Yang–Mills All-Plus
Two Loops for the Price of One

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Based on work with David Kosower (IPhT Saclay) [2205.xxxxx; ...]

Loops & Legs 2022, Ettal
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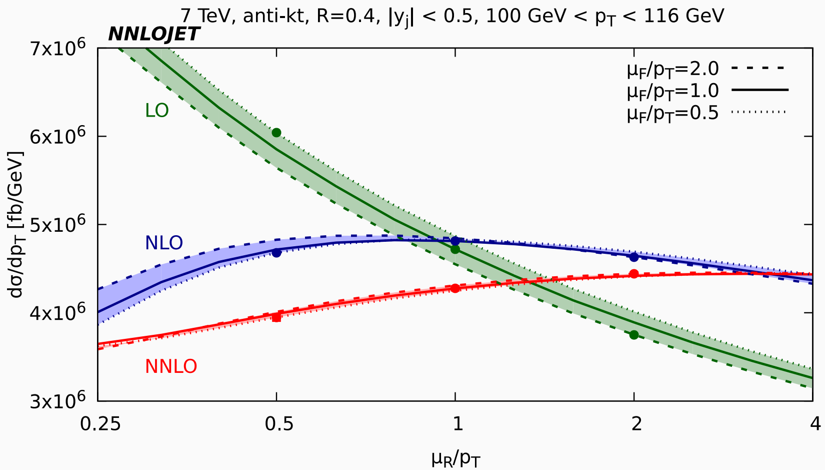
LHC: Corrections Required

Higher order corrections necessary for LHC precision measurements

– *e.g.* α_s: requires NNLO corrections for 2- and 3-jet QCD processes

Theory predictions:

- **LO**: qualitative
- **NLO**: quantitative, ~10%
- **NNLO**: "precision", accuracy of order ~3%



[Currie, Glover, Gehrmann-De Ridder, Gehrmann, Huss, Pires, 1704.00923]

Feynman diagram introduce gauge redundancies → **On-Shell Methods**

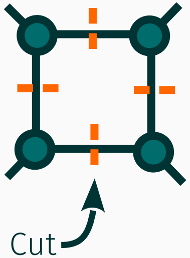
Build complicated on-shell amplitudes from simpler ones

Loop-level:
Generalized Unitarity

[BDDK, hep-ph/9403226, hep-ph/9409265; BDK, hep-ph/9708239; BCF, hep-th/0412103; ...]

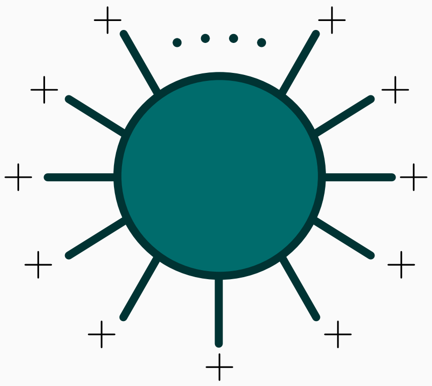
$\mathcal{A}^{(1)} = \mathrm{C}^{(1)}_{\mathrm{Box}}\, I_4^D \big[$  $\big] + \mathrm{C}^{(1)}_{\mathrm{Tri}}\, I_3^D \big[$  $\big]$

$+\, \mathrm{C}^{(1)}_{\mathrm{Bub}}\, I_2^D \big[$  $\big] + \mathrm{C}^{(1)}_{\mathrm{Tad}}\, I_1^D \big[$  $\big]$

$\mathrm{C}^{(1)}_{\mathrm{Box}} \sim \mathrm{disc}_{\mathrm{Box}} \Big[$  $\Big] \sim$ 

Generalized Unitarity Cut

Yang–Mills All-Plus



Yang–Mills Amplitudes: we consider purely gluonic case

All gluons have same helicity:
- Most symmetric case
- $A^{(0)}(1^+ \ldots n^+) = 0$

In YM: only at tree-level
With SUSY: for all loop orders

Yang–Mills All-Plus: Loop-Level Features

Closed all-n form known at **one-loop**

$$\mathrm{tr}\left(\frac{1+\gamma_5}{2}\not{i}\not{j}\not{k}\not{l}\right)$$

$$A^{(1)}(1^+\dots n^+) = -\frac{1}{3}\frac{\sum_{1\leq i<j<k<l\leq n}\langle i|jkl|i]}{\langle 12\rangle\langle 23\rangle\dots\langle (n-1)n\rangle\langle n1\rangle} + \mathcal{O}(\epsilon)$$

Conjectured [Bern, Chalmers, Dixon, Kosower, hep-ph/9312333]

Proven [Mahlon, hep-ph/9312276; BDK, hep-th/0501240]

No $\frac{1}{\epsilon}$ divergences

No branch cuts

} **Tree-like!**

Connected to $\mathcal{N}=4$ sYM MHV at **one-loop, two-loop (and beyond)**

[BDDK, hep-th/9611127; Britto, Jehu, Orta, 2011.13821]
[BDDK, hep-ph/0001001; Badger, Frellesvig, Zhang, 1310.1051; Badger, Mogull, Peraro, 1606.02244]
[Chicherin, Henn, 2202.05596, 2204.00329]

Decomposition of Two-Loop Amplitudes

[Catani, hep-ph/9802439; Sterman, Tejeda-Yeomans, hep-ph/0210130]

$$A_n^{(2)} \simeq \underbrace{\left(A_n^{(0)}\mathcal{I}_n^{(2)}\right)}_{\text{Tree} \times \text{2-Loop}} + \underbrace{\left(A_n^{(1)}\mathcal{I}_n^{(1)}\right)}_{\text{1-Loop} \times \text{1-Loop}} + \underbrace{P_n^{(2)} + R_n^{(2)}}_{\text{Finite } F^{(2)}}$$

$P_n^{(2)}$: Finite Polylogs

$R_n^{(2)}$: Rational Part

Universal IR structures $\begin{cases} \mathcal{I}_n^{(2)} \sim \frac{1}{\epsilon^4} \\ \mathcal{I}_n^{(1)} \sim \frac{1}{\epsilon^2} \end{cases}$

Decomposition of Two-Loop All-Plus

$$A_n^{(2)} \simeq \underbrace{\left(A_n^{(0)}\mathcal{I}_n^{(2)}\right)}_{\text{Tree} \times \text{2-Loop}} + \underbrace{\left(A_n^{(1)}\mathcal{I}_n^{(1)}\right)}_{\text{1-Loop} \times \text{1-Loop}} + \underbrace{P_n^{(2)} + R_n^{(2)}}_{\text{Finite } F^{(2)}} \sim \mathcal{O}\left(\frac{1}{\epsilon^2}\right)$$

