Algebraic geometry and p-adic numbers for scattering amplitude ansätze

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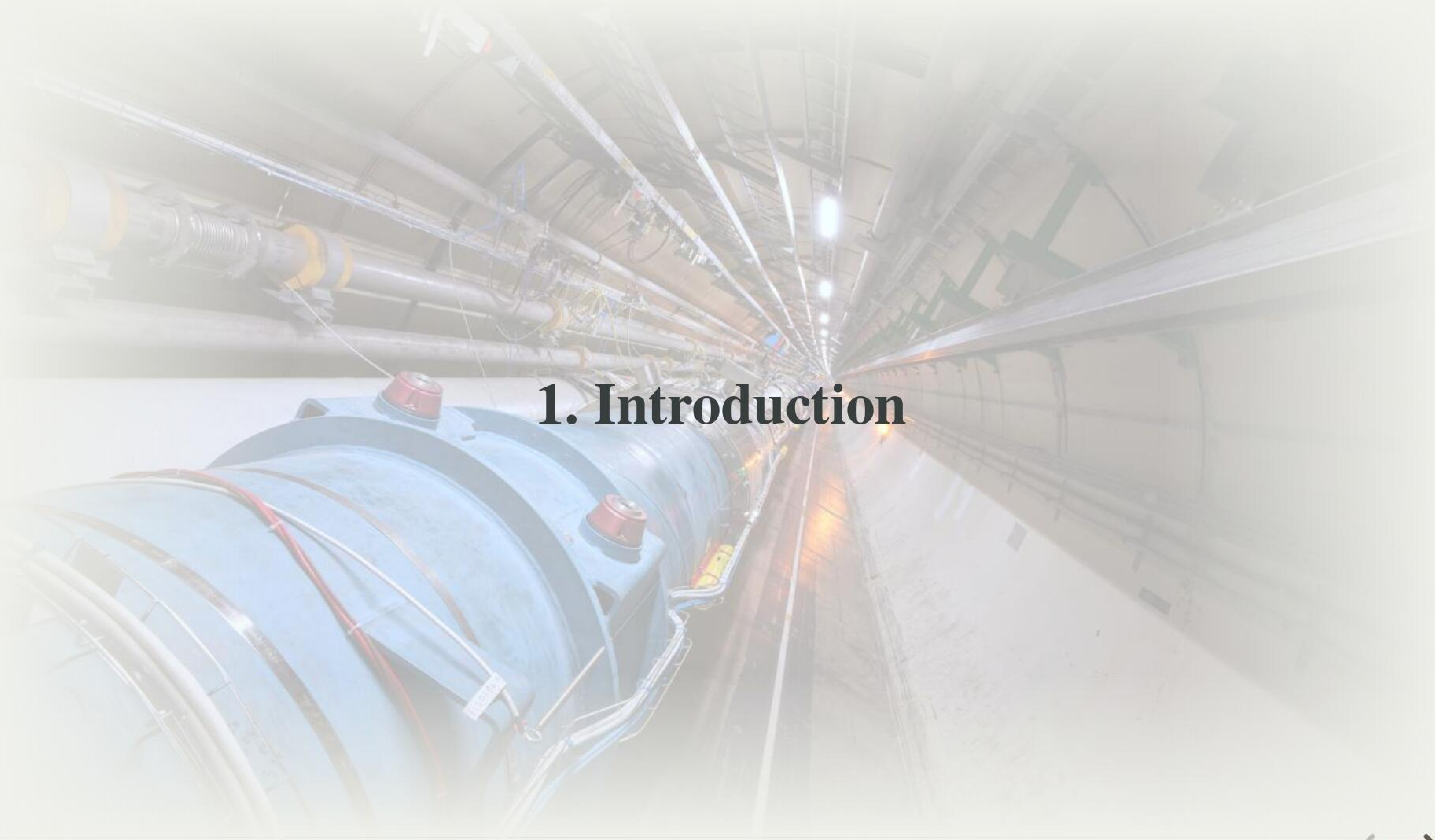


1. Introduction

The %-level fixed-order precision frontier

- Back-of-the-envelope calculation: $\alpha_S^2 \sim 0.01\ ,\ \ \alpha \sim 0.01$
- To match experimental precision need at least NNLO QCD & NLO EW corrections
- This requires $0 \to n$ at 2-loop, $0 \to n+1$ at 1-loop and $0 \to n+2$ at tree level
- Much progress recently achieved to uncover 2-loop corrections for $2 \to 3$ processes

