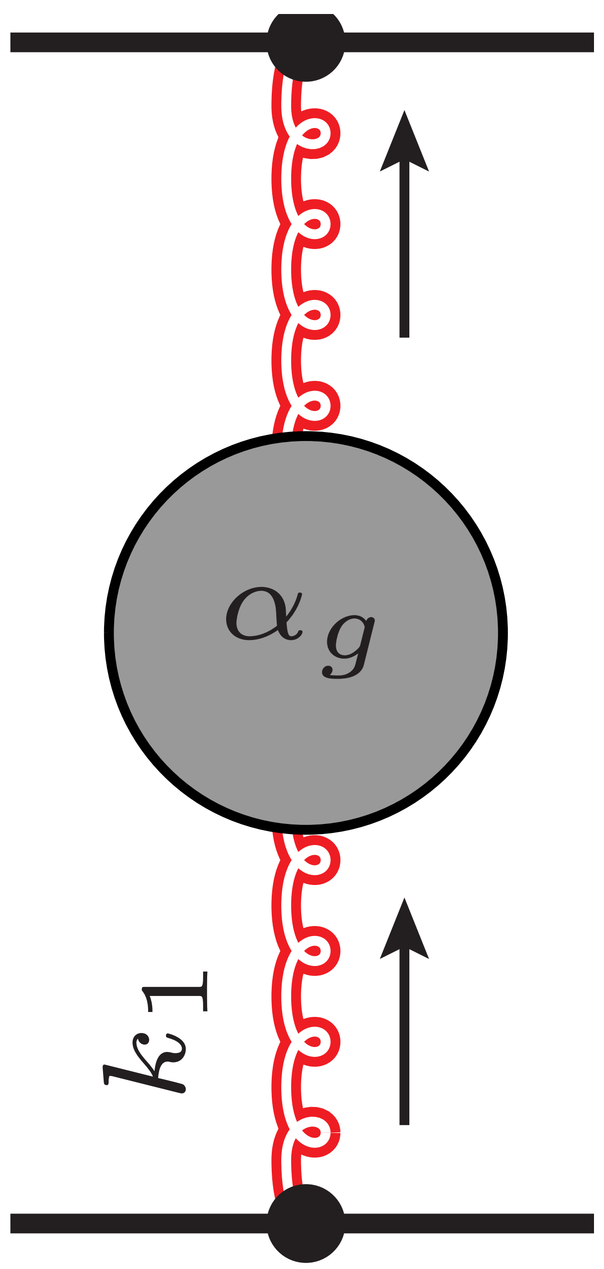

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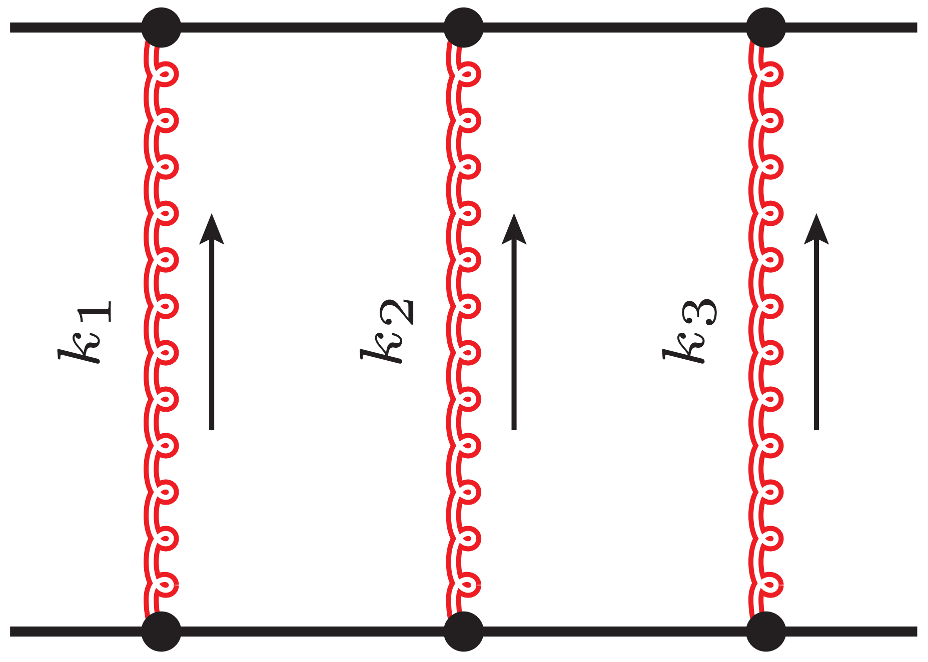
Two Reggeon exchange appears in the even amplitude. We shall not consider it.

At two loops we can start to exchange three Reggeons

sum over all possible orderings



We can calculate it based on shockwave formalism and Balitsky-JIMWLK evolution
[Caron-Huot '13]