$A_n^{(0)}$ → 0

$A_n^{(1)}$ → $\mathcal{O}(\epsilon^0)$

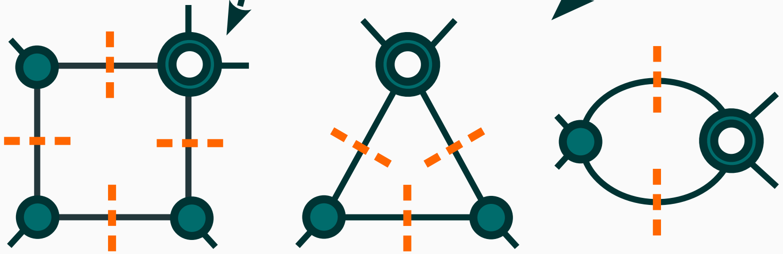
Universal IR structures $\begin{cases} \mathcal{I}_n^{(2)} \sim \frac{1}{\epsilon^4} \\ \mathcal{I}_n^{(1)} \sim \frac{1}{\epsilon^2} \end{cases}$

Polylog part of two-loop all-plus

Expressible as nested loops

$$A_n^{(2)} \simeq \underbrace{\left(A_n^{(1)}\mathcal{I}_n^{(1)}\right) + \mathcal{P}_n^{(2)}} + \underbrace{\mathcal{R}_n^{(2)}}$$

One-loop all-plus



All-n expressions available

[Dunbar, Jehu, Perkins, 1604.06631]

?

Full 4-, 5-, 6 gluon results known

[Bern, Dixon, Kosower, hep-ph/0001001; Bern, Freitas Dixon,hep-ph/0201161; Gehrmann, Lo Presti, 1511.05409; Dunbar, Jehu, Perkins, 1604.06631; Badger, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, Zoia, 1905.03733; Dunbar, Godwin, Perkins, Strong, 1911.06547; Dalgleish, Dunbar, Perkins, Strong, 2003.00897]

as well as partial 7- and n-gluon

[Dunbar, Jehu, Perkins, 1710.10071; Dunbar, Perkins, Strong, 2001.11347]

Dimensional Reconstruction

Extract analytic dependence on D_s through interpolation

$$A^{(2)} = A^{2g}_{6\mathrm{D}} + (D_s - 6) A^{sg}_{6\mathrm{D}} + (D_s - 6)^2 A^{2s}_{6\mathrm{D}}$$

Space-time dimension: number of states

Six-dimensional amplitudes

Conjecture (BMP) for all-plus at leading color:

[Badger, Mogull, Peraro, 1606.02244, 1607.00311]

$N_c^2 \ \mathrm{Tr}$

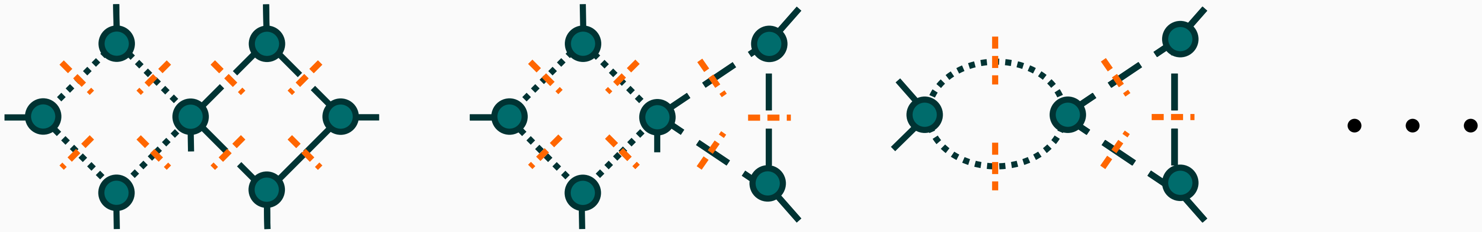
$$F^{(2)} = \underbrace{(D_s - 2) P^{(2)}}_{\text{Only polylogs}} + \underbrace{(D_s - 2)^2 R^{(2)}}_{\text{Only rational}} + \mathcal{O}(\epsilon)$$

Comparing coefficients

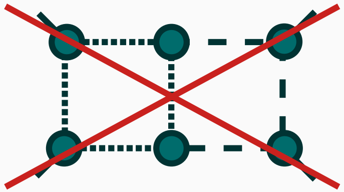
$R^{(2)}$ determined by rational part of $A^{2s}_{6\mathrm{D}}$

From $A_{6\mathrm{D}}^{2s}$ to $R_{n:1}^{(2)}(1^+ \ldots n^+)$

$A_{6\mathrm{D}}^{2s}$: Loops of different scalar flavors



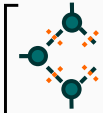
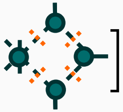
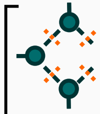
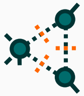
Feynman rules are flavour conserving



Forbidden

Only (one-loop)² topologies

Reformulation of leading color BMP conjecture: Two Nested One-Loop Unitarity Computations

$\mathcal{R}^{(2)} \propto \mathcal{R}^{(1)} \Big[$  $\times \mathcal{R}^{(1)} [$  $] \Big] + \mathcal{R}^{(1)} \Big[$  $\times \mathcal{R}^{(1)} [$  $] \Big] + \ldots$

Acts as “tree-amplitude”

Go further: Extend to full amplitude.

Color Decomposition of Two-Loop Amplitudes

[Dalgleish, Dunbar, Perkins, Strong, 2003.00897]

Separating color and kinematics

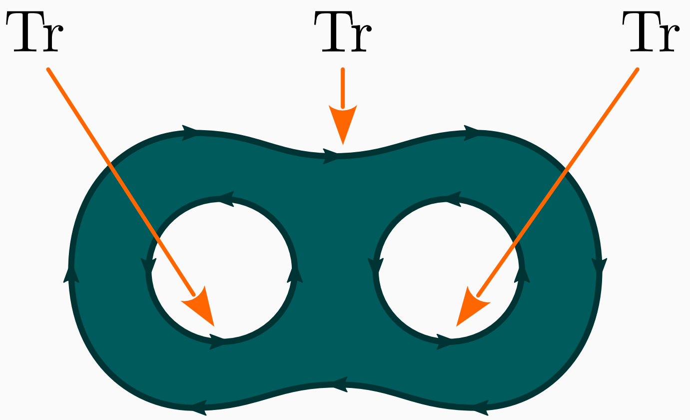
Partial amplitudes (only kinematics)

$$\mathcal{A}_n^{(2)} = N_c^2 \mathrm{Tr}\; A_{n:1}^{(2)} + N_c \mathrm{Tr}\; \mathrm{Tr}\; A_{n:i}^{(2)} + \mathrm{Tr}\; \mathrm{Tr}\; \mathrm{Tr}\; A_{n:i,j}^{(2)} + \mathrm{Tr}\; A_{n:1\mathrm{B}}^{(2)}$$