$pp \to Wjj$ [Abreu et al. '21] , $pp \to W\gamma j$ [Badger et al. '22],
$pp \to Hbb$ [Badger et al. '21] , $pp \to Wbb$ [Badger et al. '21],
$pp \to \gamma\gamma\gamma$ [Chawdhry et al. '19, Abreu et al. '20] , $pp \to jjj$ [Abreu et al. '19],
$pp \to \gamma\gamma j$ [Agarwal et al. '21, Badger et al. '21, Chawdhry et al. '21]

Organization of loop amplitude computations

- Universal ϵ-pole structure [Catani '98; Becher, Neubert '09; Gardi, Magnea '09]

$$\mathcal{A}_{n,R}^{(l)} = \sum_{l'=0}^{l-1} \mathbf{I}_{\epsilon}^{(l-l')} \mathcal{A}_{n,R}^{(l')} + \mathcal{H}_n^{(l)} + \mathcal{O}(\epsilon)$$

- Finite remainder ($\lambda, \tilde{\lambda}$: Weyl spinors)

$$\mathcal{H}_n^{(l)} = \sum_i \; c_i(\lambda, \tilde{\lambda}) \; \times \; F_i(\lambda, \tilde{\lambda})$$

$c_i(\lambda, \tilde{\lambda})$: rational functions; $F_i(\lambda, \tilde{\lambda})$: special (transcendental) functions.

- For example, at five-point $F_i(\lambda, \tilde{\lambda}) \; \rightarrow$ pentagon functions

[Gehrmann, Henn, Lo Presti '18; Chicherin, Sotnikov '20; Chicherin, Sotnikov, Zoia '22]

Problem set-up & motivation

- Standard procedure at 2-loop: analytically reconstruct C_i from finite-field ($\mathbb{F}_p$) samples [von Manteuffel-Schabinger '15, Peraro '16]

- Number of required samples grows rapidly with multiplicity of external kinematics

Process	$\mathcal{H}^{(2)}_{pp\to jjj}$	$\mathcal{H}^{(2)}_{pp\to\gamma\gamma\gamma}$	$\mathcal{H}^{(2)}_{pp\to W+2j}$	$\mathcal{A}^{(1)}_{0\to 6g} + \mathcal{O}(\epsilon)$
Ansatz size	$\mathcal{O}(10^5)$	$\mathcal{O}(10^5)$	$\mathcal{O}(10^6)$	$\mathcal{O}(10^9)$

- $\mathcal{A}^{(1)}_{0\to 6g}$ done in $\sim 10^4$ evaluations over $\mathbb{C}$ w/ constraints from singular limits [DL, Maître '19]

Can we systematize this approach and apply it directly at 2-loop numerics?

2. Geometry of spinor space

Complex & massless kinematics

- Let us consider scattering amplitudes in the analytical continutation to complex momenta

$$P_i \in \mathbb{C}^{D=4}\,, \qquad i \in (1, \dots, n)\,.$$

- The P_i live in the $(1/2, 1/2)$ representation of the Lorentz group. We have

$$\det(P_{i,\mu}\sigma^{\mu,\alpha\dot{\alpha}}) = m_i^2\,.$$

- For massless scattering, $m_i = 0 \;\Rightarrow\; \mathrm{rank}(P_i{}^{\alpha\dot{\alpha}}) = 1$ (down from 2)

$$P_i{}^{\alpha\dot{\alpha}} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} = |i\rangle[i|\,,$$

$$P_i \in \mathbb{R}^4 \Rightarrow \tilde{\lambda}^{\dot{\alpha}} = (\lambda^\alpha)^*\,, \qquad P_i \in \mathbb{C}^4 \Rightarrow \lambda^\alpha, \tilde{\lambda}^{\dot{\alpha}} \text{ independent}\,.$$

Constraints from singular limits

- Take an NMHV tree as a simplified example for a C_i

$$\mathcal{A}^{(0)}_{q^+g^+g^+\bar{q}^-g^-g^-} = \frac{\mathcal{N}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle[45][56][61]s_{345}}\,, \qquad \mathcal{N} \in \mathcal{M}_{6,\{-1,0,0,1,0,0\}}\,, \quad \dim(\mathcal{M}_{d,\vec{\phi}}) = 143\,.$$

where $\mathcal{M}_{d,\vec{\phi}}$: vector space of products of spinor brackets, d: mass dimension, $\vec{\phi}$: phase weights.