Overall factor of $\pi^2 = -(i\pi)^2$ is indicative of a *cut*

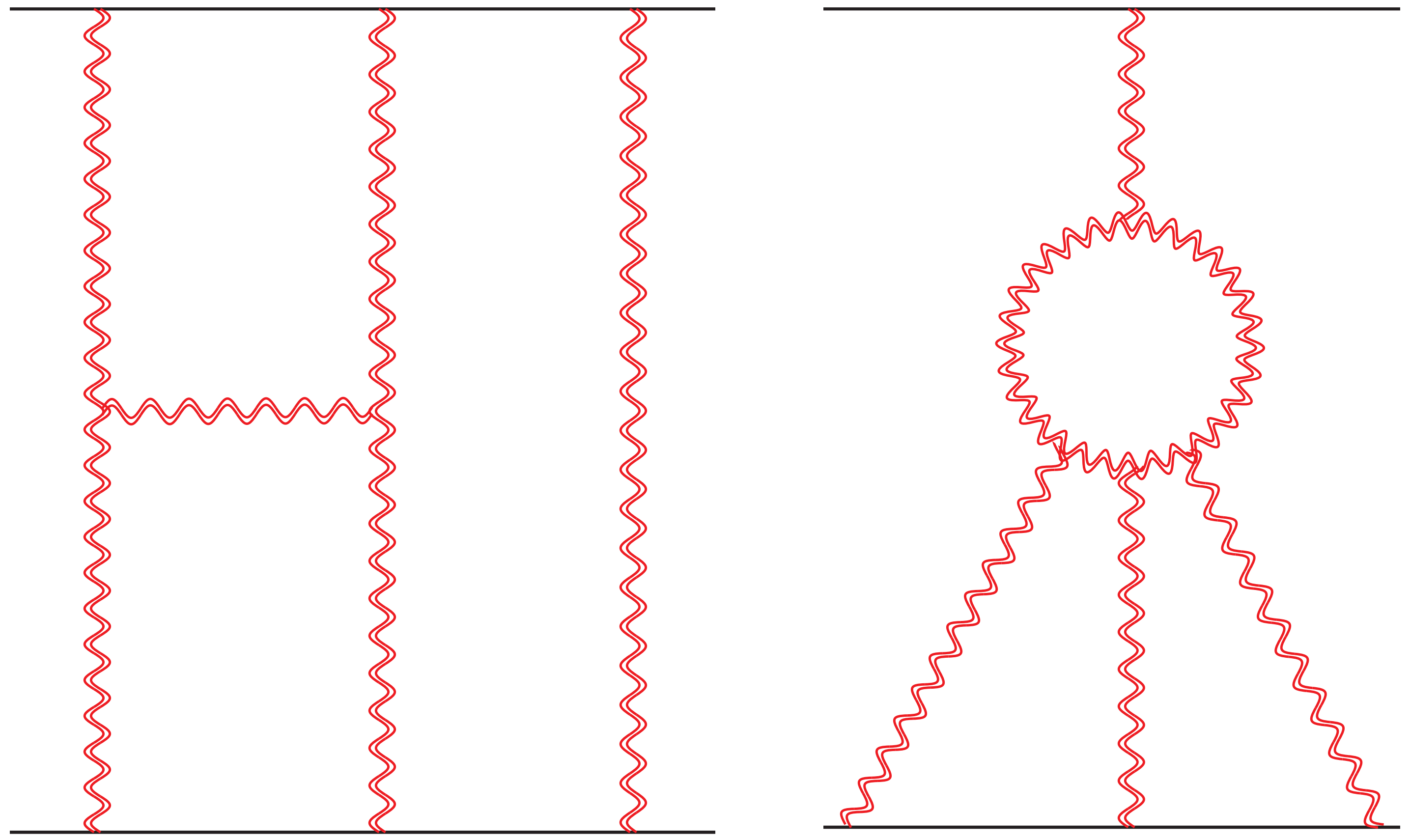
Even number of $\mathbf{T}^2_{s-u}$

$$\mathcal{M}^{(-,2,0),\,\text{MRS}}_{ij\to ij} = \pi^2\left(-\frac{1}{8\epsilon^2} + \frac{\pi^2}{48} + \frac{4}{3}\epsilon\zeta_3 + \mathcal{O}(\epsilon^2)\right)\left((\mathbf{T}^2_{s-u})^2 - \frac{1}{12}C_A^2\right)\mathcal{M}^{\text{tree}}_{ij\to ij}$$

[Caron-Huot, Gardi, Vernazza '17]

we shall call this scheme MRS = Multiple Reggeon States

MRS at three and four loops



$N_c \to \infty$

commutators are subleading in planar limit

only this gives planar contribution

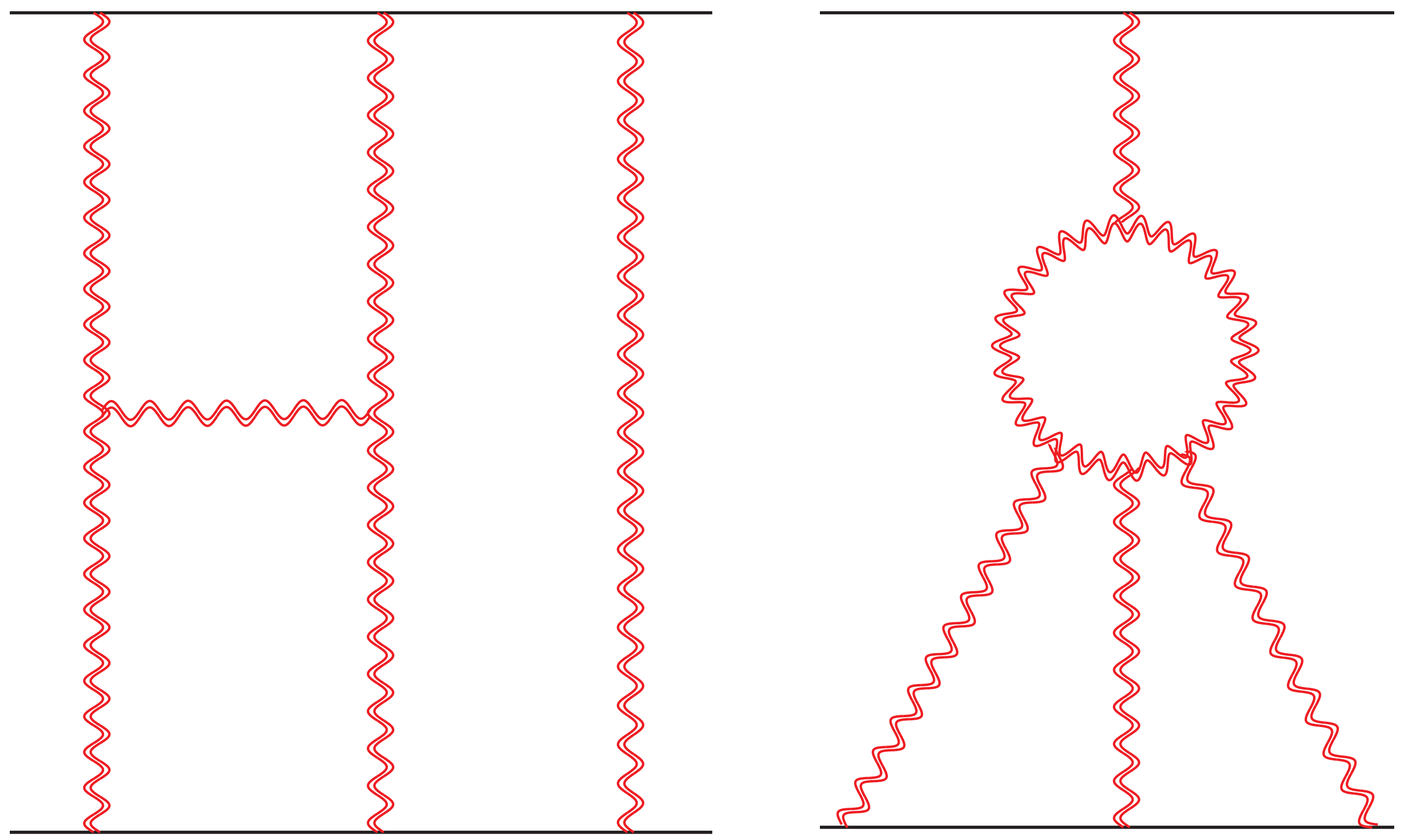
$$\mathcal{M}_{ij\to ij}^{(-,3,1),\,\mathrm{MRS}} = -\pi^2 r_\Gamma^3 \left[S_A^{(3)}(\epsilon)\mathbf{T}_{s-u}^2 \left[\mathbf{T}_{s-u}^2, \mathbf{T}_t^2\right] + S_B^{(3)}(\epsilon)\left[\mathbf{T}_{s-u}^2, \mathbf{T}_t^2\right]\mathbf{T}_{s-u}^2 + S_C^{(3)}(\epsilon) C_A^3 \right] \hat{\mathcal{M}}_{\mathrm{tree}}$$

