The complete singlet contribution to the massless quark form factor at three loops in QCD

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Based on *JHEP* 12 (2021) 095

Loops and Legs in Quantum Field Theory
April 29th 2022

TRR 257 - Particle Physics Phenomenology
after the Higgs Discovery

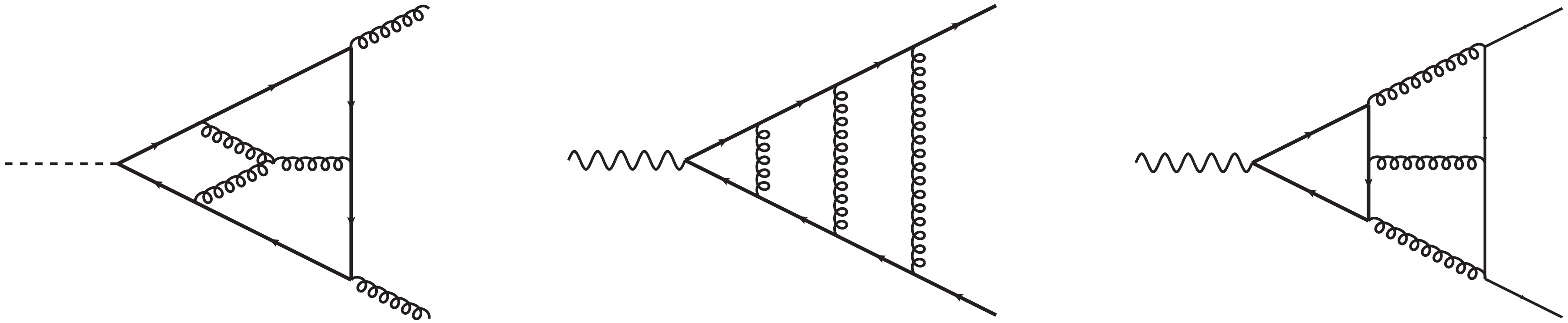
Institute for Theoretical Particle Physics and Cosmology

RWTH AACHEN UNIVERSITY

Motivation

- Form factors (FFs) are important ingredients for phenomenologically interesting processes:
  - ➢ hadronic Higgs production/decay
  - ➢ Drell-Yan
  - ➢ quark pair production in e^+e^- collisions

- FFs can be used to extract universal quantities:
  - cusp anomalous dimension
  - quark/gluon anomalous dimensions



Quark FFs (*)

[Gehrmann and Primo `21]

Axial

Pseudoscalar

[Ahmed, Gehrmann et al. `15]

[Gehrmann and Kara `14]

Scalar

Vector

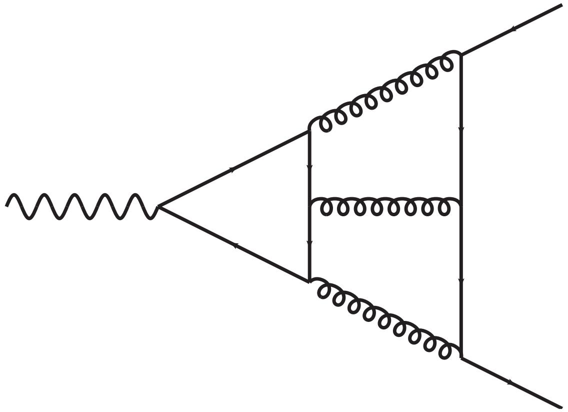
[Moch, Vermaseren et al. `05, Baikov, Chetyrkin et al. `09, Lee, Smirnov et al. `10, Gehrmann, Glover et al. `10]

[Henn, Smirnov et al. `16, Henn, Lee et al. `16, Lee, Smirnov et al. `17, Lee, Smirnov et al. `19, Henn, Korchemsky et al. `20, Manteuffel, Panzer et al. `20, Lee, Manteuffel et al. `21, Agarwal, Manteuffel et al. `21, Lee, Manteuffel et al. `22]

(*) limited to massless QCD

[Gehrmann and Primo `21]

Three-loop pure *singlet* contribution in massless QCD

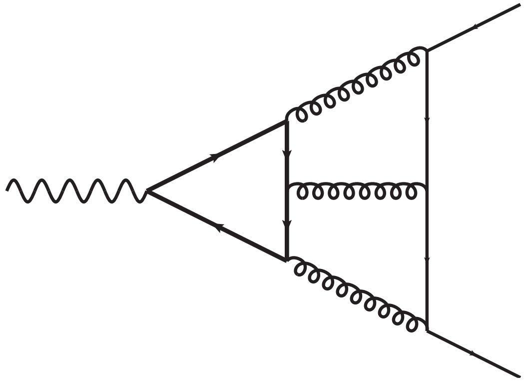


Axial

Vector

We include effects of a **massive** top quark

Axial



[Gehrmann and Primo `21]

Three-loop pure *singlet* contribution in massless QCD

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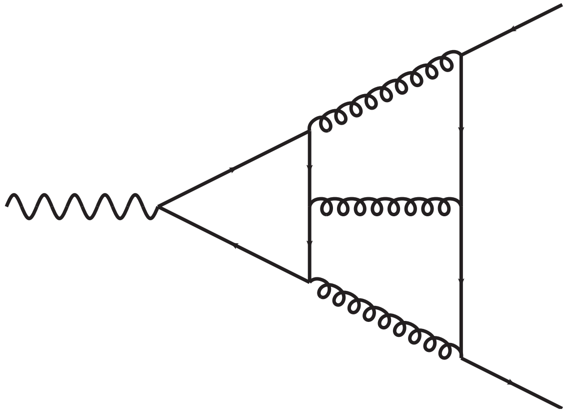
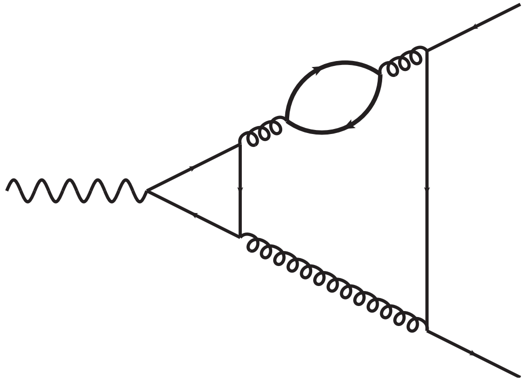




Why include top quark mass effects?

- Two reasons related to the presence of axial-anomaly type diagrams

Top-loop contribution does not decouple in the low energy limit

- Leads to non-decoupling logarithms [Collins et al. `78, Chetyrkin et al. `93/`94, Larin et al. `94/`95]
- Singlet contribution to Z boson decay rate was found to be considerable in the large top mass limit [Chetyrkin et al. `94, Larin et al. `94/`95, Baikov et al. `12]

For an appropriate renormalization scale dependence

- Purely massless contribution is not scale independent!
- Singlet axial current has non-vanishing anomalous dimension

→ Leads to non-trivial scale dependence

Outline

- Introduction (✓)
- Preliminaries
- Computation
- Renormalization
- Results

Preliminaries

- Consider 3-point vertex function with external off-shell Z boson and pair of massless quarks of flavor q with on-shell outgoing momenta p_1 and p_2 in QCD with $n_f = n_l + 1 = 6$ flavors ($2p_1 \cdot p_2 = s$)
- Lorentz tensor decomposition: $\bar{u}(p_1)\,\Gamma^\mu\,v(p_2)\,\delta_{ij} = \bar{u}(p_1)\,\big(v_q\,F^V\gamma^\mu + a_q\,F^A\gamma^\mu\gamma_5\big)\,v(p_2)\,\delta_{ij}$
- FFs can be projected out:

$$F^V = \frac{-1}{s(4-4\epsilon)}\,\mathrm{Tr}\big[\not{p}_2\gamma_\mu\not{p}_1\,\Gamma^\mu\big]$$

$$F^A = \frac{-1}{s(4-4\epsilon)}\,\mathrm{Tr}\big[\not{p}_2\gamma_\mu\gamma_5\not{p}_1\,\Gamma^\mu\big]$$

- γ_5 is treated *non-anticommuting* (we use Larin prescription)

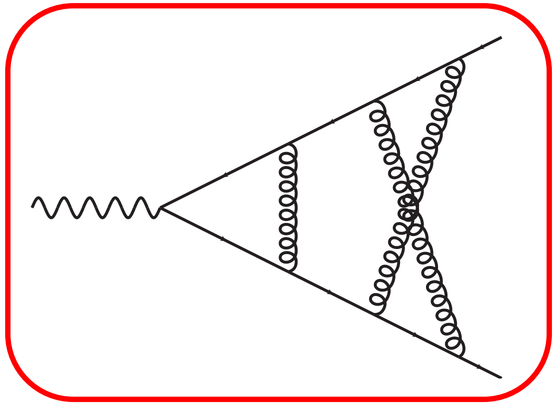
[Larin and Vermaseren `91, Larin `93]

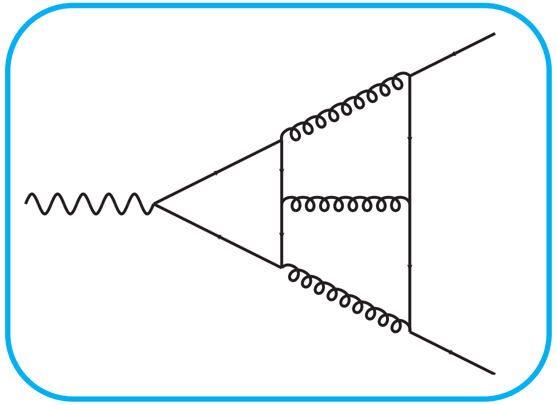
- Classify corrections to F^V and F^A based on topology of contributing Feynman diagrams:

$$F^V = F^V_{ns} + F^V_s = F^V_{ns} + \sum_f \frac{v_f}{v_q}\,F^V_{s,f}$$

$$F^A = F^A_{ns} + F^A_s = F^A_{ns} + \sum_f \frac{a_f}{a_q}\,F^A_{s,f}$$

flavor of quark coupling to Z boson



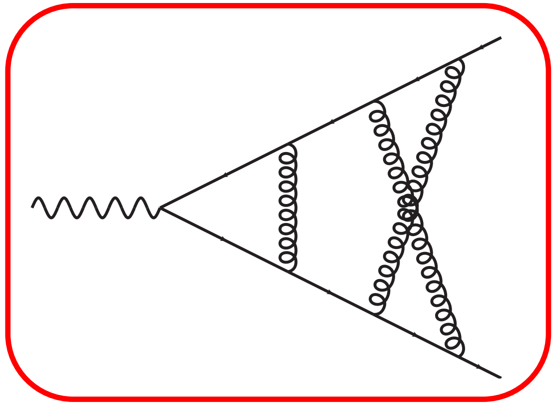


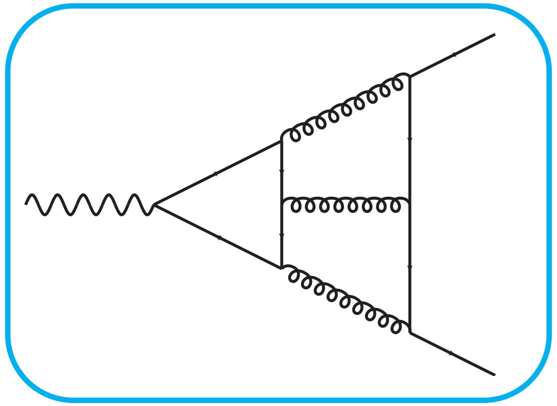
Preliminaries

$$F^V = F^V_{ns} + F^V_s = F^V_{ns} + \sum_f \frac{v_f}{v_q} F^V_{s,f}$$

$$F^A = F^A_{ns} + F^A_s = F^A_{ns} + \sum_f \frac{a_f}{a_q} F^A_{s,f}$$

- **Non-singlet** contribution starts from tree level
- With q massless and anticommuting γ_5: $F^V_{ns} = F^A_{ns}$ (chirality conservation)
- **Singlet** contribution starts from 2-loop order





Vector F_s^V

- ➢ Vanishes at 2-loop order (Furry)
- ➢ Leading 3-loop result is UV and IR finite

massless QCD: [Moch, Vermaseren et al. `05, Baikov, Chetyrkin et al. `09, Gehrmann, Glover et al. `10]

Axial F_s^A

- ➢ Contributions from weak doublets add up to zero in massless limit
- ➢ Only non-zero contribution from top-bottom doublet

$$F^A_s = \lambda_q \left(F^A_{s,b} - F^A_{s,t}\right)$$

full 2-loop [Bernreuther, Bonciani et al. `05]
massless QCD [Gehrmann and Primo `21]

$$F^V = F^V_{ns} + F^V_s = F^V_{ns} + \sum_f \frac{v_f}{v_q} F^V_{s,f}$$

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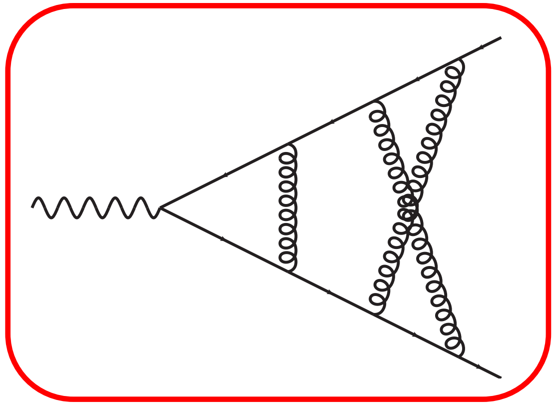