- Take $\langle 12\rangle \sim \varepsilon$ and $\langle 23\rangle \sim \varepsilon$, get two cases (soft vs. collinear):

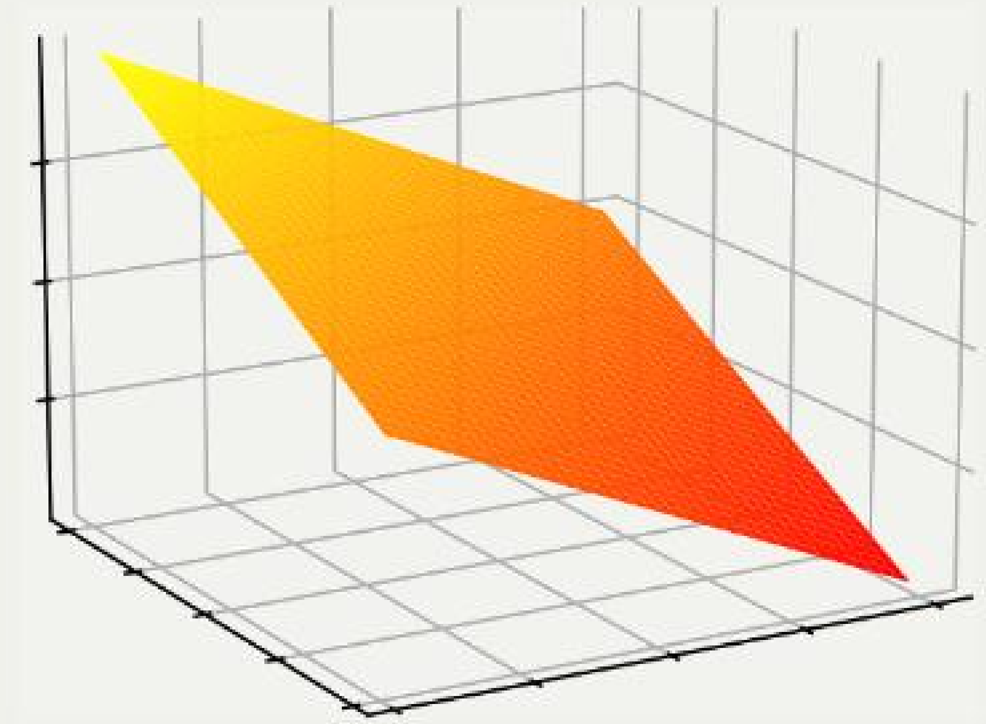
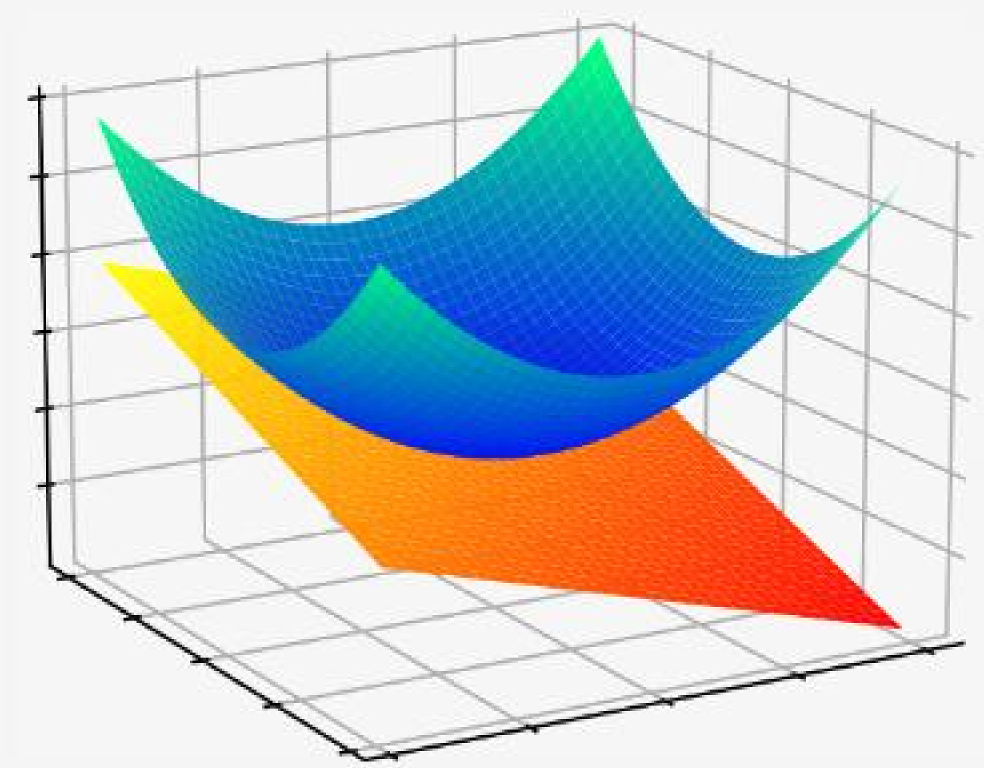
$$\begin{cases} \circ \quad |2\rangle \sim \varepsilon \quad \rightarrow \quad \mathcal{A}^{(0)}_{q^+g^+g^+\bar{q}^-g^-g^-} \sim \varepsilon^{-2} \quad \Rightarrow \quad \mathcal{N} \sim \varepsilon^0 \;:\; \text{no constaint} \\ \circ \quad \langle 12\rangle \sim \langle 23\rangle \sim \langle 13\rangle \sim \varepsilon \quad \rightarrow \quad \mathcal{A}^{(0)}_{q^+g^+g^+\bar{q}^-g^-g^-} \sim \varepsilon^{-1} \quad \Rightarrow \quad \mathcal{N} \sim \varepsilon^1 \;:\; \underline{\text{constraint}}! \end{cases}$$