[Caron-Huot, Gardi, Vernazza '17]

Functions in ϵ

Uniform transcendental weight

MRS at three and four loops



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[Caron-Huot, Gardi, Vernazza '17]

Functions in ϵ

Uniform transcendental weight

Common ratios between even and odd zetas

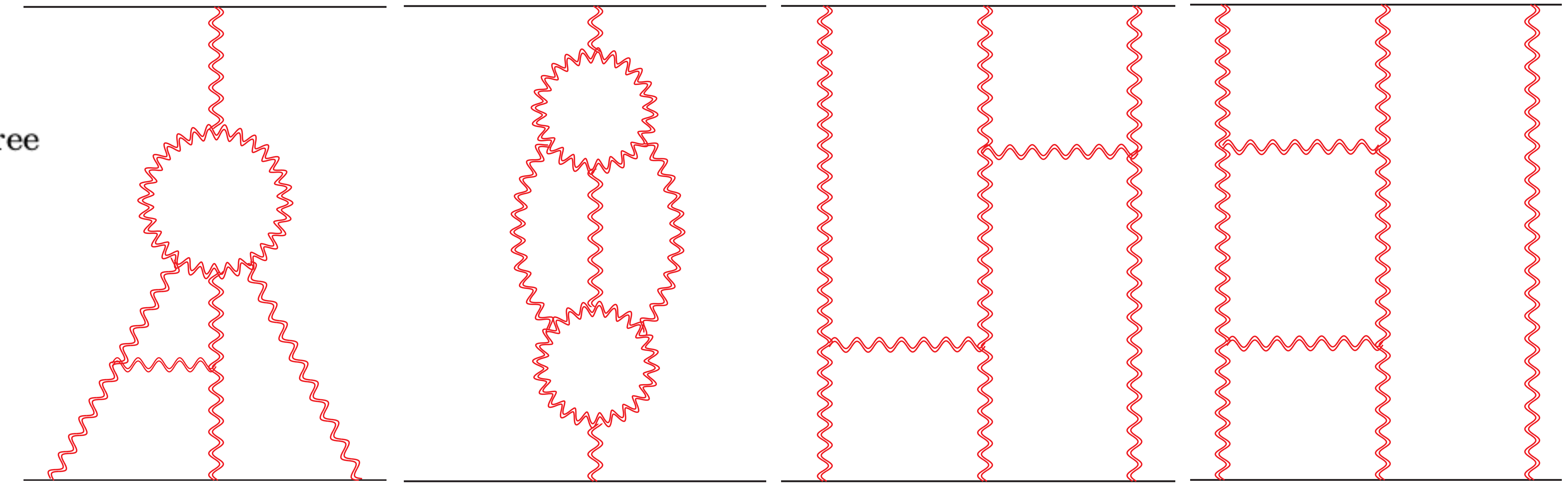
Define new zeta values to absorb them

They appear in other computations

[Baikov, Chetyrkin '19; Kotikov, Teber '19]

$$\mathcal{M}^{(-,4,2),\,\mathrm{MRS}}_{ij\to ij} = \frac{\pi^2 r_\Gamma^4}{2}\left[\frac{1}{\epsilon^4}\mathbf{K}^{(4)} + \overbrace{\left(\frac{1}{\epsilon}\zeta_3 + \frac{3}{2}\zeta_4\right)}^{\hat{\zeta}_3}\mathbf{K}^{(1)} + \mathcal{O}(\epsilon)\right]\hat{\mathcal{M}}_{\mathrm{tree}}$$

$$\begin{aligned}
\mathbf{K}^{(4)} &= \frac{1}{96}\left[\mathbf{T}^2_{s-u},\left[\mathbf{T}^2_{s-u},\mathbf{T}^2_t\right]\right]\mathbf{T}^2_t + \frac{7}{576}\mathbf{T}^2_t\left[\left(\mathbf{T}^2_{s-u}\right)^2,\mathbf{T}^2_t\right] \\
&\quad - \frac{1}{192}\left[\mathbf{T}^2_{s-u},\mathbf{T}^2_t\right]\mathbf{T}^2_t\mathbf{T}^2_{s-u} - \frac{5}{192}\mathbf{T}^2_{s-u}\left[\mathbf{T}^2_{s-u},\mathbf{T}^2_t\right]\mathbf{T}^2_t, \\
\mathbf{K}^{(1)} &= \frac{49}{48}\left[\mathbf{T}^2_{s-u},\left[\mathbf{T}^2_{s-u},\mathbf{T}^2_t\right]\right]\mathbf{T}^2_t - \frac{47}{288}\mathbf{T}^2_t\left[\left(\mathbf{T}^2_{s-u}\right)^2,\mathbf{T}^2_t\right] \\
&\quad + \frac{101}{96}\left[\mathbf{T}^2_{s-u},\mathbf{T}^2_t\right]\mathbf{T}^2_t\mathbf{T}^2_{s-u} - \frac{49}{48}\mathbf{T}^2_{s-u}\left[\mathbf{T}^2_{s-u},\mathbf{T}^2_t\right]\mathbf{T}^2_t + \frac{1}{24}\left(\frac{d_{AA}}{N_A} - \frac{C_A^4}{24}\right)
\end{aligned}$$