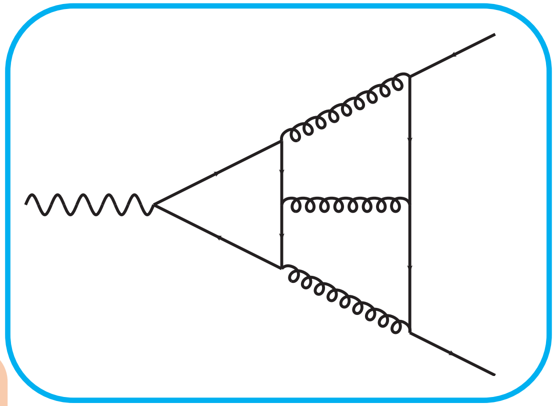

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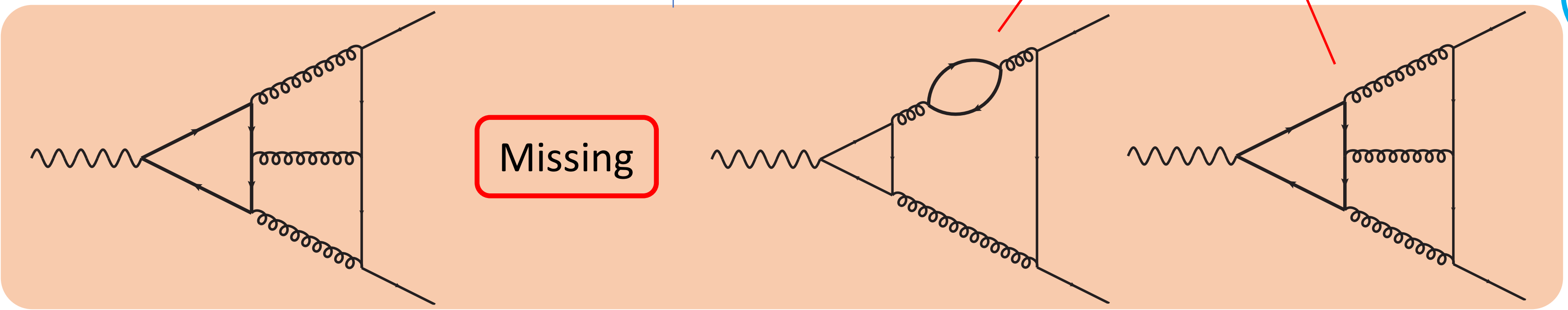

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Setup

- Expand bare FFs in $\hat{a}_s \equiv \frac{\hat{\alpha}_s}{4\pi}$

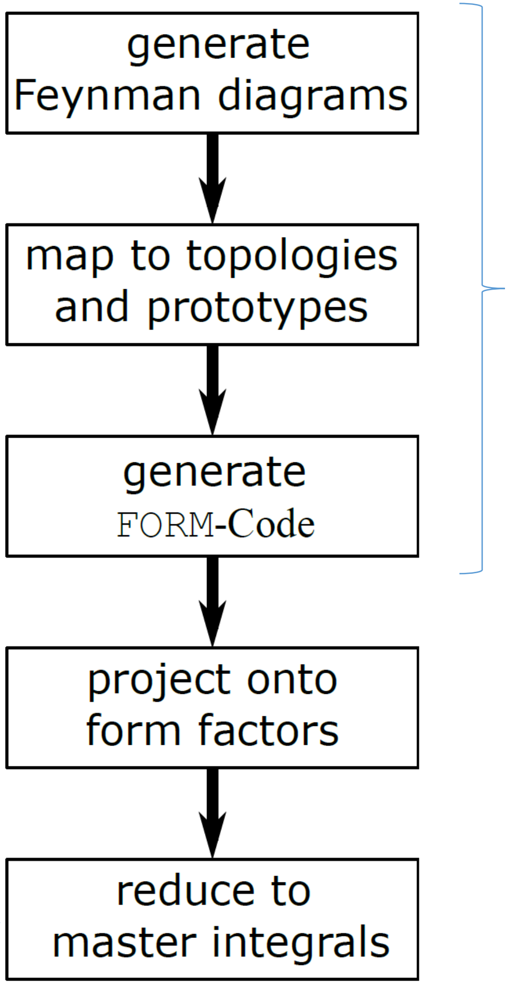
$$F^{A}_{s,b} = \sum_{n=2}^{\infty} \hat{a}_s^n \, F^{A,n}_{s,b} \qquad F^{A}_{s,t} = \sum_{n=2}^{\infty} \hat{a}_s^n \, F^{A,n}_{s,t}$$

- 2-loop order: 2 × 2 diagrams
  3-loop order: 2 × 57 diagrams

- Matching to scalar integrals

$$\mathrm{PRID}(n_1, \cdots, n_{12}) = \int \frac{d^d k_1}{i\pi^{d/2}} \int \frac{d^d k_2}{i\pi^{d/2}} \int \frac{d^d k_3}{i\pi^{d/2}} \frac{1}{D_1^{n_1} \cdots D_{12}^{n_{12}}}$$

5596 scalar integrals



DiaGen
Czakon (unpublished)

FORM
Vermaseren `00

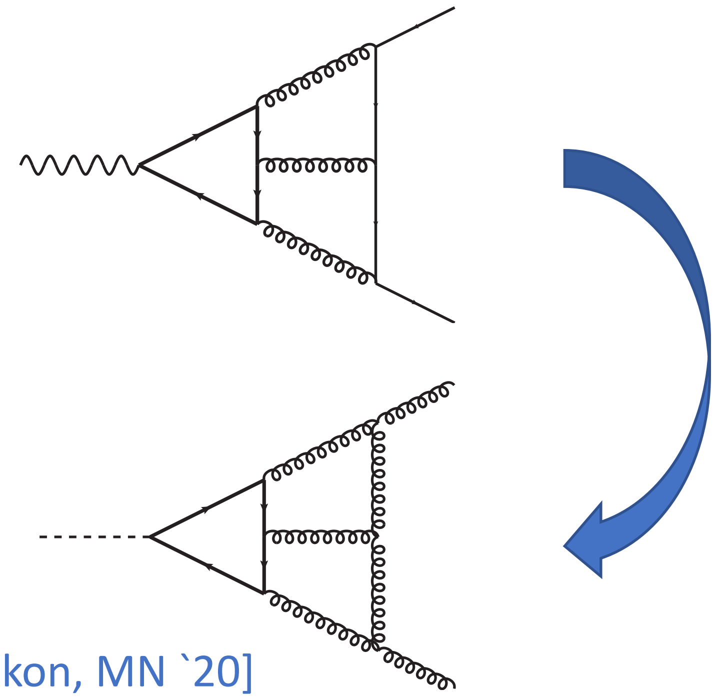
IdSolver
Czakon (unpublished)

Setup

- Expand bare FFs in $\hat{a}_s \equiv \frac{\hat{\alpha}_s}{4\pi}$

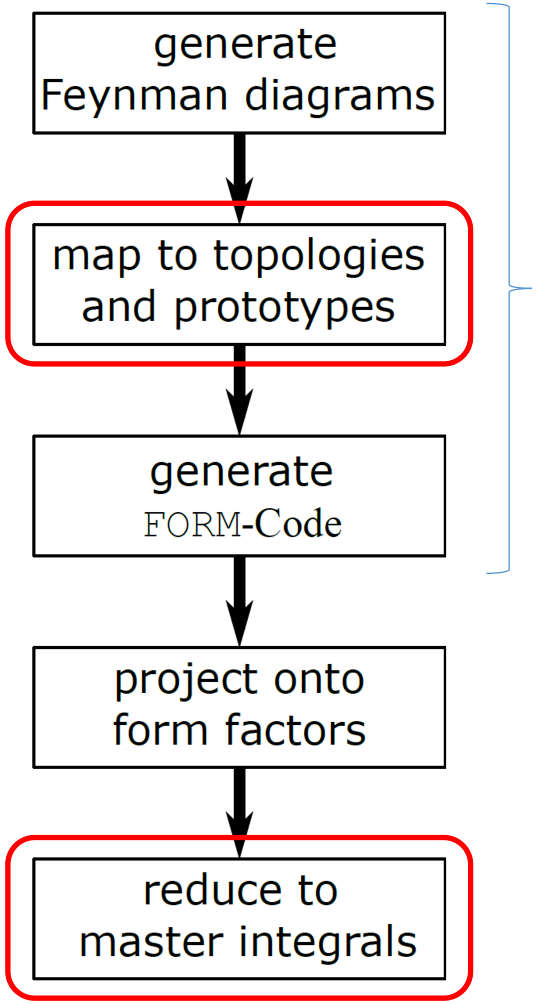
$$F_{s,b}^{A} = \sum_{n=2}^{\infty} \hat{a}_s^n \, F_{s,b}^{A,n} \qquad F_{s,t}^{A} = \sum_{n=2}^{\infty} \hat{a}_s^n \, F_{s,t}^{A,n}$$

- Note the similar structure!



map scalar integrals in $F_{s,t}^{A}$ to those in C_{ggH}

[Czakon, MN `20]



DiaGen [Czakon (unpublished)]

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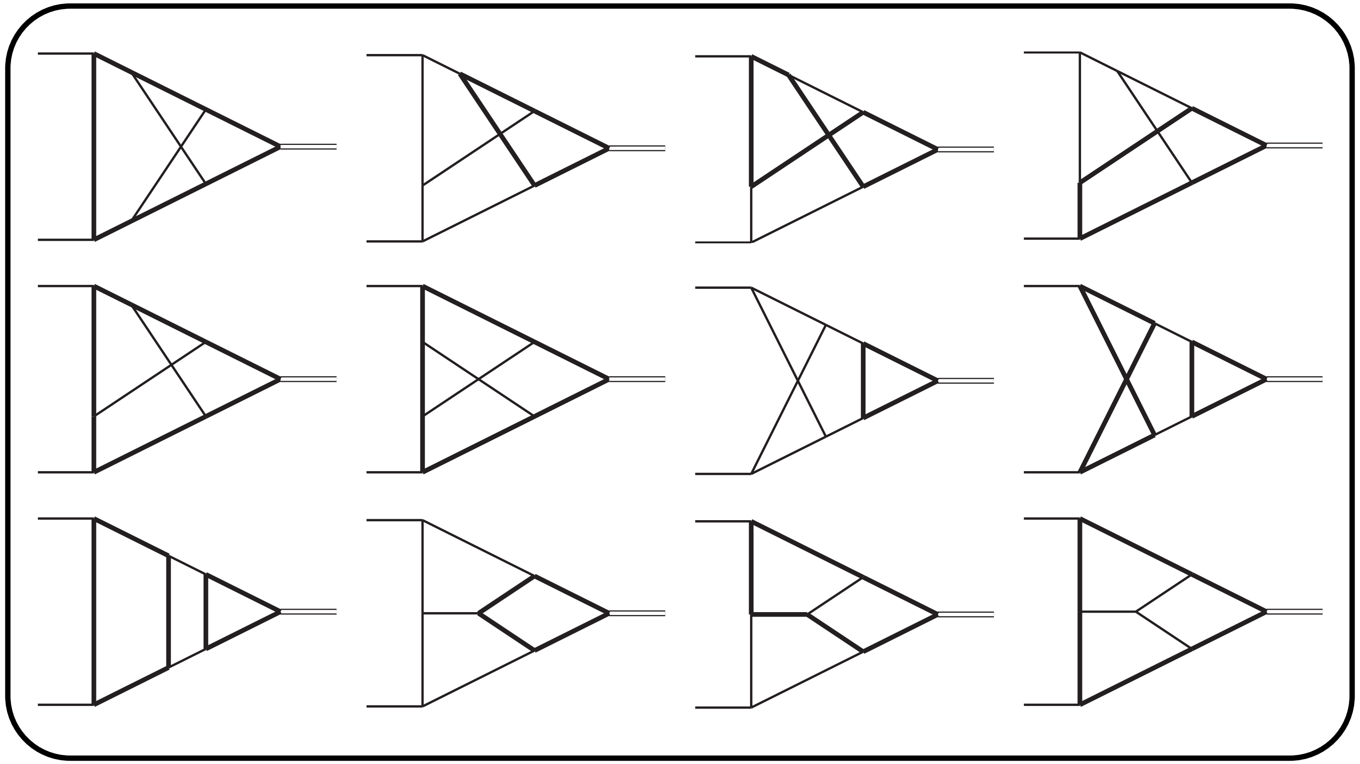

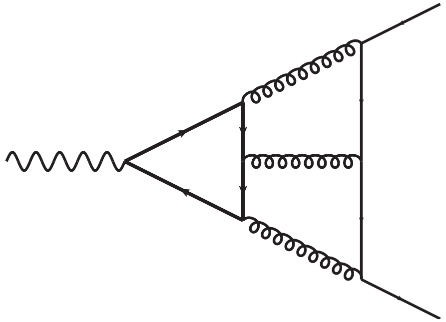

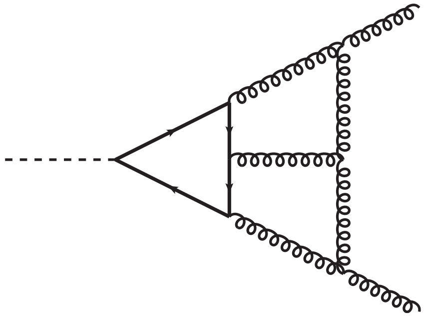

map scalar integrals in $F^{A,3}_{s,t}$ to those in C_{ggH}

[Czakon, MN `20]