Color structures (only color)

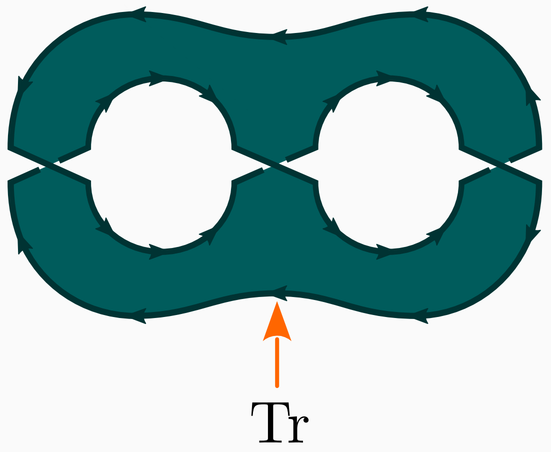
Stringy Picture for Color Traces

Three Boundaries



$$\ni \begin{cases} N_c^2 \mathrm{Tr} \\ N_c \mathrm{Tr}\ \mathrm{Tr} \\ N_c^0 \mathrm{Tr}\ \mathrm{Tr}\ \mathrm{Tr} \end{cases}$$

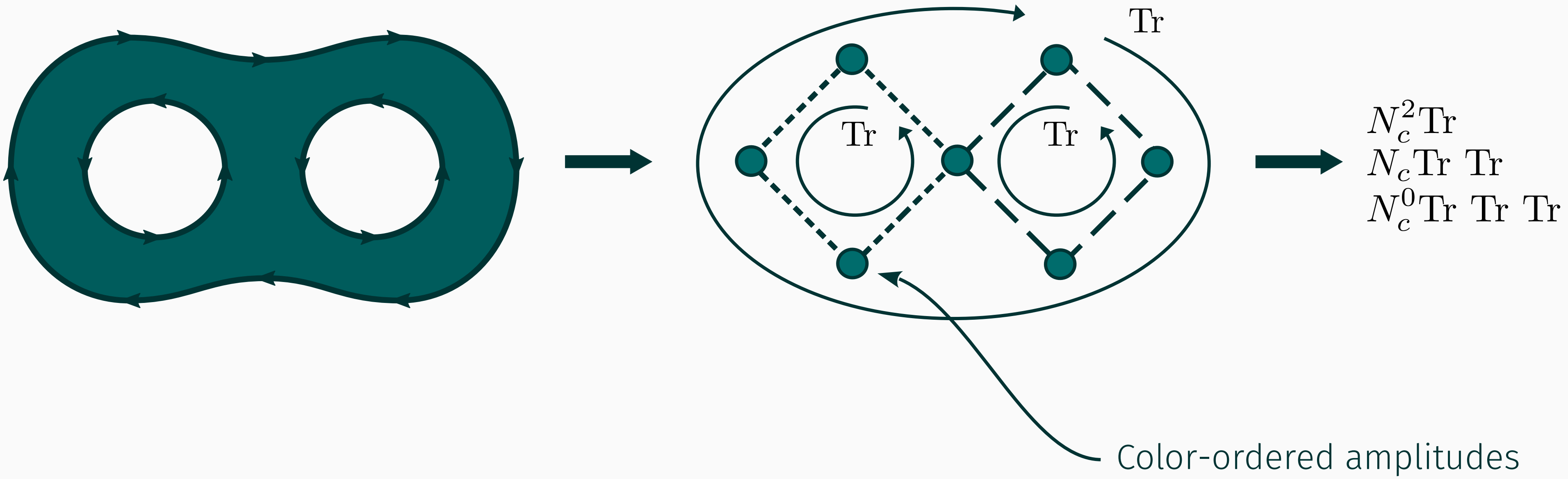
$\mathrm{Tr}(\mathbb{1}) = N_c$



Single Boundary

$$\ni \quad N_c^0 \mathrm{Tr}$$

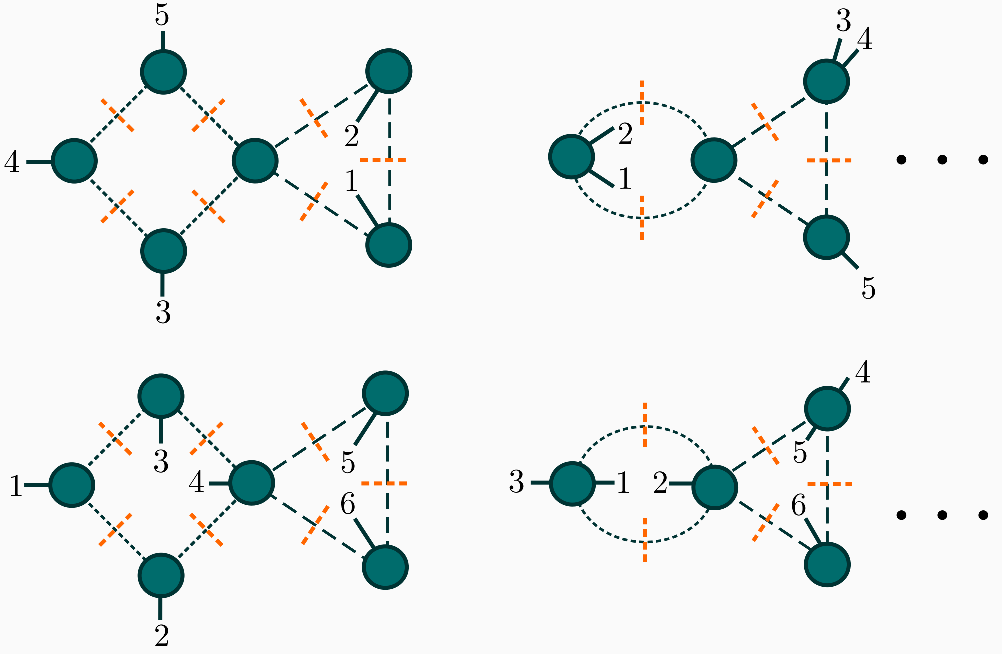
Use stringy picture as a guide to construct non-planar cuts



Examples

$\mathrm{Tr}(12)\mathrm{Tr}(345)$

$\mathrm{Tr}(12)\mathrm{Tr}(34)\mathrm{Tr}(56)$



$\longrightarrow R^{(2)}_{5:3}$

$\longrightarrow R^{(2)}_{6:2,2}$

Again (one-loop)² topologies sufficient

Verification

Numerical agreement with all known results

[Bern, Dixon, Kosower, hep-ph/0001001; Bern, Freitas, Dixon,hep-ph/0201161;
Badger, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, Zoia, 1905.03733
Dunbar, Godwin, Perkins, Strong, 1911.06547; Dalgleish, Dunbar, Perkins, Strong, 2003.00897]