The four-loop contribution is entirely nonplanar

[Falcioni, Gardi, **CM**, Vernazza '20]

Choosing a scheme

The impact factors can be found from matching to amplitudes

$$\mathcal{M}^{(-,2,0)}_{ij\to ij} = \left(C_i^{(1)}C_i^{(1)} + C_i^{(2)} + C_j^{(2)}\right)\mathcal{M}^{\text{tree}}_{ij\to ij} + \mathcal{M}^{(-,2,0),\,\text{MRS}}_{ij\to ij}$$

$$\mathcal{M}^{(-,2,0),\,\text{MRS}}_{ij\to ij} = \pi^2\left(-\frac{1}{8\epsilon^2} + \frac{\pi^2}{48} + \frac{4}{3}\epsilon\zeta_3 + \mathcal{O}(\epsilon^2)\right)\left((\mathbf{T}^2_{s-u})^2 - \frac{1}{12}C_A^2\right)\mathcal{M}^{\text{tree}}_{ij\to ij}$$

But we could move any part of the term colour proportional to the tree-level into new impact factors

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But we could move any part of the term colour proportional to the tree-level into new impact factors

What is the most "physical" scheme?

- The MRS scheme was used because it made calculational sense
- Evaluating the term in the planar limit gives $\mathcal{M}^{(-)\,\text{MRS}}_{ij\to ij}\Big|_{\text{planar}} = \frac{\pi^2 r_\Gamma^2 N_c^2}{6}\left(\frac{\alpha_s}{\pi}\right)^2\left\{S^{(2)}(\epsilon) - \left(\frac{\alpha_s}{\pi}\right) r_\Gamma N_c L\left[6\,S_C^{(3)}(\epsilon) - \frac{1}{2\epsilon}S^{(2)}(\epsilon)\right]\right\}\mathcal{M}^{\text{tree}}_{ij\to ij}$
- At odds with the term being identified as coming from Regge cuts in the complex angular momentum plane
- Naturally gives a new scheme we call the "cut" scheme

[Falcioni, Gardi, Maher, **CM**, Vernazza '21, **PRL**]

$$\mathcal{M}^{(-)}_{ij\to ij} = \underbrace{\mathcal{M}^{(-)\,\text{SRS}}_{ij\to ij} + \mathcal{M}^{(-)\,\text{MRS}}_{ij\to ij}\Big|_{\text{planar}}} + \mathcal{M}^{(-)\,\text{MRS}}_{ij\to ij}\Big|_{\text{nonplanar}}$$

$$= \mathcal{M}^{(-)\,\text{pole}}_{ij\to ij} + \mathcal{M}^{(-)\,\text{cut}}_{ij\to ij}$$

SRS = Single Reggeon State

Introducing the cut scheme

[Falcioni, Gardi, Maher, **CM**, Vernazza '21, **PRL**]

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- We define a new pole term with a new Regge trajectory and new impact factors

$$\mathcal{M}^{(-)\,\mathrm{pole}}_{ij\to ij} = \tilde{C}_i(t)\,\tilde{C}_j(t)\,e^{\tilde{\alpha}_g(t)\,C_A\,L}\,\mathcal{M}^{\mathrm{tree}}_{ij\to ij}$$

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- At two loops we compute the nonplanar piece of the MRS term.
  Then keeping the whole amplitude invariant we have a relation between the impact factors in both schemes

$$\tilde{C}^{(2)}_{i/j} = C^{(2)}_{i/j} + N_c^2 (r_\Gamma)^2 \frac{\pi^2}{12} S^{(2)}(\epsilon)$$

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- We can do the same at three loops, finding a relation between the three-loop Regge trajectories

$$\tilde{\alpha}^{(3)}_g = \alpha^{(3)}_g - (r_\Gamma)^3 N_c^2 \frac{\pi^2}{18}\left(S^{(3)}_A(\epsilon) - S^{(3)}_B(\epsilon)\right)$$

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- At four loops (and beyond) the MRS term was already nonplanar
  We have exhausted all possible parameters at NNLL to absorb planar contributions into the pole term
  As such we know the four-loop Regge cut in any gauge theory! It is a universal result

What can we extract?

Two-loop results

Two-loop results

- Using the cut scheme we can extract two-loop Regge trajectory and impact factors

$$\mathcal{M}_{ij\to ij}^{(-,2,1),\,\mathrm{pole}} = C_A \left[\alpha_g^{(2)} + \alpha_g^{(1)} \left(C_i^{(1)} + C_i^{(1)}\right)\right] \mathcal{M}_{ij\to ij}^{\mathrm{tree}}$$

Note the two-loop Regge trajectory in both schemes are equal.
It is a NLL quantity

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It is a NLL quantity

- We expand the amplitudes known to $\mathcal{O}(\epsilon^2)$ in the high-energy limit and compare [Ahmed, Henn, Mistlberger '19]

we define the cusp-subtracted Regge trajectory $\hat{\tilde{\alpha}}_g = \tilde{\alpha}_g - K$ $\qquad K(\alpha_s(\mu^2)) \equiv -\frac{1}{4}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\gamma_K(\alpha_s(\lambda^2))$

$$\begin{aligned}\hat{\tilde{\alpha}}_g^{(2)} = {} & C_A\left(\frac{101}{108} - \frac{\zeta_3}{8}\right) - \frac{7n_f}{54} + \epsilon\left[C_A\left(\frac{607}{324} - \frac{67\zeta_2}{144} - \frac{33\zeta_3}{8} - \frac{3\zeta_4}{16}\right) + n_f\left(-\frac{41}{162} + \frac{5\zeta_2}{72} + \frac{3\zeta_3}{4}\right)\right] \\ & + \epsilon^2\left[C_A\left(\frac{911}{243} - \frac{101\zeta_2}{108} - \frac{1139\zeta_3}{108} - \frac{2321\zeta_4}{384} + \frac{41\zeta_5}{8} + \frac{71\zeta_2\zeta_3}{24}\right) + n_f\left(\frac{7\zeta_2}{54} + \frac{85\zeta_3}{54} + \frac{211\zeta_4}{192} - \frac{122}{243}\right)\right] + \mathcal{O}(\epsilon^3)\end{aligned}$$