Setup

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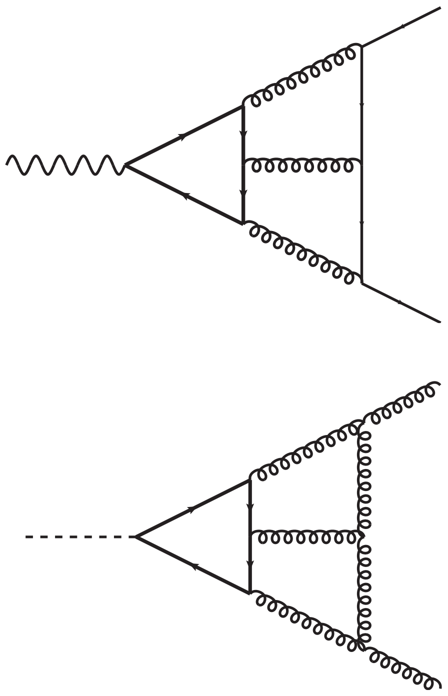
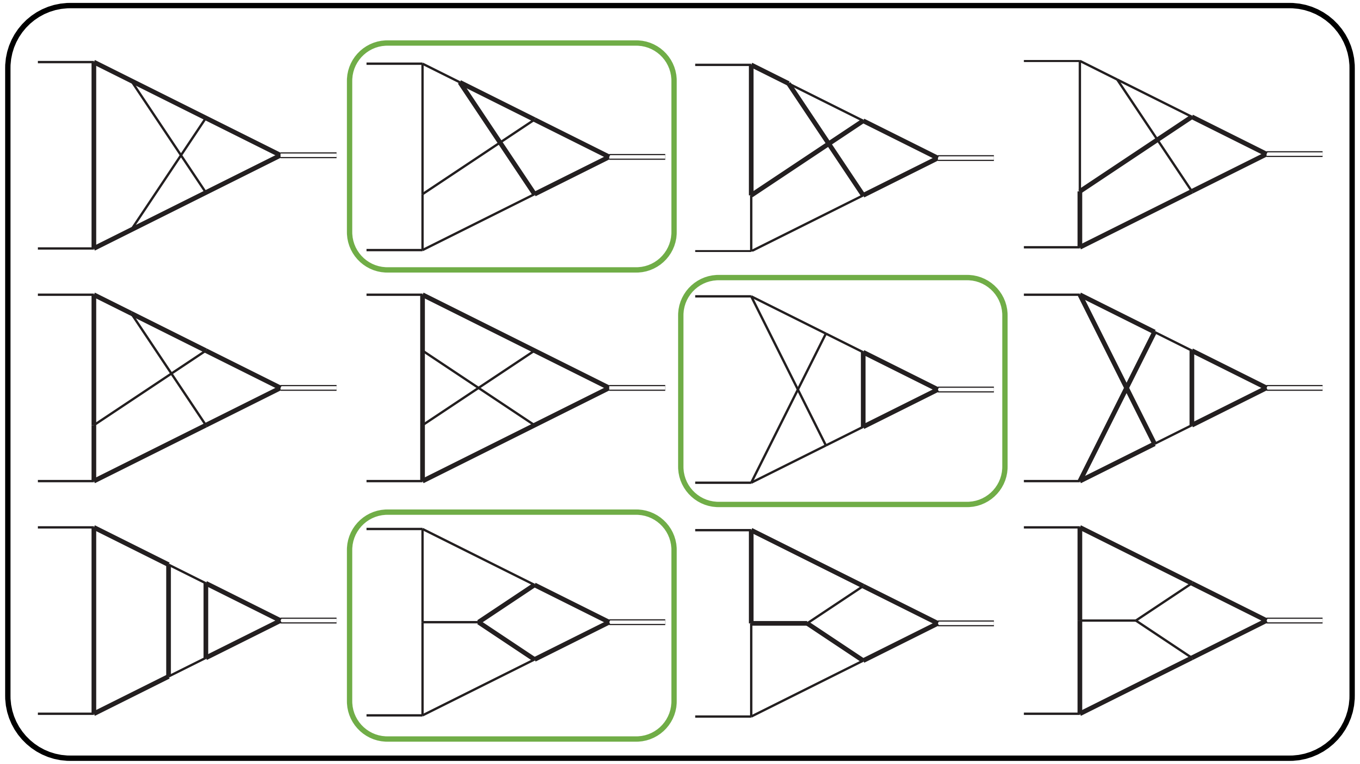


map scalar integrals in $F^{A,3}_{s,t}$ to those in C_{ggH}

➢ Reduction already done!



[Czakon, MN `20]

Setup

[Heinrich, Huber et al. `09,
Gehrmann, Glover et al. `10]

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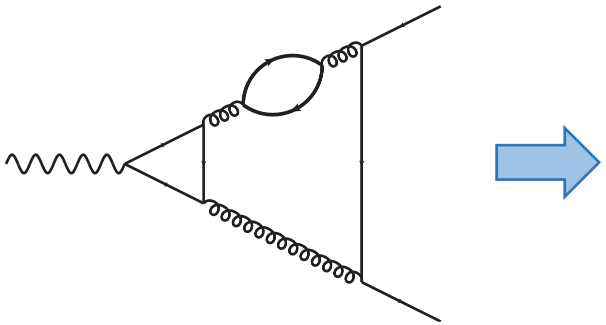
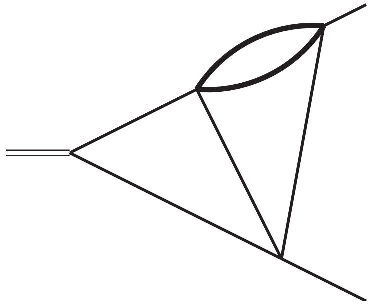
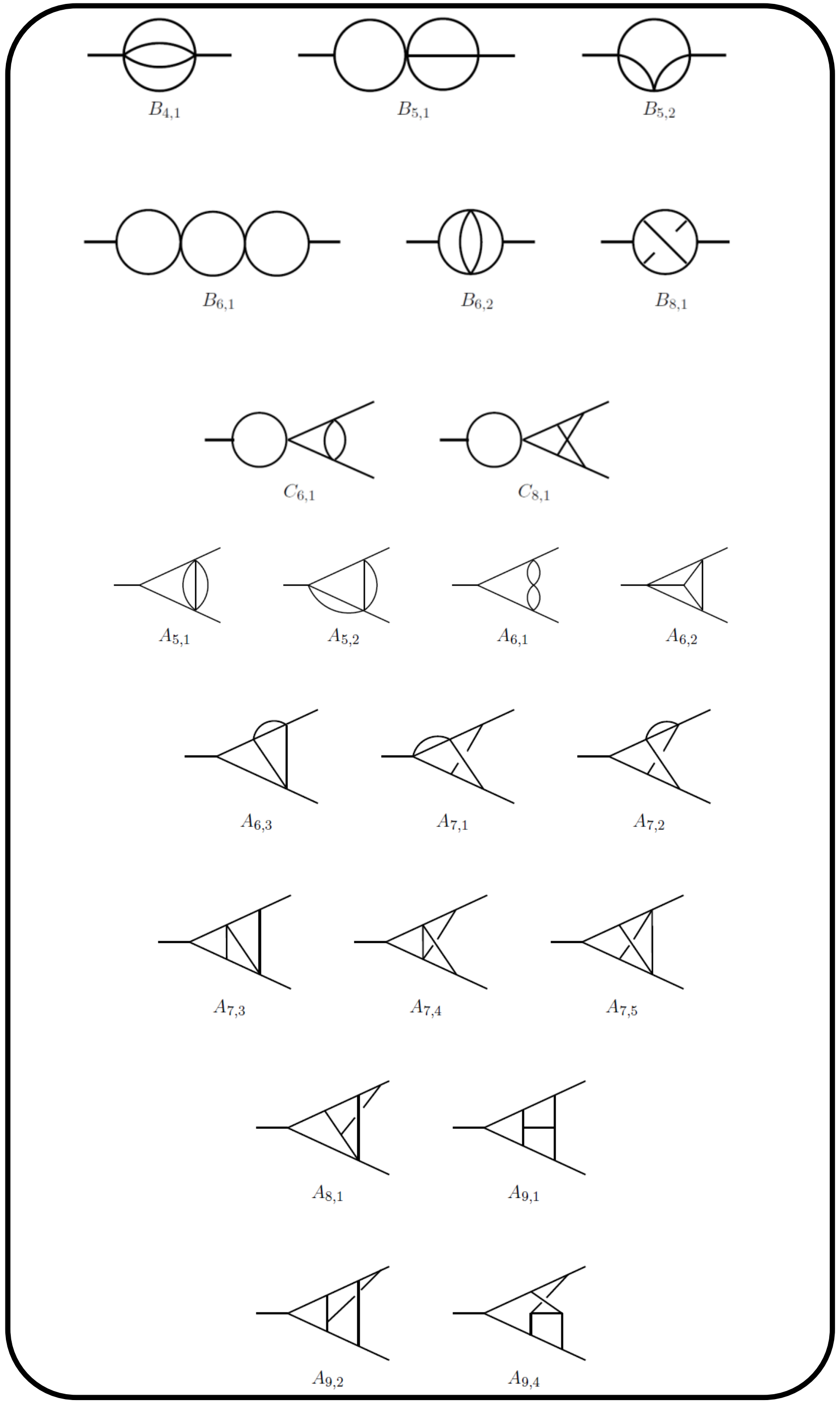
- Purely massless contribution to $F_{s,b}^{A,3}$ (✓) [Gehrmann and Primo `21]

- Also include 4 diagrams with top-quark loops



1 new master integral (not present in C_{ggH})



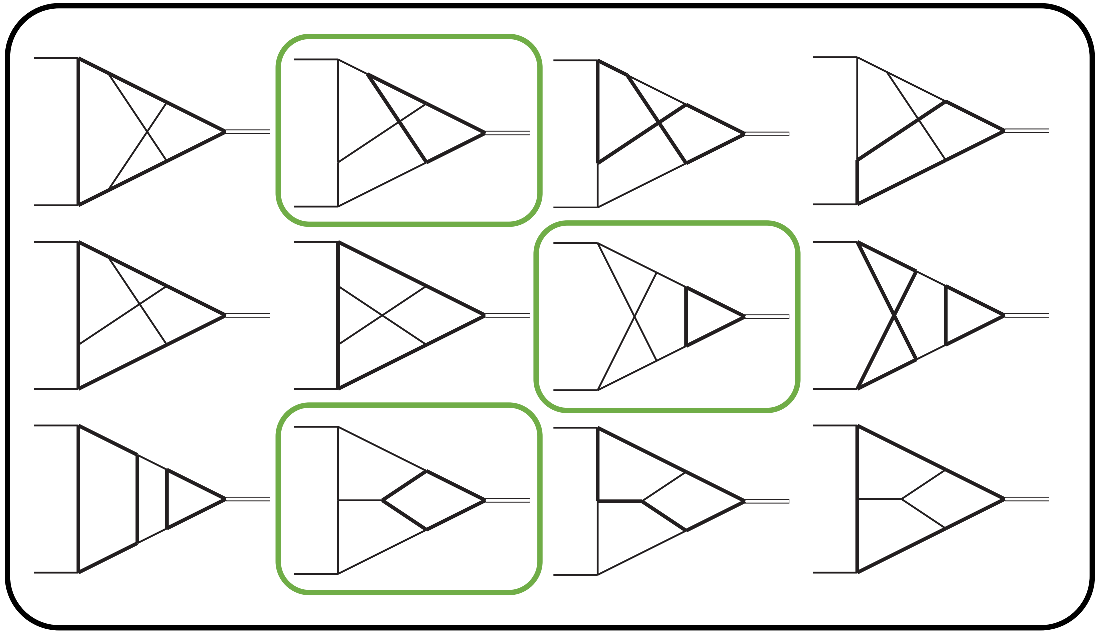


Master Integrals

- MIs with massive internal lines were computed by solving the differential equations numerically (✓)

- Massless MIs are known (✓)

- Compute the only unknown MI analytically via differential equations
  - ➢ Variable transformation $\frac{s}{m^2} = -\frac{(1-x)^2}{x}$
  - ➢ Letters: $\{x, x \pm 1\}$
  - ➢ Use CANONICA to find ε-form [Meyer `17/`18]
  - ➢ Fix boundary condition in the large mass limit



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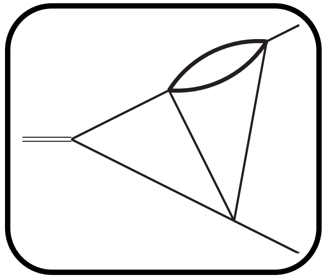