Generation and evaluation of cuts automated in Mathematica ← optimized series expansion

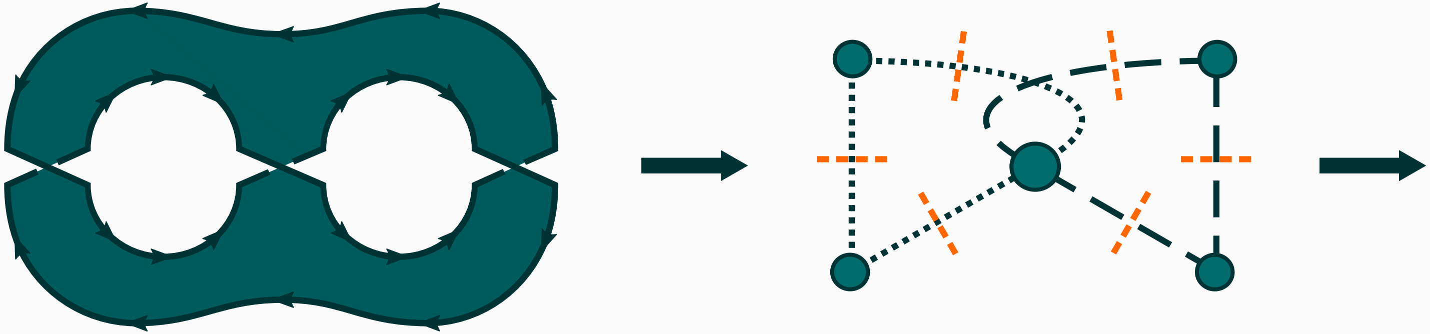
- Rational kinematics ⇒ exact results
- Analytics (n-point momentum twistor parametrization)

Number of cuts required:

n	4	5	6	7
$R^{(2)}_{n:1}$	28	165	894	2891
$R^{(2)}_{n:3}$	108	1026	5832	22400
$R^{(2)}_{n:4}$	—	—	6615	29064
$R^{(2)}_{n:2,2}$	—	—	25344	129120

Literature results not available (for 22400, 29064, 129120)

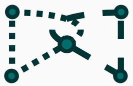
Subleading single trace $R^{(2)}_{n:1\mathrm{B}}$



$N_c^0 \mathrm{Tr}$

All-n conjecture for $R^{(2)}_{n:1\mathrm{B}}$ exists: High multiplicity checks

[Dunbar, Perkins, Strong, 2001.11347]

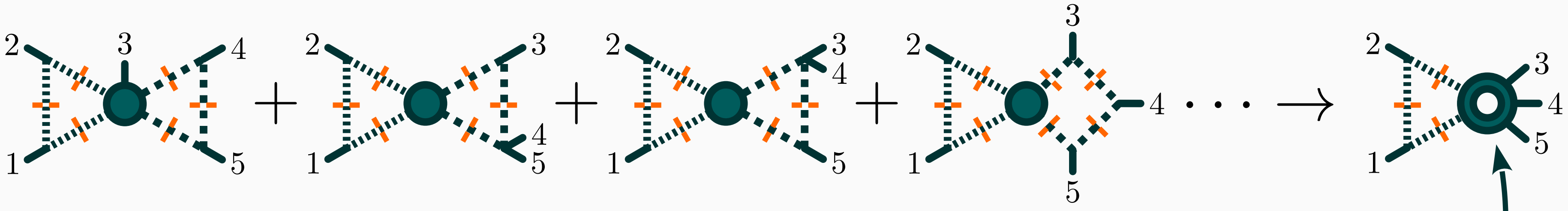
 only

All boxes and bubbles vanish

One-loop squared construction numerically verified up to **9 gluons!**

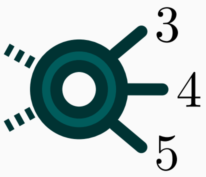
	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$
$R^{(2)}_{n:1\mathrm{B}}$	168	1925	16068	89257	400053	(77247)

Reorganization of cuts leads to
Explicit One-Loop Picture for Rational Part



Scalar one-loop amplitude
(rather its rational part)

Multiple routes
to obtain



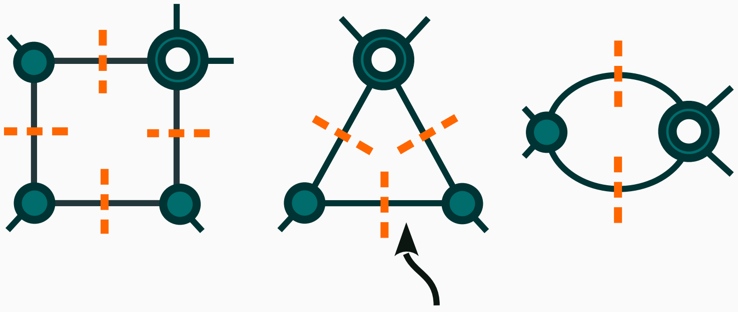
Unitarity

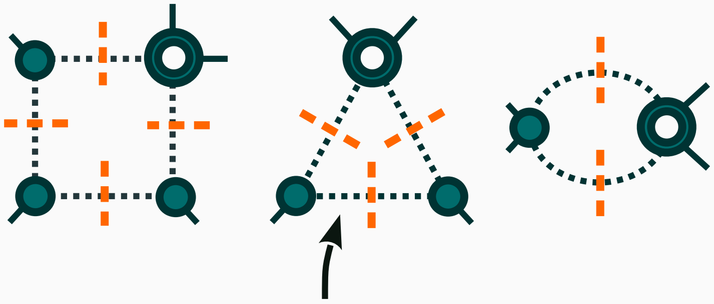
Complex
Recursion

Closed analytic
forms

Conjecture: One-loop construction extends to **Entire Full-Colour Two-Loop All-Plus Amplitude**

$$A_n^{(2)} \simeq \underbrace{\left(A_n^{(1)}\mathcal{I}_n^{(1)}\right) + \mathcal{P}_n^{(2)}} + \underbrace{\mathcal{R}_n^{(2)}}$$





Summary

Cut-constructible part of two-loop all-plus amplitudes:

- Loops can be nested, to be evaluated one-by-one

Rational terms:

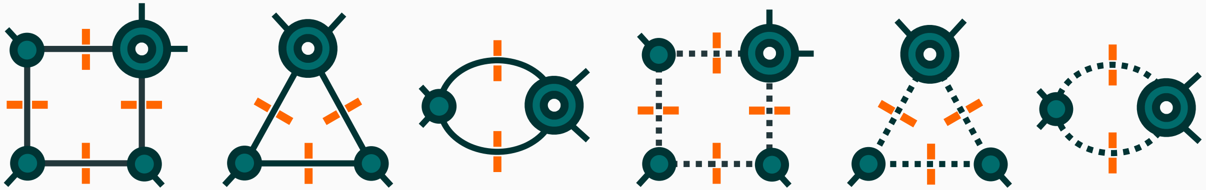
- Nested loop picture also applicable

$$\mathcal{R}^{(2)} \propto \mathcal{R}^{2\mathrm{s}(2)}_{6\mathrm{D}} = \mathcal{R}^{(1)}\Big[\mathcal{R}^{(1)}\Big]$$