- Order ϵ and above are new results [Falcioni, Gardi, Maher, **CM**, Vernazza '21, **PRL**]
- Finite at two loops, as discussed earlier
- Agrees with the planar limit of [Del Duca, Marzucca, Verbeek '21]

Two-loop impact factors

$$\mathcal{M}_{ij\to ij}^{(-,2,0),\,\text{pole}} = \left(C_i^{(1)}C_i^{(1)} + \tilde{C}_i^{(2)} + \tilde{C}_j^{(2)}\right)\mathcal{M}_{ij\to ij}^{\text{tree}}$$

$$\mathcal{M}_{ij\to ij}^{(-,2,0)\,\text{cut}} = \pi^2 (r_\Gamma)^2 S^{(2)}(\epsilon)\left[(\mathbf{T}_{s-u}^2)^2 - \frac{C_A^2}{4}\right]\mathcal{M}_{ij\to ij}^{\text{tree}}$$

- We extract them to $\mathcal{O}(\epsilon^2)$
- $\mathcal{O}(\epsilon)$ and above are new results [Falcioni, Gardi, Maher, **CM**, Vernazza '21, **PRL**]
- Agrees with the planar limit of [Del Duca, Marzucca, Verbeek '21]
- Divergences are given by the collinear anomalous dimension [Del Duca, Falcioni, Magnea, Vernazza '14]

There is no cut in the planar limit!
We are considering the cut
contribution for the first time!

The three-loop Regge trajectory

The three-loop Regge trajectory

- We can also do the same at three loops

$$\mathcal{M}_{ij\to ij}^{(-,3,1)\,\text{pole}} = N_c \left[\tilde{\alpha}_g^{(3)}(t) + \alpha_g^{(2)}(t)\left(C_i^{(1)} + C_j^{(1)}\right) + \alpha_g^{(1)}(t)\left(C_i^{(1)}C_j^{(1)} + \tilde{C}_i^{(2)} + \tilde{C}_j^{(2)}\right)\right]\mathcal{M}_{ij\to ij}^{\text{tree}}$$

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- We can also do the same at three loops

$$\mathcal{M}_{ij\to ij}^{(-,3,1)\,\mathrm{pole}} = N_c\left[\tilde{\alpha}_g^{(3)}(t) + \alpha_g^{(2)}(t)\left(C_i^{(1)} + C_j^{(1)}\right) + \alpha_g^{(1)}(t)\left(C_i^{(1)}C_j^{(1)} + \tilde{C}_i^{(2)} + \tilde{C}_j^{(2)}\right)\right]\mathcal{M}_{ij\to ij}^{\mathrm{tree}}$$

- The relevant three-loop amplitudes have been computed recently [Caola, Chakraborty, Gambuti, von Manteuffel, Tancredi '21]

$$\hat{\tilde{\alpha}}_g^{(3)} = C_A^2\left(\frac{297029}{93312} - \frac{799\zeta_2}{1296} - \frac{833\zeta_3}{216} - \frac{77\zeta_4}{192} + \frac{5}{24}\zeta_2\zeta_3 + \frac{\zeta_5}{4}\right) + C_A n_f\left(\frac{103\zeta_2}{1296} + \frac{139\zeta_3}{144} - \frac{5\zeta_4}{96} - \frac{31313}{46656}\right)$$
$$+ C_F n_f\left(\frac{19\zeta_3}{72} + \frac{\zeta_4}{8} - \frac{1711}{3456}\right) + n_f^2\left(\frac{29}{1458} - \frac{2\zeta_3}{27}\right) + \mathcal{O}(\epsilon),$$

[Falcioni, Gardi, Maher, **CM**, Vernazza '21, **PRL**; see also Caola, Chakraborty, Gambuti, von Manteuffel, Tancredi '21]

also Giulio Gambuti's talk

- Finite, thus has poles governed by the cusp anomalous dimension

  Also for $n_f = 0$ there are no N_c-subleading corrections, maximally non-Abelian

  Is it an eikonal quantity? Can we define it in terms of Wilson lines?

- Agrees with the planar limit of [Del Duca, Marzucca, Verbeek '21]

Infrared Constraints at four loops

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- Infrared factorisation $$\mathcal{M} = \mathbf{Z} \cdot \mathcal{H}$$ $$\mathbf{Z} = \mathbb{P} \exp \left\{ -\frac{1}{2} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \mathbf{\Gamma} \right\}$$
- Soft anomalous dimension in high-energy limit $$\mathbf{\Gamma} = \frac{1}{2} \gamma_K \left[L \mathbf{T}_t^2 + i\pi \mathbf{T}_{s-u}^2 \right] + \Gamma_i + \Gamma_j + \mathbf{\Delta}$$
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$$\mathbf{\Delta} = \sum_{n=0}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^n \sum_{m=0}^{n} L^m \mathbf{\Delta}^{(n,m)}$$

- Explicit results in the high-energy limit, in any gauge theory

NLL 4-loop [Caron-Huot '13]

$$\mathbf{\Delta}^{(4,3)} = -i\pi\frac{\zeta_3}{24}\left[\mathbf{T}_t^2, \left[\mathbf{T}_t^2, \mathbf{T}_{s-u}^2\right]\right]\mathbf{T}_t^2$$

NNLL 4-loop [Falcioni, Gardi, **CM**, Vernazza '20]

$$\mathrm{Re}\left[\mathbf{\Delta}^{(4,2)}\right] = \zeta_2\zeta_3\left\{\frac{1}{4}\mathbf{T}_t^2\left[\mathbf{T}_t^2, (\mathbf{T}_{s-u}^2)^2\right] + \frac{3}{4}\left[\mathbf{T}_{s-u}^2, \mathbf{T}_t^2\right]\mathbf{T}_t^2\mathbf{T}_{s-u}^2 + \frac{d_{AA}}{N_A} - \frac{C_A^4}{24}\right\}$$