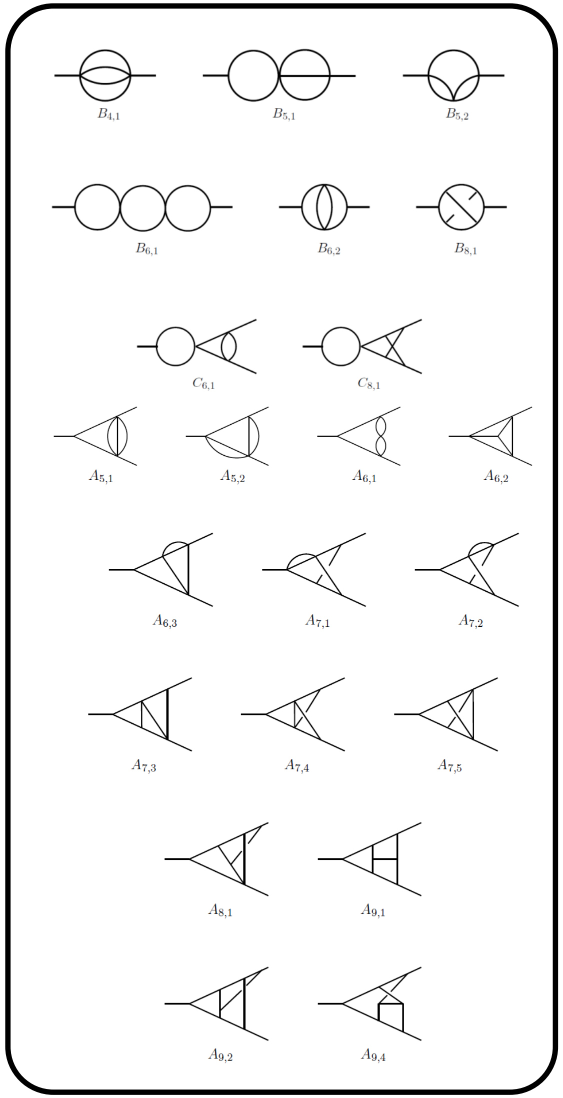

[Heinrich, Huber et al. `09, Gehrmann, Glover et al. `10]

Solving DEs Numerically

$$\frac{\mathrm{d}M_i(z,\epsilon)}{\mathrm{d}z} \equiv \sum_j A_{ij}(z,\epsilon)\, M_j(z,\epsilon)$$

- Construct a system of first-order linear differential equations with rational function coefficients A_{ij}

Solving DEs Numerically

- Construct a system of first-order linear differential equations with rational function coefficients A_{ij}
- Insert truncated ε-expansions for the MIs

$$\frac{\mathrm{d}M_i(z,\epsilon)}{\mathrm{d}z} \equiv \sum_j A_{ij}(z,\epsilon)\, M_j(z,\epsilon)$$

$$M_i(z,\epsilon) \equiv \sum_{l=0}^{\overline{n}_i-\underline{n}_i} \epsilon^{\underline{n}_i+l}\, I_{\underline{k}_i+l}(z)$$

$$\frac{\mathrm{d}I_k(z)}{\mathrm{d}z} \equiv \sum_l B_{kl}(z)\, I_l(z)$$

Solving DEs Numerically

$$\frac{\mathrm{d}M_i(z,\epsilon)}{\mathrm{d}z} \equiv \sum_j A_{ij}(z,\epsilon)\, M_j(z,\epsilon)$$

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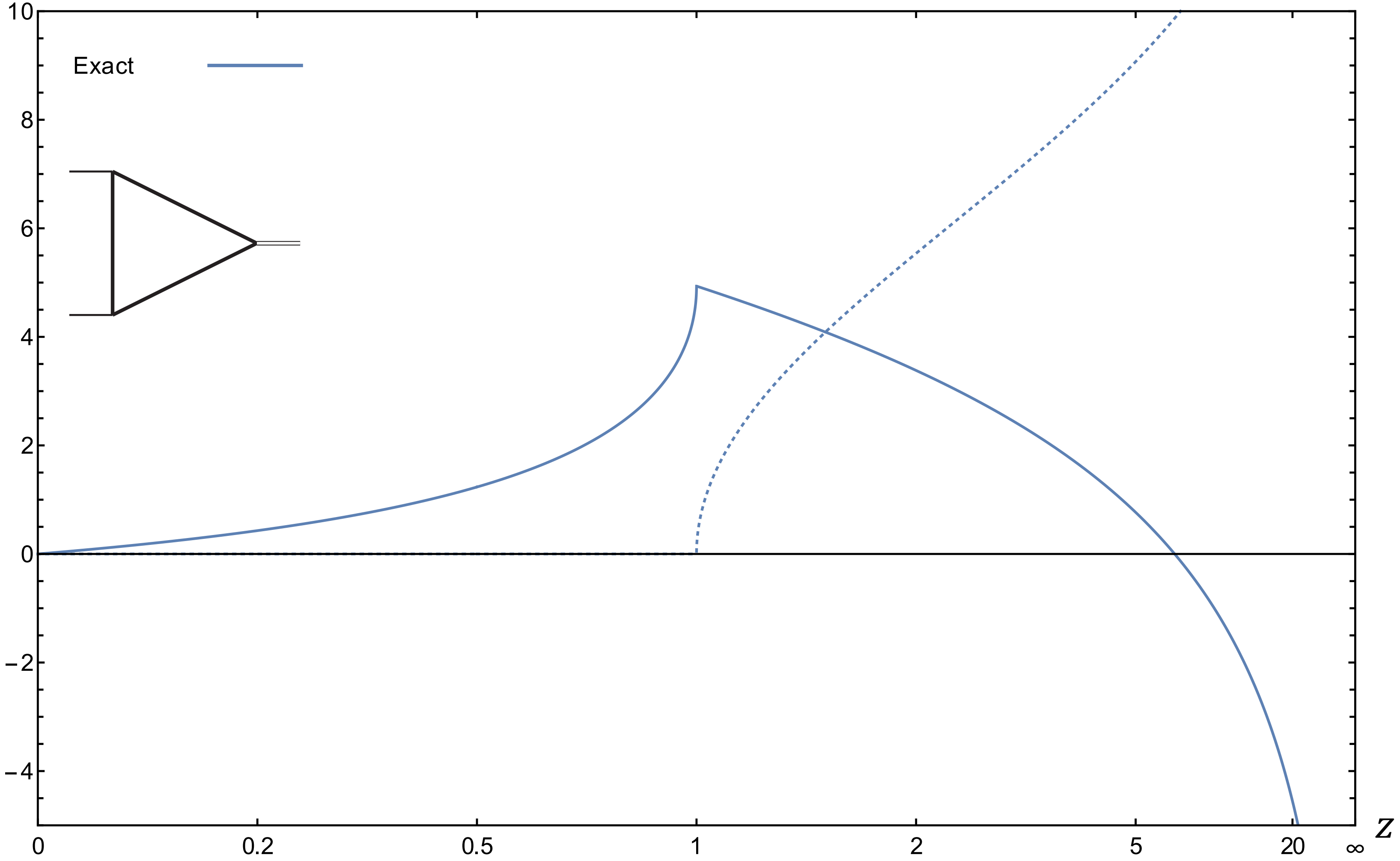
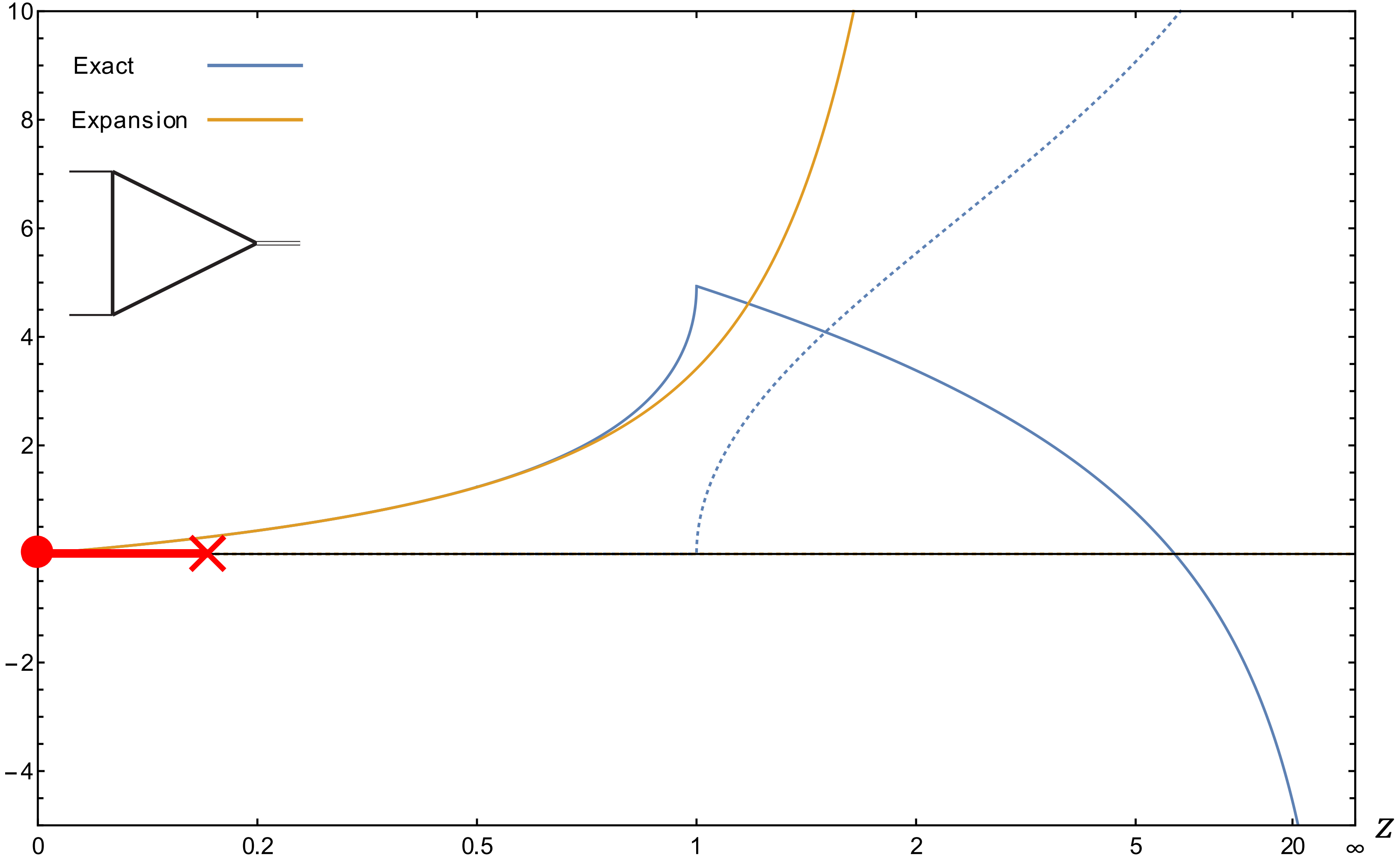


Solving DEs Numerically

- Provide initial conditions for I_k to start numerical evolution
  - Via deep expansion around $z = 0$
  - Large mass expansion fixes unknown coefficients



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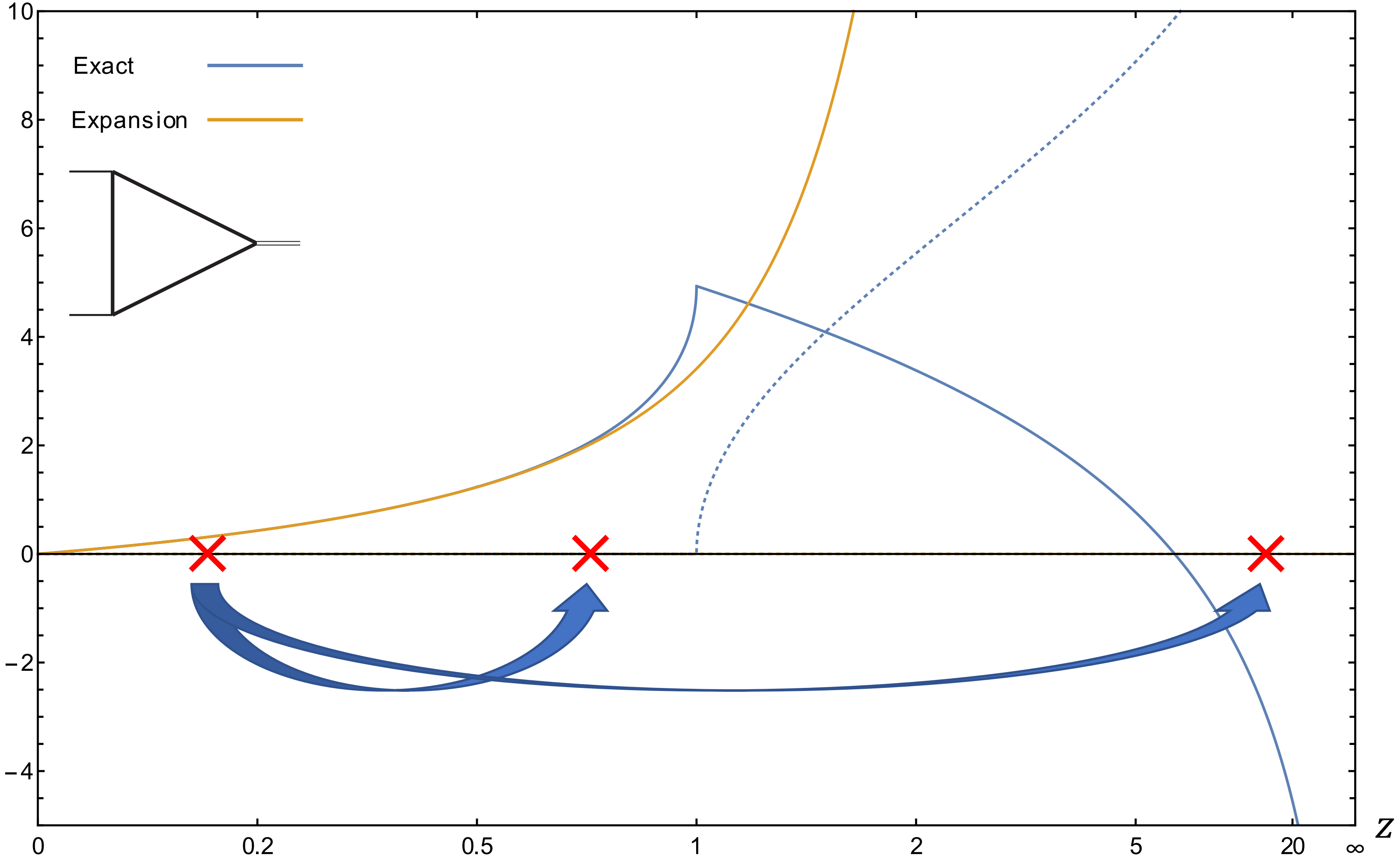
$$I_k(z) \equiv \sum_{l=\underline{l}_k}^{\infty} \sum_{m=\underline{m}_k}^{\overline{m}_k} c_{klm}\, z^l \ln^m z$$