Full-Colour All-Plus Amplitude

- Formulate as an effective one-loop unitarity computation

$$\mathcal{A}^{(2)}(1^+ \ldots n^+) =$$

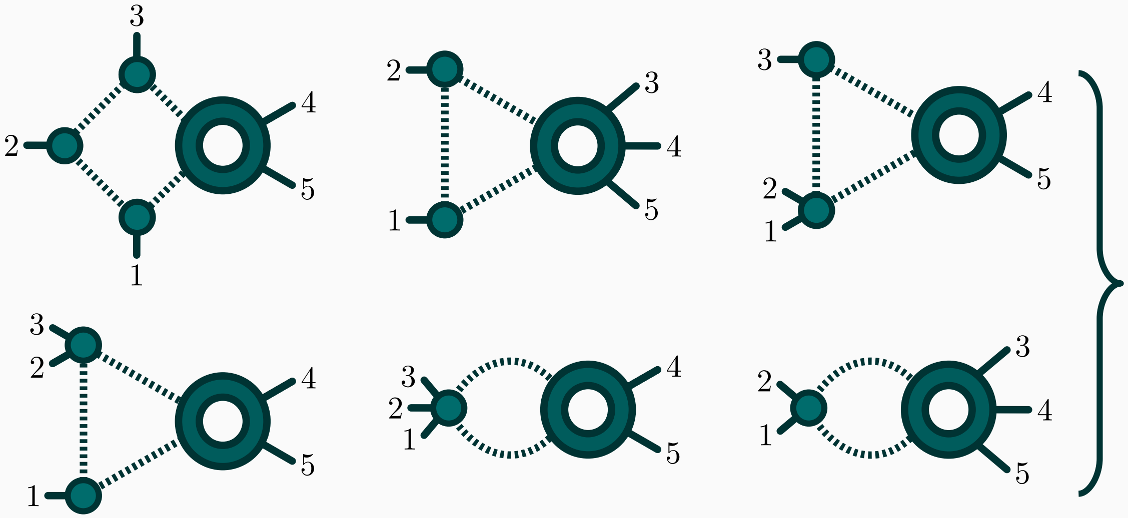


Future Directions

- **High-Multiplicity:** As a start, prove $\mathcal{R}^{(2)}_{n:1\mathrm{B}}$ conjecture
- **Extension to other helicities:** How much survives for single-minus?
- **Extension to gravity:** All-Plus graviton amplitudes
- **Conformal properties:** Explore mechanism of conformal symmetry breaking of all-plus at two-loop level

Backup

Example:



$$\sum_{\text{cyc.}} \longrightarrow R^{(2)}_{5:1}(1^+2^+3^+4^+5^+)$$

Momentum Twistor Parametrization

$$Z = \begin{pmatrix} 1 & 0 & y_1 & y_2 & y_3 & \dots & y_{n-3} & y_{n-2} \\ 0 & 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 0 & 0 & \frac{b_{n-4}}{a_2} & \tilde{b}_{n-5} & \dots & \tilde{b}_1 & 1 \\ 0 & 0 & 1 & 1 & \tilde{c}_{n-4} & \dots & \tilde{c}_2 & \tilde{c}_1 \end{pmatrix}$$

$$y_k = \sum_{i=1}^{k} \prod_{j=1}^{i} \frac{1}{a_j},$$

$$\tilde{b}_k = \tilde{b}_{k-1} + a_{n-k}(\tilde{b}_{k-1} - \tilde{b}_{k-2}) + b_k,$$

$$\tilde{c}_k = \tilde{c}_{k-1} + a_{n-k+1}(\tilde{c}_{k-1} - \tilde{c}_{k-2}) + \frac{b_{k-1}}{b_{n-4}}(c_k - 1),$$

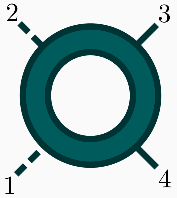
$$a_1 = s_{12}$$

$$a_{k>1} = -\frac{\langle k, k+1\rangle \langle k+2, 1\rangle}{\langle 1, k\rangle \langle k+1, k+2\rangle},$$

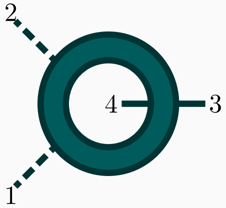
$$b_{n-4} = \frac{s_{23}}{s_{12}},$$

$$b_k = \frac{\langle n-k|n-k+1|2]}{\langle n-k|1|2]},$$

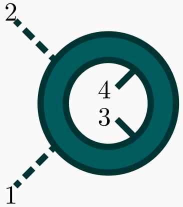
$$c_k = -\frac{\langle 1|3|n-k+2]}{\langle 1|2|n-k+2]}$$



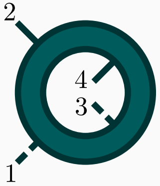
$$R^{(1)}_{4:1}(1^\varphi 2^\varphi 3^+ 4^+) = -\frac{1}{3}\frac{s_{13}-s_{34}}{\langle 34\rangle^2},$$



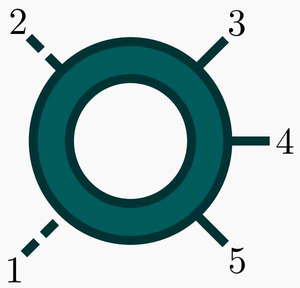
$$R^{(1)}_{4:2}(1^\varphi 2^\varphi 3^+; 4^+) = \frac{[23]}{\langle 23\rangle}$$



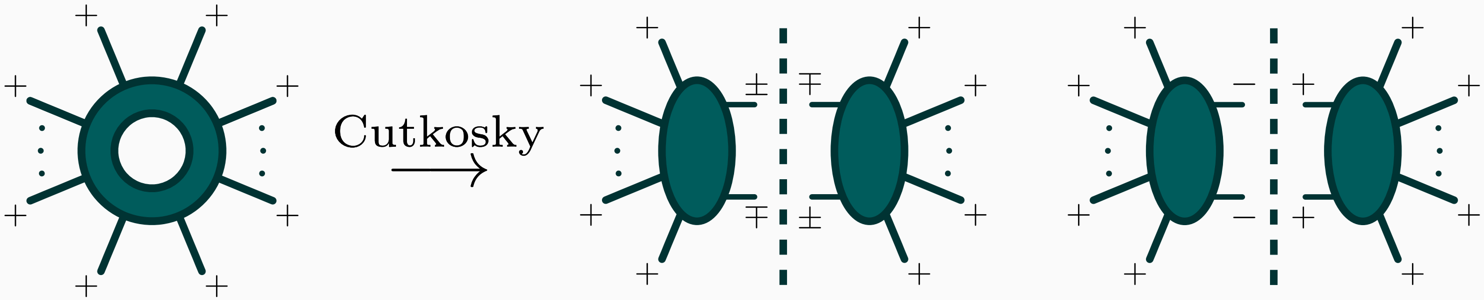
$$R^{(1)}_{4:3}(1^\varphi 2^\varphi; 3^+ 4^+) = -\frac{[34]}{\langle 34\rangle}$$