- There exists an ansatz for $\mathbf{\Delta}^{(4)}$ in general kinematics [Becher, Neubert '19]
- Matching gives asymptotic limits of the four-loop functions [Falcioni, Gardi, Maher, **CM**, Vernazza '21]

Further Constraints

[Falcioni, Gardi, Maher, **CM**, Vernazza '21]

Let us revisit the two exponents

soft anomalous dimension | cut-scheme Regge trajectory

$$-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\mathbf{\Gamma} \quad \leftrightarrow \quad C_A\tilde{\alpha}_g(t)L$$

infrared factorisation | Regge pole

The poles of the Regge trajectory are given by the cusp in the adjoint

For the two exponents to match we need the linear term of the soft anomalous dimension to be the cusp

$$\left.\frac{d}{dL}\mathbf{\Gamma}\right|_{L=0}\mathcal{M}^{\text{tree}}_{ij\to ij} = \Gamma^{\text{cusp}}_A\mathcal{M}^{\text{tree}}_{ij\to ij}$$

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The three-loop soft anomalous dimension is bootstrapable [Almelid, Duhr, Gardi, McLeod, White '17]

We have found useful constraints for a potential four-loop bootstrap

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Thank you!

Extra Slides

(Very) brief introduction to Regge poles and cuts

[Caron-Huot, Gardi, Vernazza '17]

Amplitude via dispersion relations

$$\mathcal{M}(s,t) = \frac{1}{\pi}\int_0^\infty \frac{d\hat{s}}{\hat{s}-s-i0} D_s(\hat{s},t) + \frac{1}{\pi}\int_0^\infty \frac{d\hat{u}}{\hat{u}+s+t-i0} D_u(\hat{u},t)$$

Mellin transformed discontinuities

$$D_s(s,t) = \frac{1}{2i}\int_{\gamma-i\infty}^{\gamma+i\infty} dj\, a_j^s(t) \left(\frac{s}{-t}\right)^j$$

j is the spin of the particle in the t-channel

Even and odd amplitudes at leading power

$$\mathcal{M}^{(+)}(s,t) = i\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{\sin(\pi j)} \cos\left(\tfrac{\pi j}{2}\right) a_j^{(+)}(t)\, e^{jL},$$

$$\mathcal{M}^{(-)}(s,t) = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dj}{\sin(\pi j)} \sin\left(\tfrac{\pi j}{2}\right) a_j^{(-)}(t)\, e^{jL},$$

Regge poles

$$a_j^{(-)}(t) \simeq \frac{1}{j-1-\alpha(t)}$$

Regge trajectory

$j = 1 + \mathcal{O}(\alpha_s)$

gluon

$$\mathcal{M}^{(-)}(s,t)|_{\text{Regge pole}} \simeq \frac{\pi}{\sin\frac{\pi\,\alpha(t)}{2}} \frac{s}{t} e^{L\,\alpha(t)} + \dots$$

Regge cuts

$$a_j^{(-)}(t) \simeq \frac{1}{(j-1-\alpha(t))^{1+\beta(t)}}$$

$$\mathcal{M}^{(-)}(s,t)|_{\text{Regge cut}} \simeq \frac{\pi}{\sin\frac{\pi\,\alpha(t)}{2}} \frac{s}{t} \frac{1}{\Gamma(1+\beta(t))} L^{\beta(t)} e^{L\,\alpha(t)} + \text{subleading logs.}$$

Specific colour structures in t-channel orthonormal basis

$$c_{qq}^{[8]} = \frac{2}{\sqrt{N_c^2-1}} (t^b)^{a_4}{}_{a_1} (t^b)^{a_3}{}_{a_2},$$

$$c_{qq}^{[1]} = \frac{1}{N_c} \delta^{a_4}{}_{a_1} \delta^{a_3}{}_{a_2},$$

$$c_{gg}^{[8_a]} = \frac{1}{N_c} \frac{1}{\sqrt{N_c^2-1}} f^{a_1 a_4 b} f^{a_2 a_3}{}_b,$$

$$c_{gg}^{[10+\bar{10}]} = \sqrt{\frac{2}{(N_c^2-4)(N_c^2-1)}} \left[\frac{1}{2} (\delta^{a_1}{}_{a_2} \delta^{a_3}{}_{a_4} - \delta^{a_3}{}_{a_1} \delta^{a_4}{}_{a_2}) - \frac{1}{N_c} f^{a_1 a_4 b} f^{a_2 a_3}{}_b \right],$$

$$c_{qg}^{[8_a]} = \sqrt{\frac{2}{N_c(N_c^2-1)}} (t^b)^{a_4}{}_{a_1} i f^{a_2 a_3 b}.$$

Formulating highly energetic partons as Wilson lines

[Caron-Huot '13; Caron-Huot, Gardi, Vernazza '17]

"eikonal approximation"

Fourier conjugate of t

$$U(z_\perp) = \mathbf{P}\exp\left[ig_s \mathbf{T}^a \int_{-\infty}^{\infty} dx^+ A_+^a(x^+, x^- = 0, z_\perp)\right]$$

regulate rapidity divergences by tilting Wilson-line off the light cone

$$\eta = \frac{1}{2}\log\frac{p_+}{p_-}$$

$$T^a_{i,L} = [\mathbf{T}^a U(z_i)]\frac{\delta}{\delta U(z_i)}$$

$$T^a_{i,R} = [U(z_i)\mathbf{T}^a]\frac{\delta}{\delta U(z_i)}$$

our parton is a collection of such Wilson lines

evolves according to Balitsky-JIMWLK

$$\frac{d}{d\eta}|\psi_i\rangle = -H|\psi_i\rangle \qquad H = \frac{\alpha_s}{2\pi^2}\int d^d z_i d^d z_j d^d z_0 \frac{z_{0i}\cdot z_{0j}}{(z_{0i}^2 z_{0j}^2)^{1-\epsilon}}\left\{T^a_{i,L}T^a_{j,L} + T^a_{i,R}T^a_{j,R} - U^{ab}_{\text{ad}}(z_0)\left(T^a_{i,L}T^b_{j,R} + T^a_{j,L}T^b_{i,R}\right)\right\}$$