Solving DEs Numerically

- Transport initial I_k in the parameter space
  - ➢ Circumvent singular points of DEs with numerical evolution in the complex plane



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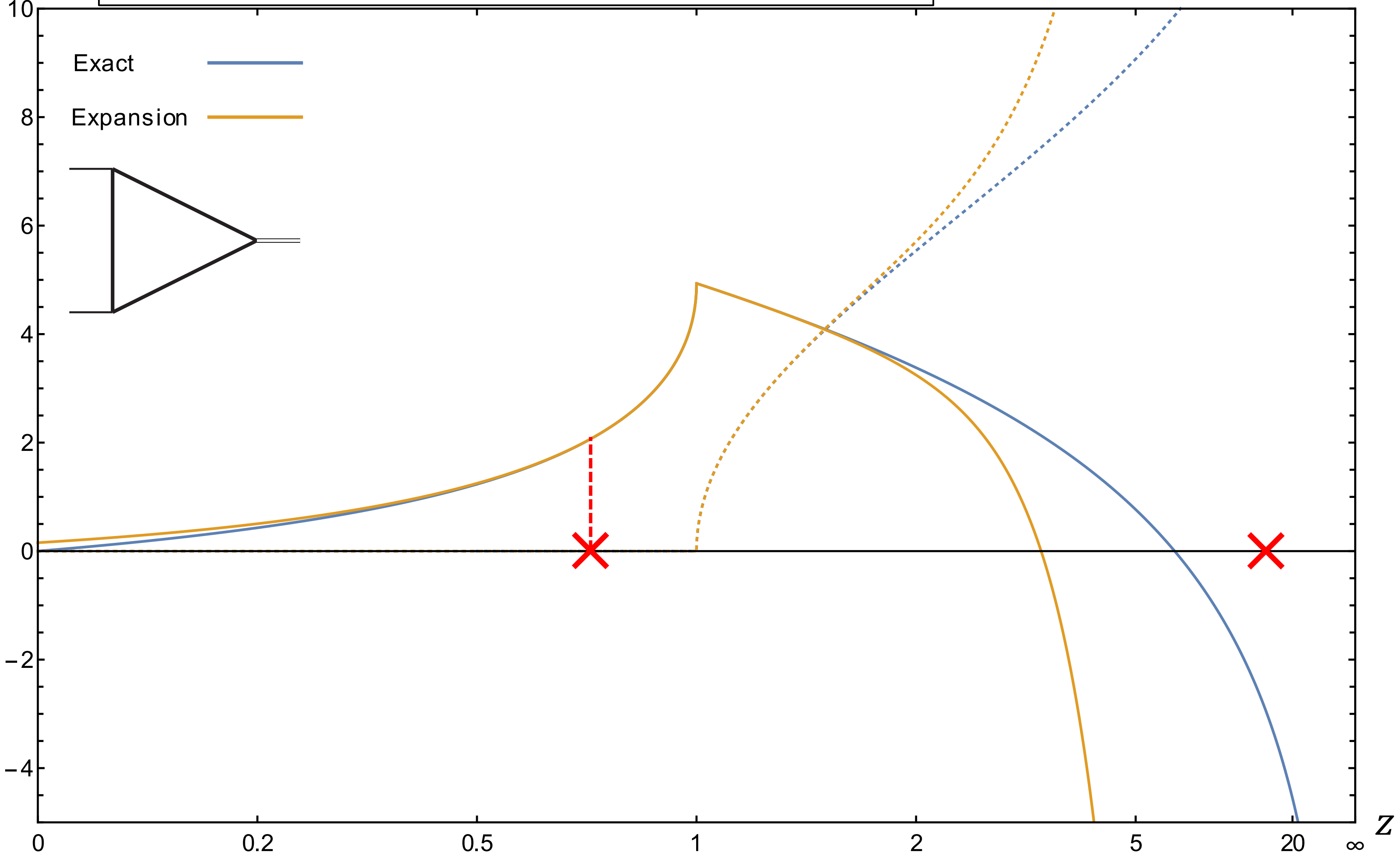
Solving DEs Numerically

- Expand DEs around $y = 0$ and match the expansion with previously obtained high-precision values to access special limits

$$I_k(z(y)) \equiv \sum_l F_{kl}(y)\, c_l \implies c_k = \sum_l \left(F^{-1}\right)_{kl}(y)\, I_l(z(y))$$



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$$I_k(z(y)) \equiv \sum_{l=\underline{l}_k}^{\infty} \sum_{m=\underline{m}_k}^{\overline{m}_k} c_{klm}\, y^l \ln^m y$$

Solving DEs Numerically

- Repeat until desired domains are covered

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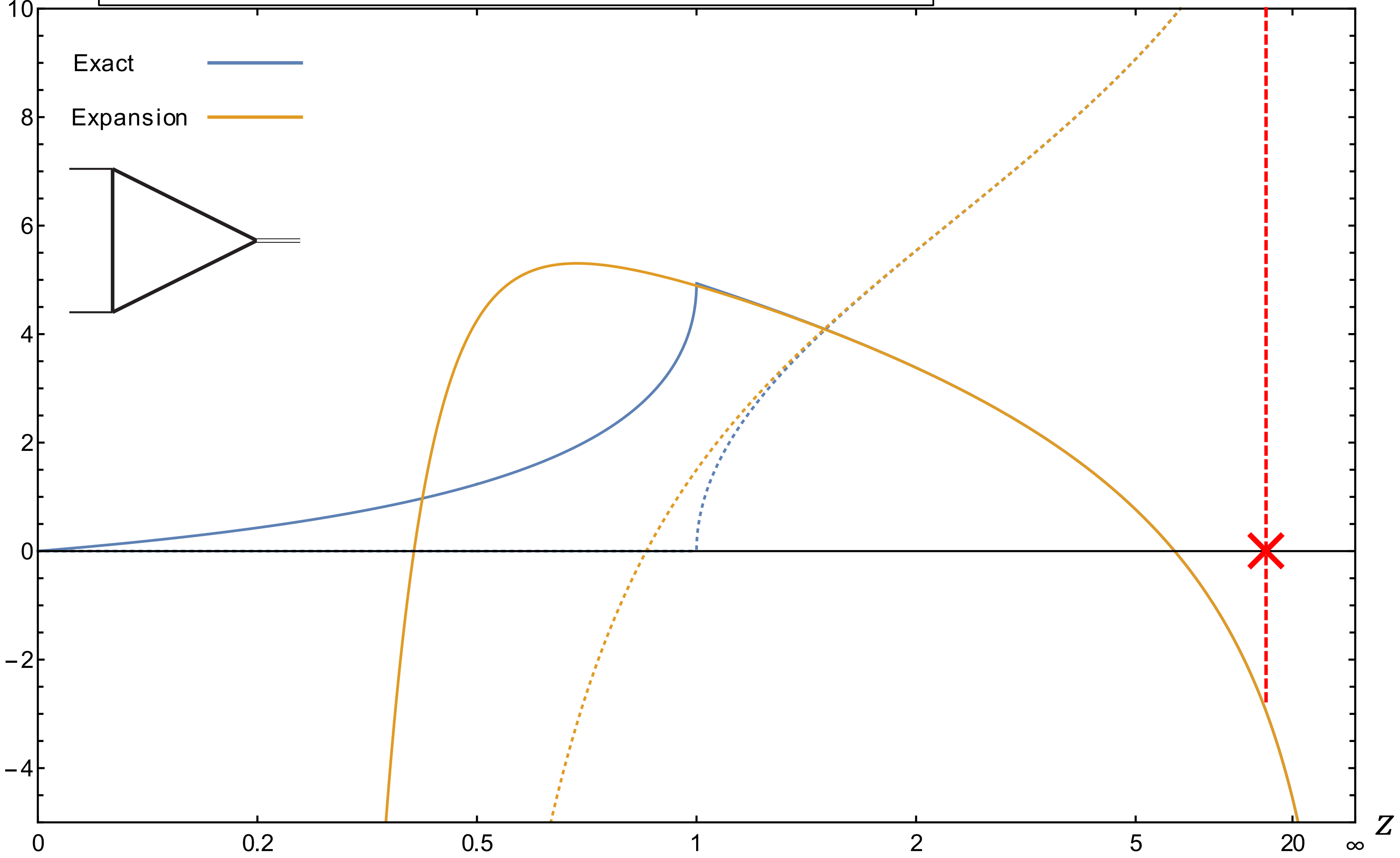

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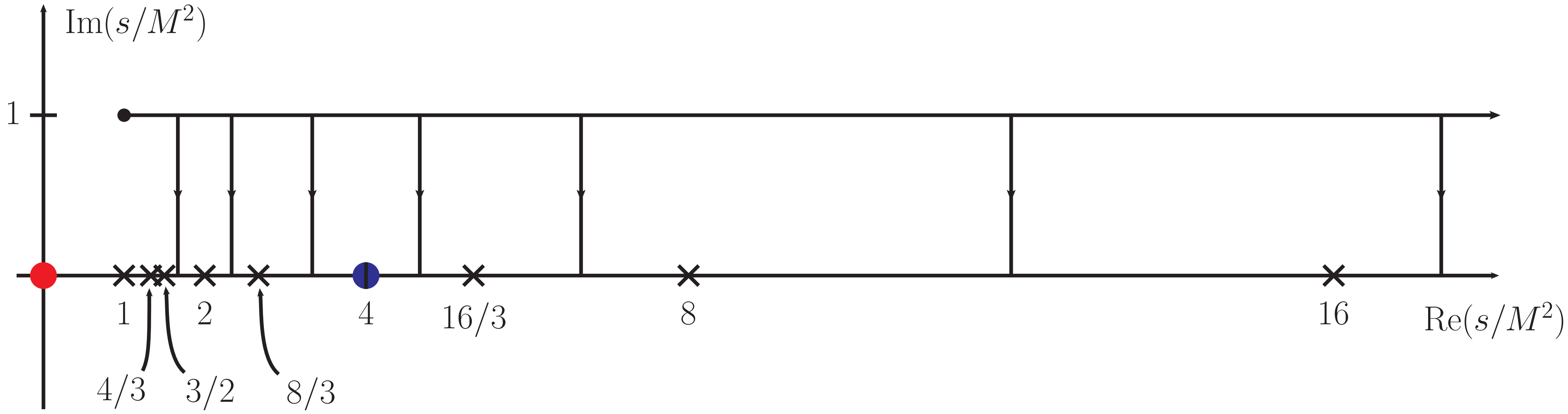
- In practice, the following integration contour was used for the computation of the MIs:



- Require a local error of $\mathcal{O}(10^{-40})$
- Collect $2 \cdot 10^5$ numerical samples with at least 20 correct digits
- Allows expansions in kinematic limits to high orders in small parameter
- Note: This method was originally developed for [Czakon, Fiedler et al. `15]

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- Preliminaries (✓)
- Computation (✓)
- Renormalization
- Results

Renormalization Formulae

- Individual contributions to F_s^A can be renormalized independently [Chetyrkin and Kühn `93]

$$\mathbf{F}^A_{s,b}(a_s, m_t, \mu) = Z_{ns}\, Z_2\, F^A_{s,b}(\hat{a}_s, \hat{m}_t) \,+\, Z_s\, Z_2 \Big(F^A_{ns}(\hat{a}_s, \hat{m}_t) + \sum_{i=1}^{n_f} F^A_{s,i}(\hat{a}_s, \hat{m}_t) \Big)$$

$$\mathbf{F}^A_{s,t}(a_s, m_t, \mu) = Z_{ns}\, Z_2\, F^A_{s,t}(\hat{a}_s, \hat{m}_t) \,+\, Z_s\, Z_2 \Big(F^A_{ns}(\hat{a}_s, \hat{m}_t) + \sum_{i=1}^{n_f} F^A_{s,i}(\hat{a}_s, \hat{m}_t) \Big)$$

- $\hat{a}_s\ S_\epsilon = Z_{a_s}(\mu^2)\, a_s(\mu^2)\, \mu^{2\epsilon}$
- $\widehat{m}_t = Z_m m_t$ (on-shell)
- On-shell wavefunction renormalization $Z_2 \neq 1$
- $$Z_s = a_s^2\, C_F \left(\frac{3}{\epsilon} + \frac{3}{2} \right) + a_s^3 \Big(C_A\, C_F \Big(-\frac{22}{3}\frac{1}{\epsilon^2} + \frac{109}{9}\frac{1}{\epsilon} - \frac{163}{27} + 26\,\zeta_3 \Big) + C_F^2 \Big(-\frac{18}{\epsilon} + \frac{23}{2} - 24\,\zeta_3 \Big) + C_F\, n_f \Big(\frac{4}{3}\frac{1}{\epsilon^2} + \frac{2}{9}\frac{1}{\epsilon} + \frac{88}{27} \Big) \Big) + \mathcal{O}(a_s^4)$$