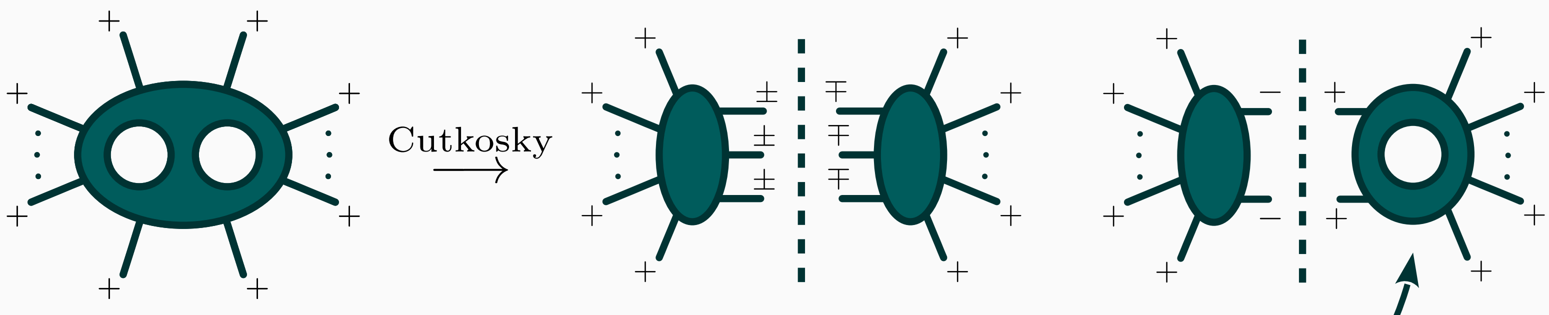
$$R^{(1)}_{4:3}(1^\varphi 2^+; 3^\varphi 4^+) = -2\frac{[24]}{\langle 24\rangle}$$



$$R^{(1)}_{5:1}(1^\varphi 2^\varphi 3^+ 4^+ 5^+)$$
$$= \frac{1}{3}\Big[-\frac{m^2 \langle 35\rangle [3|12|3] [35]^3}{s_{12}\langle 45\rangle [3|2(3+4)|5] (s_{12}\langle 4|2|3] - \langle 34\rangle [3|12|3])}$$
$$+ \frac{m^2 \langle 5|2|3] [35]^3}{\langle 45\rangle [3|2(3+4)|5] (s_{12}\langle 4|2|3] - \langle 34\rangle [3|12|3])} - \frac{m^2 \langle 4|1|5] [3|2(3+4)|5] [34]}{{s_{15}}^2 \langle 34\rangle \langle 45\rangle [3|(1+2)1|5]}$$
$$- \frac{m^2 [3|42|3] [35]^2 [45]}{\langle 34\rangle \langle 45\rangle [3|(1+2)1|5] [3|2(3+4)|5] [34]} + \frac{2 s_{12} [34] [3|42|3]}{s_{23}\langle 34\rangle \langle 5|4|3]^2}$$
$$- \frac{[3|21|3] [3|2(3+4)|5] [34]}{s_{12}\langle 5|4|3] (s_{12}\langle 4|2|3] - \langle 34\rangle [3|12|3])} - \frac{s_{23}\langle 4|1|5]^2 [34]}{s_{15}\langle 34\rangle^2 \langle 45\rangle [3|(1+2)1|5]} + \frac{s_{15} [34] [3|42|3]}{s_{23}\langle 34\rangle \langle 5|4|3]^2}$$
$$+ \frac{\langle 4|1|5]^2 \langle 5|1(3+4)2|3] [34]^2}{{s_{15}}^2 \langle 34\rangle \langle 45\rangle \langle 5|4|3] [3|(1+2)1|5]} + \frac{\langle 4|1|5] [3|2(3+4)|5] [34]}{s_{15}\langle 34\rangle \langle 45\rangle [3|(1+2)1|5]} - \frac{2 [3|12|3] [34]^2}{s_{23}\langle 5|4|3]^2}\Big].$$