Target Projectile

The amplitude is then written as $\mathcal{M}_{ij\to ij} \sim \langle\psi_j|e^{-HL}|\psi_i\rangle$

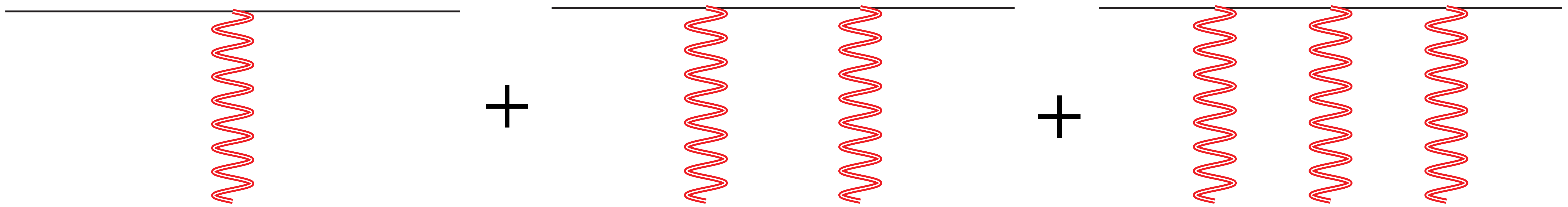
Target and projectile separated by some $z_\perp^2 \leftrightarrow t$

Evolve so that they are at equal rapidity

Expand Wilson line in *Reggeons*

Reggeon field

$$U = \exp\left[ig_s \mathbf{T}^a W^a\right] \sim$$



Only odd/even number of Reggeons contribute to the odd/even amplitude

In this framework the NLL divergences have been resummed and finite parts known to very high loop order

[Caron-Huot, Gardi, Reichel, Vernazza '17, '20]

$$\hat{\mathcal{M}}^{(+)}_{\text{NLL}}\Big|_s = \frac{i\pi}{L(C_A - \mathbf{T}_t^2)}\left(1 - R(\epsilon)\frac{C_A}{C_A - \mathbf{T}_t^2}\right)^{-1} \times \left[\exp\left\{\frac{B_0(\epsilon)}{2\epsilon}\frac{\alpha_s}{\pi}L(C_A - \mathbf{T}_t^2)\right\} - 1\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(\text{tree})} + \mathcal{O}(\epsilon^0).$$

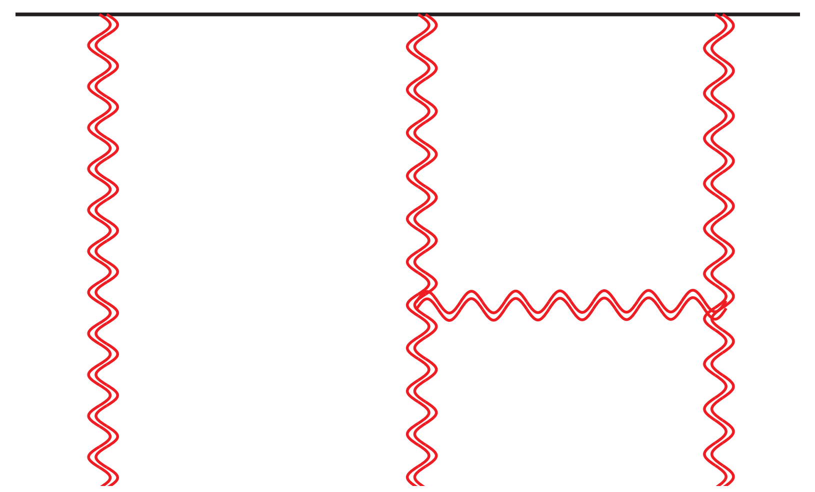
Can we achieve the same at NNLL?

Explicit momentum-space Hamiltonians

The diagonal transitions are

dresses one Reggeon with the trajectory

adds a *rung* between two Reggeons



$$H_{k\to k} = -\int d^dp\, C_A \alpha_g(p) W^a(p) \frac{\delta}{\delta W^a(p)}$$

$$+\,\alpha_s \int d^dq d^dp_1 d^dp_2\, H_{22}(q;p_1,p_2) W^x(p_1+q) W^y(p_2-q) (F^x F^y)^{ab} \frac{\delta}{\delta W^a(p_1)} \frac{\delta}{\delta W^b(p_1)}$$

with kernel $H_{22}(q;p_1,p_2) = \dfrac{(p_1+p_2)^2}{p_1^2 p_2^2} - \dfrac{(p_1+q)^2}{p_1^2 q^2} - \dfrac{(p_2-q)^2}{q^2 p_2^2}$

Source of the difficulty at NNLL. Three Reggeons spoil the symmetry between colour and kinematics, which is there for two Reggeons (NLL).

The off-diagonal transitions are

$$H_{1\to 3} = \alpha_s^2 \int d^dp_1 d^dp_2 d^dp\ \mathrm{tr}\left[F^a F^b F^c F^d\right] W^b(p_1) W^c(p_2) W^d(p_3) H_{13}(p_1,p_2,p_3) \frac{\delta}{\delta W^a(p)}$$

with kernel $H_{13}(p_1,p_2,p_3) = \dfrac{r_\Gamma}{3\epsilon}\left[\left(\dfrac{\mu^2}{(p_1+p_2+p_3)^2}\right)^\epsilon + \left(\dfrac{\mu^2}{p_2^2}\right)^\epsilon - \left(\dfrac{\mu^2}{(p_1+p_2)^2}\right)^\epsilon - \left(\dfrac{\mu^2}{(p_2+p_3)^2}\right)^\epsilon\right]$