[Ahmed, Chen at al. `21]

Z_{ns} can be found in [Larin and Vermaseren `91]

Renormalization Formulae

- Individual contributions to F_s^A can be renormalized independently [Chetyrkin and Kühn `93]

$$\mathbf{F}^A_{s,b}(a_s, m_t, \mu) = Z_{ns}\, Z_2\, F^A_{s,b}(\hat{a}_s, \hat{m}_t) + Z_s\, Z_2 \Big(F^A_{ns}(\hat{a}_s, \hat{m}_t) + \sum_{i=1}^{n_f} F^A_{s,i}(\hat{a}_s, \hat{m}_t) \Big)$$

$$\mathbf{F}^A_{s,t}(a_s, m_t, \mu) = Z_{ns}\, Z_2\, F^A_{s,t}(\hat{a}_s, \hat{m}_t) + Z_s\, Z_2 \Big(F^A_{ns}(\hat{a}_s, \hat{m}_t) + \sum_{i=1}^{n_f} F^A_{s,i}(\hat{a}_s, \hat{m}_t) \Big)$$

$\mathcal{O}(a_s^2)$ $\mathcal{O}(a_s^2)$ $1+\mathcal{O}(a_s)$ $\mathcal{O}(a_s^2)$

$1+\mathcal{O}(a_s^2)$ $1+\mathcal{O}(a_s^2)$

$1+\mathcal{O}(a_s)$ $\mathcal{O}(a_s^2)$

- Expanded to $\mathcal{O}(a_s^3)$ last term can be dropped: $Z_{ns}\, Z_2\, F^A_{s,b}(\hat{a}_s, \hat{m}_t) + Z_s\, Z_2\, F^A_{ns}(\hat{a}_s, \hat{m}_t)$
  - Note: Z_2 does not contribute at 3-loop order
  - Note: "Physical" combination requires only non-singlet axial current renormalization

$$\mathbf{F}^A_{s,b}(a_s, m_t) - \mathbf{F}^A_{s,t}(a_s, m_t) = Z_{ns}\, Z_2 \left(F^A_{s,b}(\hat{a}_s, \hat{m}_t) - F^A_{s,t}(\hat{a}_s, \hat{m}_t) \right)$$

Finite Remainder

- The UV-renormalized FFs still contain IR divergences, starting from 3-loop order, regularized as poles in ε

➢ Factorize IR singularities and define the finite remainder

$$\begin{aligned}\mathcal{F}^{A}_{s,b}(a_s, m_t, \mu) &= I_{q\bar{q}}\,\mathbf{F}^{A}_{s,b}(a_s, m_t, \mu) \\ &= a_s^2\,\mathcal{F}^{A,2}_{s,b}(\mu) + a_s^3\,\mathcal{F}^{A,3}_{s,b}(m_t, \mu) + \mathcal{O}(a_s^4) \\ \mathcal{F}^{A}_{s,t}(a_s, m_t, \mu) &= I_{q\bar{q}}\,\mathbf{F}^{A}_{s,t}(a_s, m_t, \mu) \\ &= a_s^2\,\mathcal{F}^{A,2}_{s,t}(m_t, \mu) + a_s^3\,\mathcal{F}^{A,3}_{s,t}(m_t, \mu) + \mathcal{O}(a_s^4)\end{aligned}$$

$$I_{q\bar{q}} = 1 - 2\,a_s \left(\frac{\mu^2}{-s - i0^+}\right)^{\epsilon} \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\, C_F \left(\frac{1}{\epsilon^2} + \frac{3}{2\epsilon}\right) + \mathcal{O}(a_s^2)$$

[Catani `98]

➢ Alternatively, define finite remainder in $\overline{MS}$ scheme [Becher and Neubert `09]

$$\mathcal{F}^{A}_{s,b}(a_s, m_t, \mu) = I_{q\bar{q}}\,\mathbf{F}^{A}_{s,b}(a_s, m_t, \mu) = I_{q\bar{q}}\,Z_{q\bar{q}}\,\mathcal{F}'^{A}_{s,b}(a_s, m_t, \mu)$$

$$I_{q\bar{q}}\,Z_{q\bar{q}} = 1 + a_s\,C_F \left(-\ln^2 \frac{\mu^2}{-s - i0^+} - 3\ln\frac{\mu^2}{-s - i0^+} + \frac{\pi^2}{6}\right) + \mathcal{O}(a_s^2)$$

RG Equations

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \mathbf{F}^A_{s,b}(a_s, m_t, \mu) = \gamma_s \Big(\mathbf{F}^A_{ns}(a_s, m_t, \mu) + \sum_{i=1}^{n_f} \mathbf{F}^A_{s,i}(a_s, m_t, \mu) \Big)$$

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \mathbf{F}^A_{s,t}(a_s, m_t, \mu) = \gamma_s \Big(\mathbf{F}^A_{ns}(a_s, m_t, \mu) + \sum_{i=1}^{n_f} \mathbf{F}^A_{s,i}(a_s, m_t, \mu) \Big)$$

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \Big(\mathbf{F}^A_{s,b}(a_s, m_t, \mu) - \mathbf{F}^A_{s,t}(a_s, m_t, \mu) \Big) = 0$$

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \mathcal{F}^A_{s,b}(a_s, m_t, \mu) = \gamma_s \Big(\mathcal{F}^A_{ns}(a_s, m_t, \mu) + \sum_{i=1}^{n_f} \mathcal{F}^A_{s,i}(a_s, m_t, \mu) \Big)$$

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- Note that
  - Z_{ns} is scale independent
  - Z_s has non-zero anomalous dimension:

$$\mu^2 \frac{\mathrm{d}Z_s}{\mathrm{d}\mu^2} = \frac{1}{n_f} \gamma_S Z_S \equiv \gamma_s \big(Z_{ns} + n_f Z_s \big)$$

  - Expanded to $\mathcal{O}(a_s^3)$, the sum can be neglected

RG Equations

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \mathbf{F}^A_{s,b}(a_s, m_t, \mu) = \gamma_s \Big(\mathbf{F}^A_{ns}(a_s, m_t, \mu) + \sum_{i=1}^{n_f} \mathbf{F}^A_{s,i}(a_s, m_t, \mu) \Big)$$

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \mathbf{F}^A_{s,t}(a_s, m_t, \mu) = \gamma_s \Big(\mathbf{F}^A_{ns}(a_s, m_t, \mu) + \sum_{i=1}^{n_f} \mathbf{F}^A_{s,i}(a_s, m_t, \mu) \Big)$$

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \Big(\mathbf{F}^A_{s,b}(a_s, m_t, \mu) - \mathbf{F}^A_{s,t}(a_s, m_t, \mu) \Big) = 0$$

Again, the "physical" combination has vanishing anomalous dimension

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \mathcal{F}^A_{s,b}(a_s, m_t, \mu) = \gamma_s \Big(\mathcal{F}^A_{ns}(a_s, m_t, \mu) + \sum_{i=1}^{n_f} \mathcal{F}^A_{s,i}(a_s, m_t, \mu) \Big)$$

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \mathcal{F}^A_{s,t}(a_s, m_t, \mu) = \gamma_s \Big(\mathcal{F}^A_{ns}(a_s, m_t, \mu) + \sum_{i=1}^{n_f} \mathcal{F}^A_{s,i}(a_s, m_t, \mu) \Big)$$

RG Equations

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \mathbf{F}^A_{s,b}(a_s, m_t, \mu) = \gamma_s \left(\mathbf{F}^A_{ns}(a_s, m_t, \mu) + \sum_{i=1}^{n_f} \mathbf{F}^A_{s,i}(a_s, m_t, \mu) \right)$$

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \mathbf{F}^A_{s,t}(a_s, m_t, \mu) = \gamma_s \left(\mathbf{F}^A_{ns}(a_s, m_t, \mu) + \sum_{i=1}^{n_f} \mathbf{F}^A_{s,i}(a_s, m_t, \mu) \right)$$

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \left(\mathbf{F}^A_{s,b}(a_s, m_t, \mu) - \mathbf{F}^A_{s,t}(a_s, m_t, \mu) \right) = 0$$

Same structure since $I_{q\bar{q}}$ is scale independent

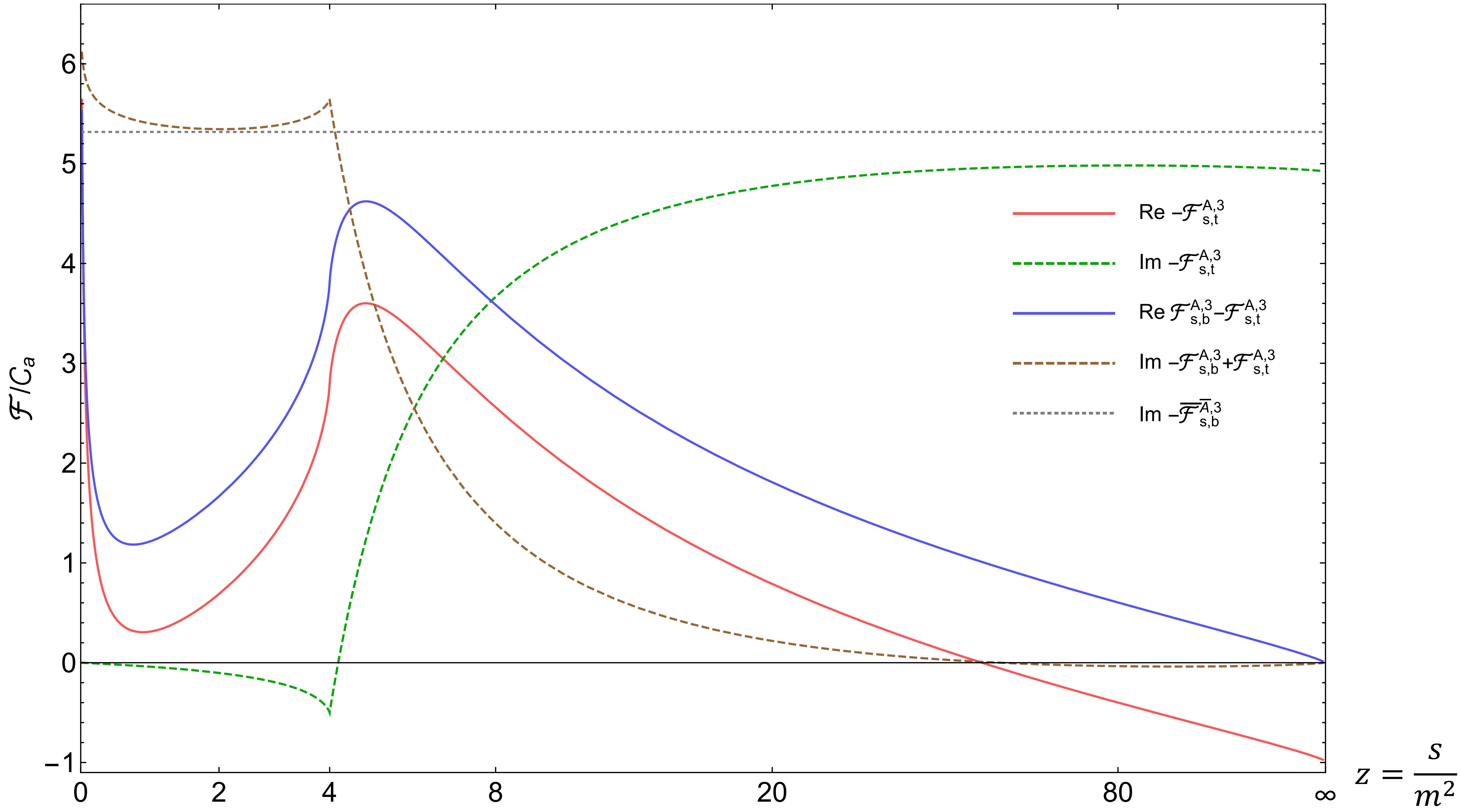
$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \mathcal{F}^A_{s,b}(a_s, m_t, \mu) = \gamma_s \left(\mathcal{F}^A_{ns}(a_s, m_t, \mu) + \sum_{i=1}^{n_f} \mathcal{F}^A_{s,i}(a_s, m_t, \mu) \right)$$

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \mathcal{F}^A_{s,t}(a_s, m_t, \mu) = \gamma_s \left(\mathcal{F}^A_{ns}(a_s, m_t, \mu) + \sum_{i=1}^{n_f} \mathcal{F}^A_{s,i}(a_s, m_t, \mu) \right)$$