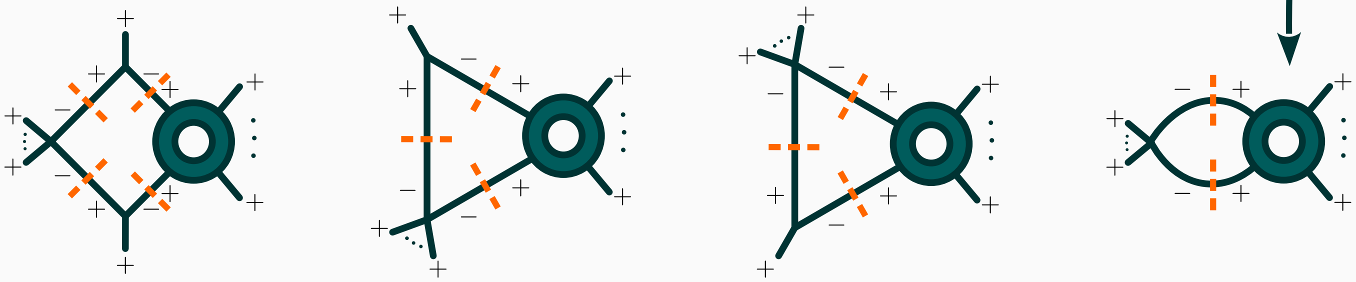

Cutkosky

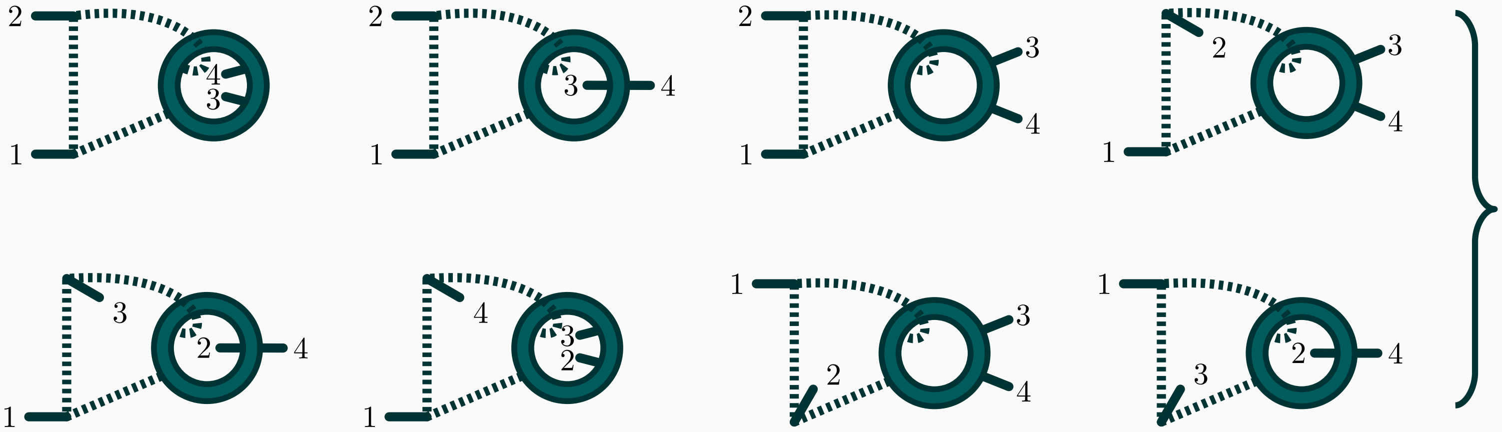
Two-loop Unitarity Cuts



One-Loop Generalized Cuts

No further cuts possible





$$\xrightarrow{\sum_{\text{cyc.}}} R^{(2)}_{4:1\mathrm{B}}$$

Spinor-Helicity Variables

Spinor-helicity variables encode both helicity and kinematics

Kinematics: $\langle ij\rangle = -\sqrt{s_{ij}}e^{\phi_{ij}}$ $\quad [ij] = \sqrt{s_{ij}}e^{-\phi_{ij}}$ $\quad \langle ij\rangle\,[ji] = s_{ij}$

Helicity (e.g. gluons): $\varepsilon^-_\mu(p) = \frac{1}{\sqrt{2}}\frac{\langle p|\sigma_\mu|q]}{[pq]}$ $\quad \varepsilon^+_\mu(p) = -\frac{1}{\sqrt{2}}\frac{[p|\sigma_\mu|q\rangle}{\langle pq\rangle}$

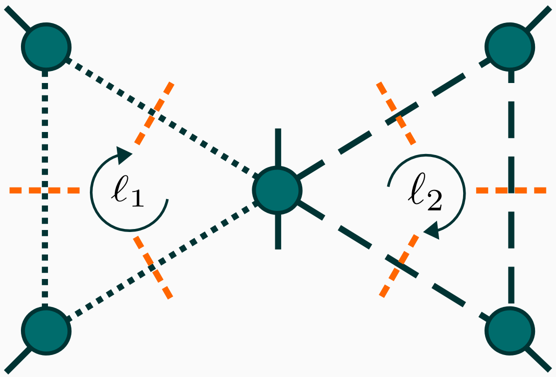
Natural variables for amplitudes, allow compact expressions

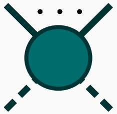
Famous Example: Parke-Taylor Amplitude

$$A^{(0)}(1^+\ldots i^-\ldots j^-\ldots n^+) = \frac{\langle ij\rangle^4}{\langle 12\rangle\,\langle 23\rangle\ldots\langle n1\rangle}$$

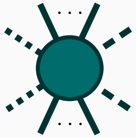
In on-shell approach, proof needs half a page

Only One-Loop Ingredients Needed





Known for arbitrary number of gluons
[Ferrario, Rodrigo, Talavera, hep-th/0602043]



Computable via recursion

$\ell_{\square}, \ell_{\triangle}, \ell_{\circ}$ Parametrization following [Badger, 0806.4600]

$\mathrm{C}_{\square}[\mu^4], C_{\triangle}[\mu^2], \mathrm{C}_{\circ}[\mu^2]$ Residues via series expansion in Mathematica. Parameter integrals known
[Forde, 0704.1835, Kilgore, 0711.5015]

$\mathrm{I}_{\square}[\mu^4], I_{\triangle}[\mu^2], \mathrm{I}_{\circ}[\mu^2]$ Integrals are well known, *e.g.* [Badger, 0806.4600]

Loop-Level BCFW On-Shell Recursion

Tree-level BCFW: Only factorizable single poles

$$A^{(0)}(z) = \frac{g^{(0)}(z)}{(z-z_i)} + \mathcal{O}\left((z-z_i)^0\right)$$

Factorizing,
i.e. originate from on-shell propagator

Loop-level BCFW: Pole structure more complicated

$$A^{(1)}(z) = \frac{f^{(1)}(z)}{(z-z_i)^2} + \frac{g^{(1)}(z)}{(z-z_i)} + \mathcal{O}\left((z-z_i)^0\right)$$

$$\frac{f^{(1)}(z_i)}{(z-z_i)^2} + \frac{f^{(1)\prime}(z_i)}{(z-z_i)} + \mathcal{O}((z-z_i)^0)$$

Factorizing and non-factorizing

$$\mathrm{Split}_+^{(1)}(1^+2^+) \sim \frac{[12]}{\langle 12\rangle^2}$$

"Pole-under-pole" term
[Dunbar, Ettel, Perkins, 1003.3398]

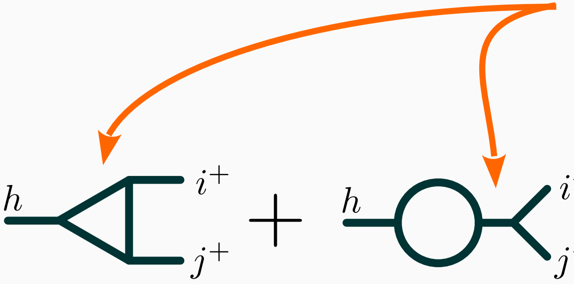
Not obtainable from factorization

Computing Missing-Pole Pieces

Origin of double poles:
[Bern, Chalmers, hep-ph/9503236]

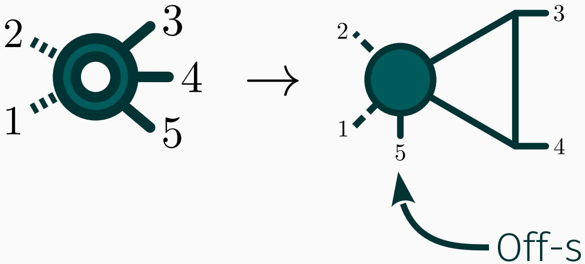
Extra poles from triangle integral and propagator

$$\mathrm{Split}^{(1)}_{+}(i^{+}j^{+}) \sim$$



Existing approach:
Augmented Recursion (off-shell)

[Dunbar, Ettle, Perkins, 1003.3398;
Alston, Dunbar, Perkins, 1208.0190, 1507.08882]

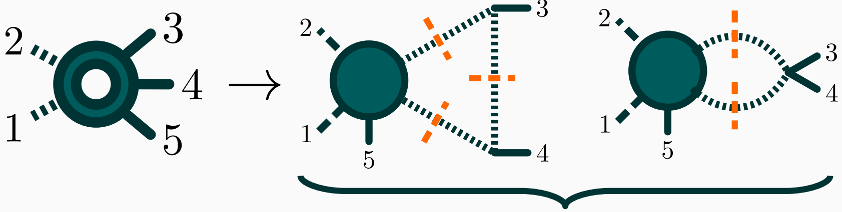


Used at two-loop for all-plus rational terms
[Dunbar, Jehu, Perkins, 1603.07514; ...]