The odd NNLL amplitude

$$\mathcal{M}_{ij\to ij} \sim \langle \psi_j | e^{-HL} | \psi_i \rangle$$

Expand in terms of the coupling and logarithms

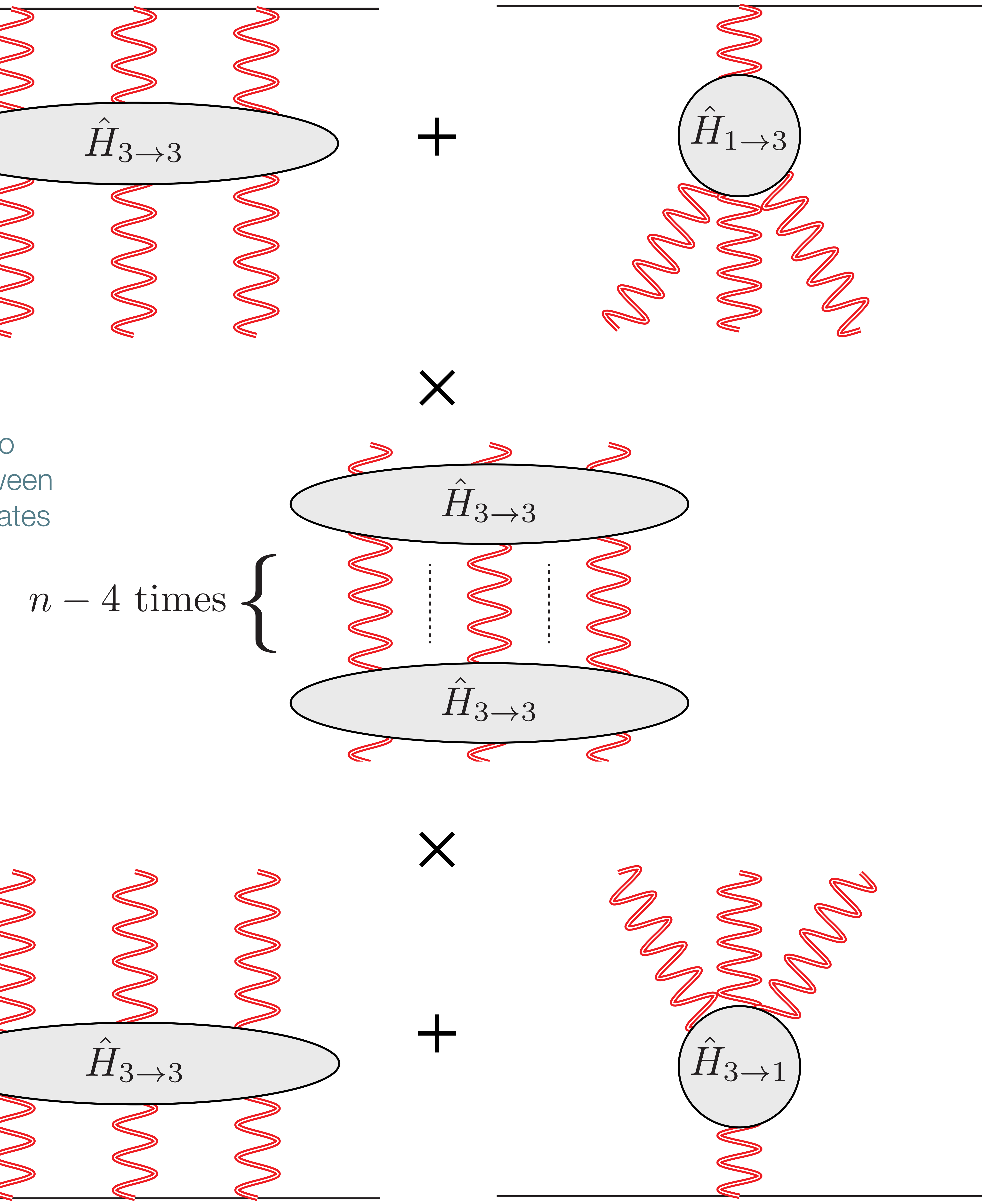
$$|\psi_i\rangle = \sum_{k=1}^{\infty} g_s^k \, |\psi_{i,k}\rangle \qquad\qquad \hat{H}_{m\to n} \sim g_s^{|m-n|}$$

Expand Hamiltonian to describe transitions between different Reggeon (W) states

We get the all-order expression for the n-loop amplitude

$$\hat{\mathcal{M}}^{(-),\mathrm{NNLL}}_{ij\to ij} \sim \left(\frac{\alpha_s}{\pi}\right)^n \Big(\langle \psi_{j,3} | \, \hat{H}_{3\to 3} + \langle \psi_{j,1} | \, \hat{H}_{3\to 1} \Big)$$
$$\times \, \hat{H}^{n-4}_{3\to 3}$$
$$\times \Big(\hat{H}_{3\to 3} \, |\psi_{j,3}\rangle + \hat{H}_{1\to 3} \, |\psi_{j,1}\rangle \Big)$$
$$+ \, \text{explicit two-loop information}$$

Only require Leading Order Balitsky-JIMWLK Hamiltonian and results from two-loop amplitudes (impact factors)



NLL pole parameters

Two-loop Regge trajectory

[Fadin, Kotsky, Fiore ’95; Fadin, Fiore, Kotsky ’96;
Fadin, Fiore, Quartarolo ’96; Blumlein, Ravindran, van Neerven '98]

$$\alpha_g^{(2)} = -\frac{b_0}{16\epsilon^2} + \frac{1}{\epsilon}\left[C_A\left(\frac{67}{144} - \frac{\zeta_2}{8}\right) - \frac{5n_f}{72}\right] + C_A\left(\frac{101}{108} - \frac{\zeta_3}{8}\right) - \frac{7n_f}{54} + \mathcal{O}(\epsilon)$$

One-loop impact factors

[Fadin, Fiore ’92; Fadin, Lipatov ’93; Fadin, Fiore, Quartarolo ’94;
Lipatov ’97; Del Duca, Schmidt ’98; Bern, Del Duca, Schmidt ‘98]

$$C_q^{(1)} = -\frac{C_F}{2\epsilon^2} - \frac{3C_F}{4\epsilon} + C_A\left(\frac{85}{72} + \frac{3\zeta_2}{4}\right) + C_F\left(\frac{\zeta_2}{4} - 2\right) - \frac{5n_f}{36} + \mathcal{O}(\epsilon)$$

$$C_g^{(1)} = -\frac{C_A}{2\epsilon^2} - \frac{b_0}{4\epsilon} + C_A\left(\zeta_2 - \frac{67}{72}\right) + \frac{5n_f}{36} + \mathcal{O}(\epsilon)$$