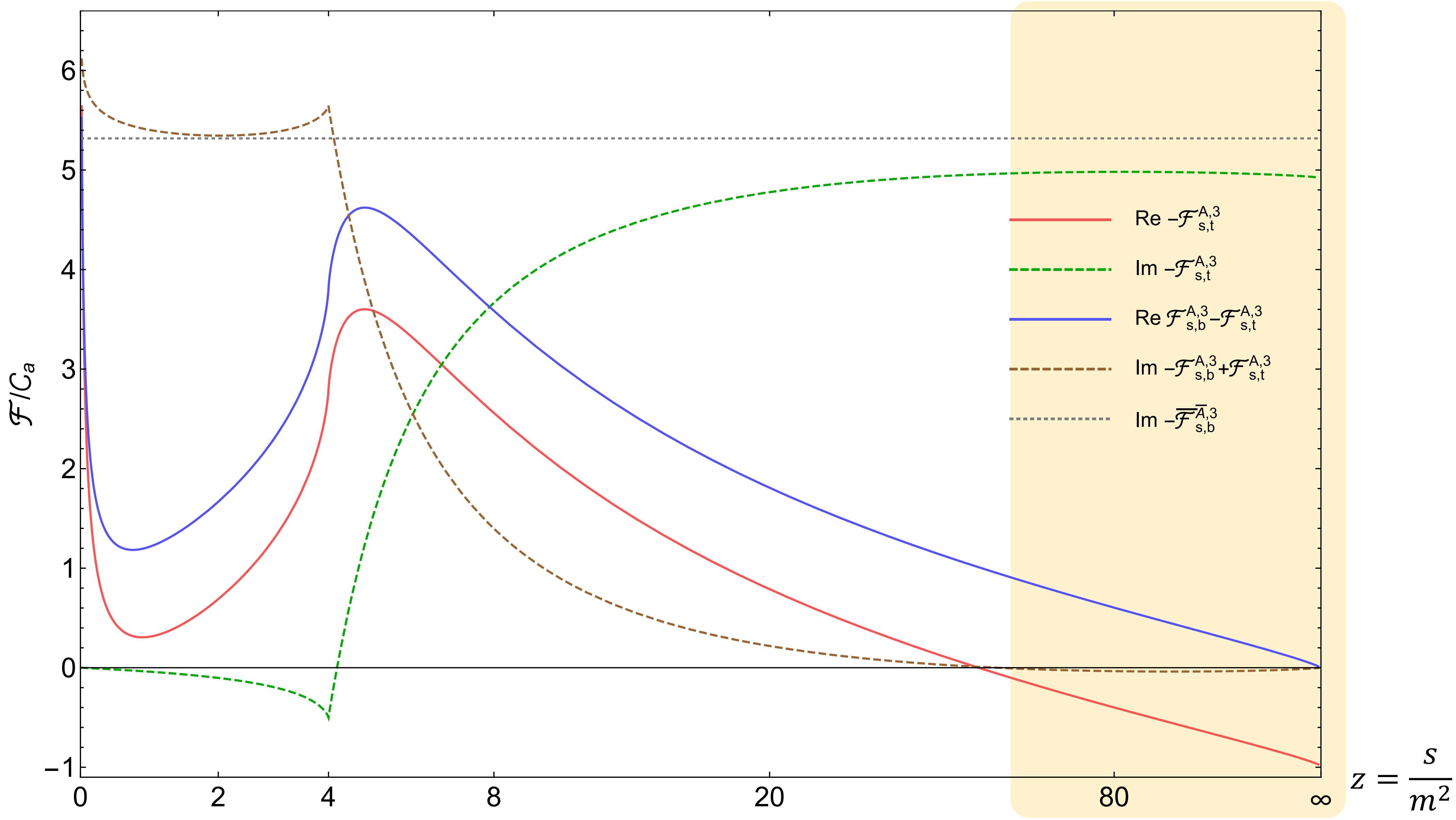
Outline

- Introduction (✓)
- Preliminaries (✓)
- Computation (✓)
- Renormalization (✓)
- Results

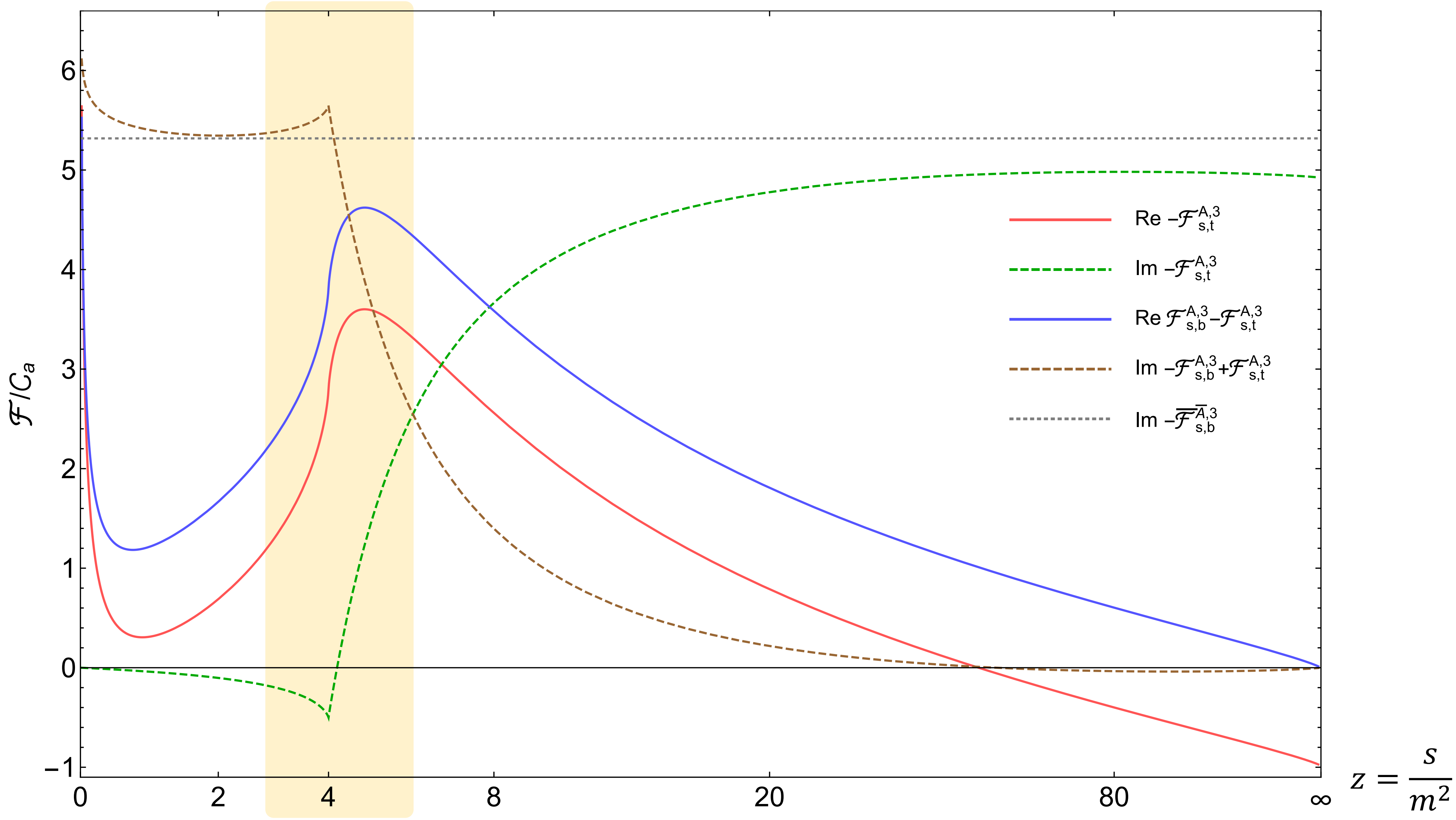
Results



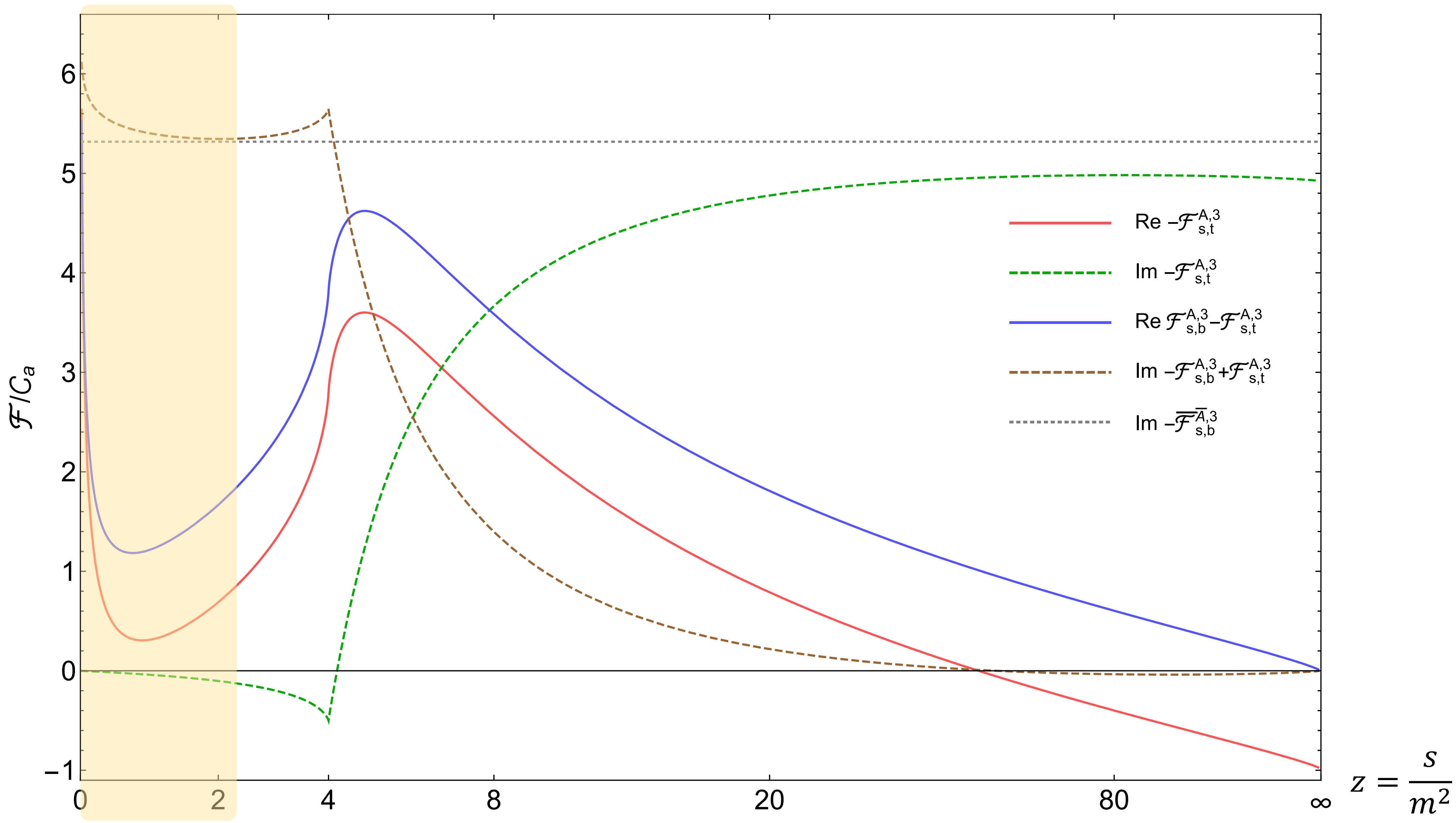
- $C_a = \mathrm{Re}\{$[Gehrmann and Primo `21]$\}$
- $\mu^2 = s$



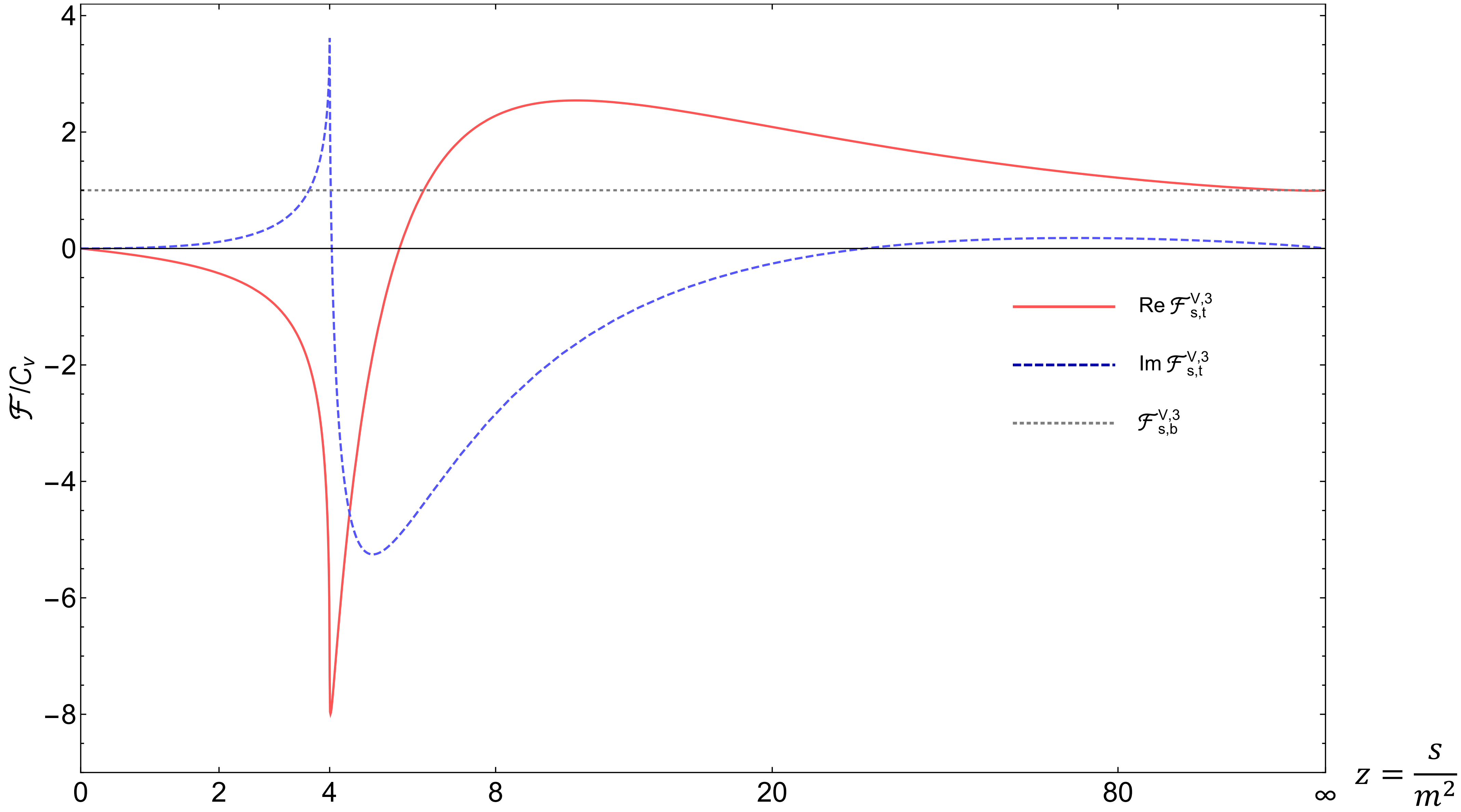
- Strong check: $\mathcal{F}^{A}_{s,t}(a_s, x) \to \mathcal{F}^{A}_{s,b}(a_s, x)$ in high energy limit $\Leftrightarrow$ F^A_S vanishes with 6 massless quarks
- Green dashed does not overlap with dotted gray, because of 6 massless quarks in gluon self-energy insertion while the reference has 5 (same reason for the red curve not approaching exactly -1)