Off-shell current

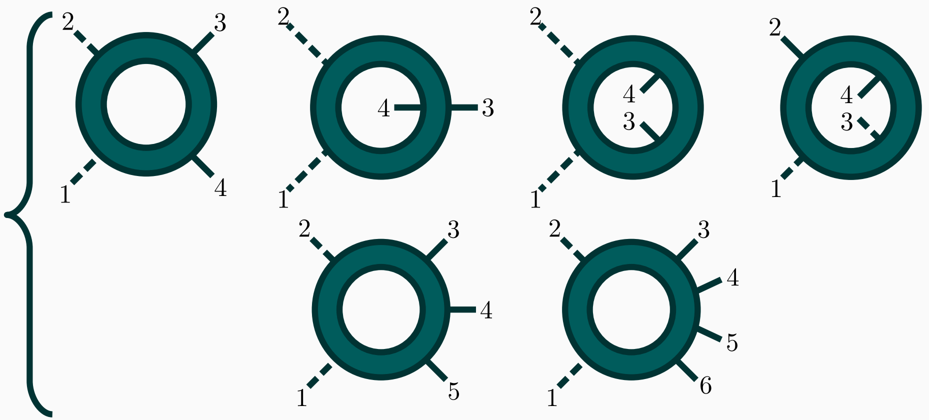
Our approach:
Unitarity (on-shell)

Only two cuts needed:



Double and single poles are exactly missing part
(*i.e.* "pole-under-pole" and non-factorizable poles)

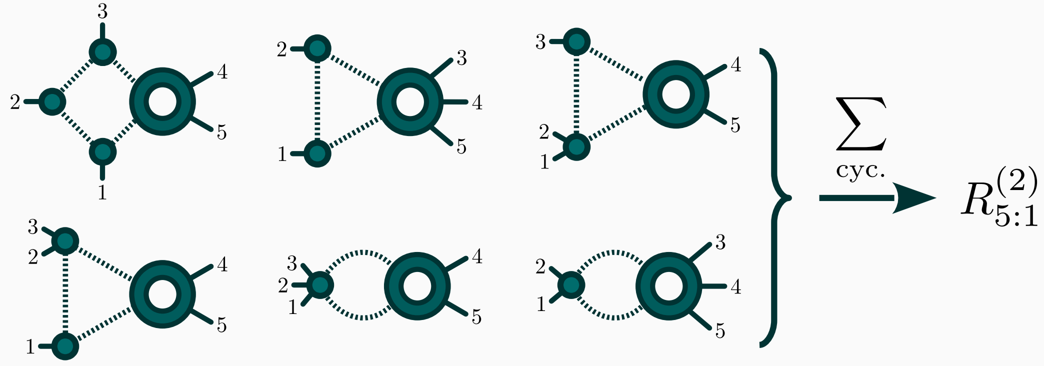
From unitarity:
Obtained analytic expressions for “missing” pole terms, verified numerically



Used to verify two-loop All-Plus results for

$$R^{(2)}_{4:1} \quad R^{(2)}_{4:1\text{B}} \quad R^{(2)}_{4:3} \quad R^{(2)}_{5:1}$$

Example:



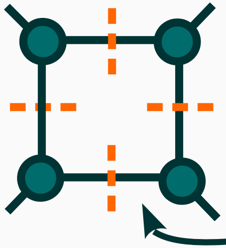
Sidenote: Why Are Rational Terms "Hard"?

Generalized unitarity cuts probe branch cuts

$$A_n^{(2)} \simeq \underbrace{\left(A_n^{(0)}\mathcal{I}_n^{(2)}\right) + \left(A_n^{(1)}\mathcal{I}_n^{(1)}\right) + P_n^{(2)}} + R_n^{(2)}$$

No branch cuts

Polylogs, etc. → Branch cuts in 4D kinematics
"Cut-constructible"

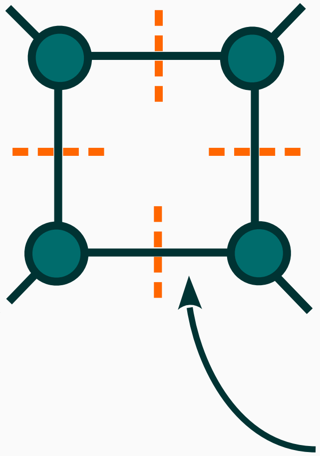


Product of four-dimensional amplitudes

Internal and external states and momenta four-dimensional

However, rational part $R_n^{(2)}$ *is* constructible from D-dimensional branch cuts

D-dimensional Generalized Unitarity



Product of D-dimensional tree amplitudes

External states and momenta left four-dimensional

Internal D-dimensional states and momenta

Connection of all-plus with $\mathcal{N}=4$ sYM MHV Amplitude

At amplitude level:

$$A^{(1)}(1^+\dots n^+) = -2\epsilon(1-\epsilon)(4\pi)^2 \left[\frac{A^{(1)}_{\mathcal{N}=4}(1^+\dots i^-\dots j^-\dots n^+)}{\langle ij\rangle^4}\right]\Bigg|_{\epsilon\to\epsilon-2}$$

Dime

Dimension Shift
$4-2\epsilon \ \to\ 8-2\epsilon$

First conjectured in [hep-th/9611127]
Now proven to all orders in ϵ [2011.13821]

At integrand level: simple replacement

At one-loop: $\delta^{(8)}(Q) \longrightarrow (D_s-2)\mu^4$

At two-loop: Partial $\delta^{(8)}(Q) \to F_1 = (D_s-2)\left(\mu_1^2\mu_2^2 + (\mu_1^2+\mu_2^2)^2 + 2\mu_{12}(\mu_1^2+\mu_2^2)\right) + 16(\mu_{12}^2 - \mu_1^2\mu_2^2)$

Conjectured in [Badger, Frellesvig, Zhang, 1310.1051; Badger, Mogull, Peraro, 1606.02244]