- Typical behaviour due to Coulomb effect at threshold
  - Real part varies smoothly
  - Imaginary part experiences a sharp turn



- Non-decoupling mass logarithms become visible in low energy limit



- Vector part only features a power-suppressed logarithmic behavior

Result at Low Energies

- In this region it is more sensible to renormalize the perturbative coupling constant such that the heavy quark is decoupled:

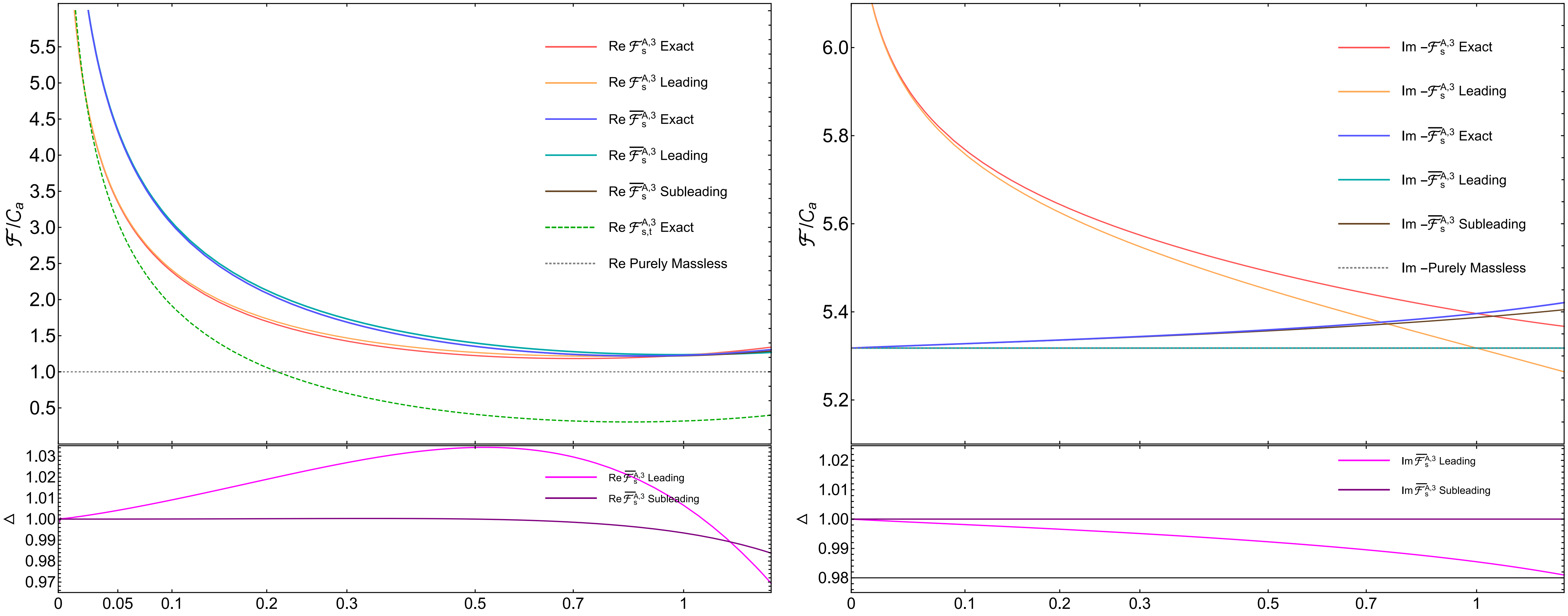
$$a_s = \zeta_\alpha \, \bar{a}_s \qquad\qquad \zeta_\alpha = 1 + \bar{a}_s \, \frac{2}{3} \, \ln \frac{\mu^2}{m_t^2} + \mathcal{O}(\bar{a}_s^2)$$

- Re-expand:

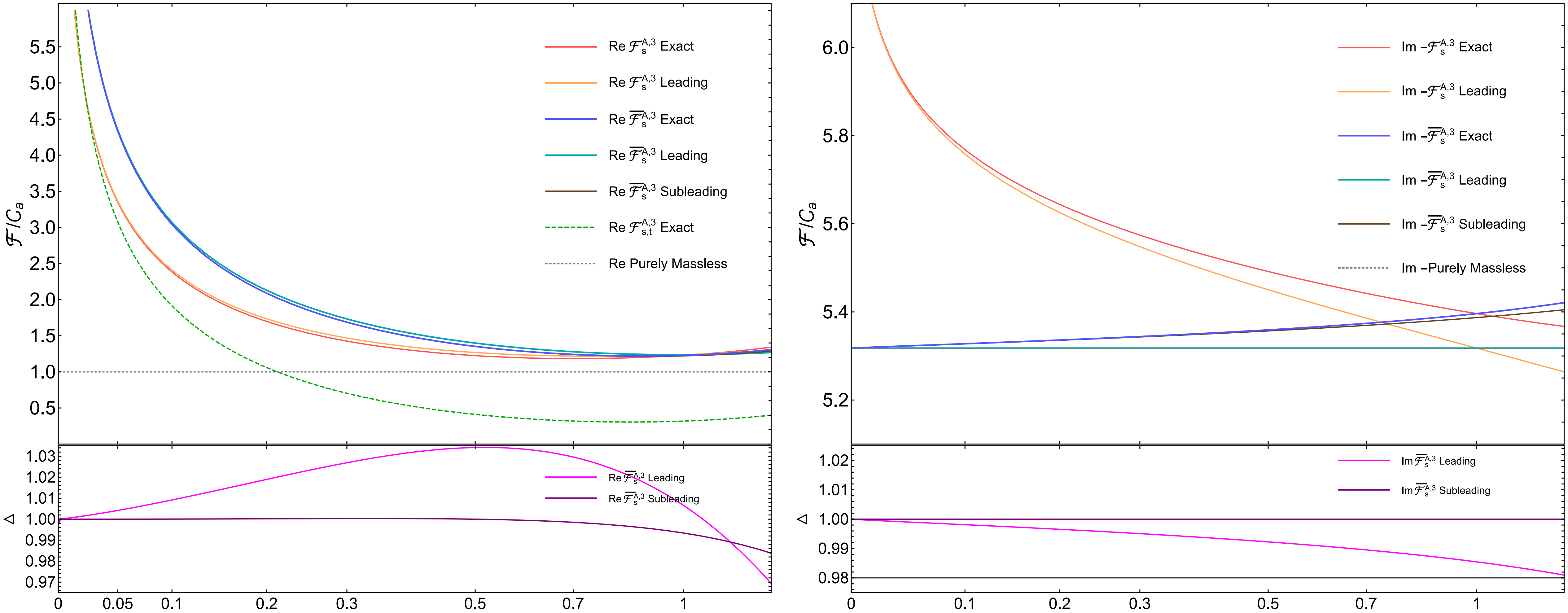
$$\begin{aligned}
\bar{\mathcal{F}}^A_{s,b}(\bar{a}_s, m_t, \mu) &= \mathcal{F}^A_{s,b}(a_s = \zeta_\alpha \, \bar{a}_s, m_t, \mu) \\
&= \bar{a}_s^2 \, \bar{\mathcal{F}}^{A,2}_{s,b}(\mu) + \bar{a}_s^3 \, \bar{\mathcal{F}}^{A,3}_{s,b}(m_t, \mu) + \mathcal{O}(\bar{a}_s^4) \\
\bar{\mathcal{F}}^A_{s,t}(\bar{a}_s, m_t, \mu) &= \mathcal{F}^A_{s,t}(a_s = \zeta_\alpha \, \bar{a}_s, m_t, \mu) \\
&= \bar{a}_s^2 \, \bar{\mathcal{F}}^{A,2}_{s,t}(m_t, \mu) + \bar{a}_s^3 \, \bar{\mathcal{F}}^{A,3}_{s,t}(m_t, \mu) + \mathcal{O}(\bar{a}_s^4)
\end{aligned}$$

- $\Rightarrow$

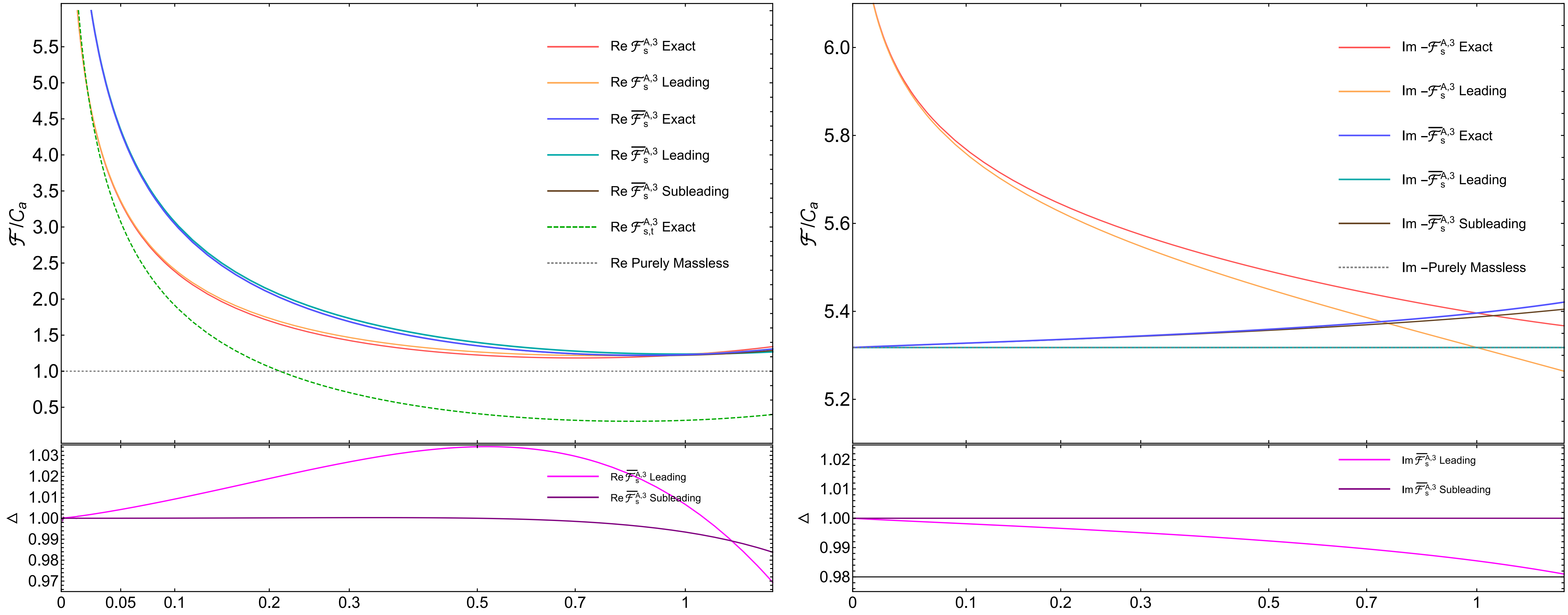
$$\bar{\mathcal{F}}^{A,3}_s(x) \equiv \bar{\mathcal{F}}^{A,3}_{s,b}(x) - \bar{\mathcal{F}}^{A,3}_{s,t}(x) = \sum_{n=0}^{\infty} \sum_{m=\underline{m}_n}^{\overline{m}_n} c_{n,m} \, x^n \, \ln^m x$$



- Compare leading and sub-leading large mass approximation with exact result in range (0, 1.33), corresponding to $\sqrt{s} \in (0, 200) GeV$ at $m_t = 173 GeV$



- Accuracy for leading approximation deviating at most 3%
- Including the sub-leading term, the deviation is at most 1%



- Note: The logarithmic enhancement in the imaginary part is removed by decoupling

➢ In leading power approximation: $\mathrm{Im}\left[\bar{\mathcal{F}}_s^{A,3}(x)\right]$ is just a constant (given by the purely massless result)

Conclusions

- We determined numerically the finite remainder of the singlet contribution to the massless quark FF with exact top quark mass dependence for the axial and vector part
- This ingredient should be included for an appropriate renormalization scale dependence and for the non-decoupling mass logarithms
- The result provides one of the missing ingredients needed to push the theoretical predictions of Z-mediated Drell-Yan processes to the third order in QCD coupling
  - ➢ See for example [Duhr and Mistlberger `21]