- This constraint drops the ansatz size from 143 to 84: fixed 59 parameters with 1 evalution!

Why is there branching and how do we control it?

Crash course in algebraic geometry

- Notation employed for polynomial rings: $\mathbb{F}[x_1, \ldots, x_n]$, e.g. $r = \mathbb{R}[x, y, z]$

Algebra ∼ Ideals	**Geometry ∼ Varieties**
$K = \langle x - y - z \rangle_r = \{a\,(x - y - z) : a \in r\}$ In general: $\langle q_1, \ldots, q_k \rangle_A = \left\{ \sum_{i=1}^{k} a_i q_i : a_i \in A \right\}$	 $= V(K)$
$J = \langle x^3 - x^2y - x^2z + xy^2 - xz - y^3 - y^2z + yz + z^2 \rangle_r$ $= \langle (x - y - z)(x^2 + y^2 - z) \rangle_r$ $= \langle (x - y - z) \rangle_r \cap \langle (x^2 + y^2 - z) \rangle_r$ $= K \cap L$	 $= V(J) = V(K) \cup V(L)$

Towards the domain of the $C_i\left(\lambda, \tilde{\lambda}\right)$

- Let us start from the polynomial ring (the spinors are understood to be taken component-wise)

$$S_n = \mathbb{F}\left[|1\rangle, [1|, \ldots, |n\rangle, [n|\right] \; .$$

- $\mathbb{F}$ may be any of $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{F}_p$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, … (see later!).

- Define the momentum-conservation ideal as (it has 4 generators, written as a 2×2 tensor)

$$J_{\Lambda_n} = \Big\langle \sum_i |i\rangle[i| \Big\rangle_{S_n} \; .$$

- Two polynomials p and q should be equivalent if they differ by an element of J_{Λ_n}

$$p - q \in J_{\Lambda_n} \quad \Rightarrow \quad p \sim q \, .$$

Singular varieties

- What we need is not a polynomial ring but a *quotient ring*

$$R_n = S_n / J_{\Lambda_n} \; .$$

- We dub ideals of R_n generated by denominator factors ***singular ideals*** .
  A ***singular variety*** is a variety associated to a singular ideal. It is a sub-variety of $V(J_{\Lambda_n})$.

- The example we saw earlier can be mathematically described by the following decomposition:

$$\big\langle \langle 12\rangle, \langle 23\rangle \big\rangle_{R_6} = \big\langle |2\rangle \big\rangle_{R_6} \cap \big\langle \langle 12\rangle, \langle 23\rangle, \langle 13\rangle \big\rangle_{R_6}$$

- There exist algorithms to compute minimal decompositions of ideals/varieties.

Five-point irreducible varieties

- Singular ideals with one generator: 2 independent ones, both irreducible

$$\langle\langle 12\rangle\rangle_{R_5} \quad \text{and} \quad \langle\langle 1|2+3|1]\rangle_{R_5}$$

with permutations & $\lambda \leftrightarrow \tilde{\lambda}$ swap we get 35 in total. These are the possible poles.

- Singular ideals with two generators: 11 reducible, 10 irreducible

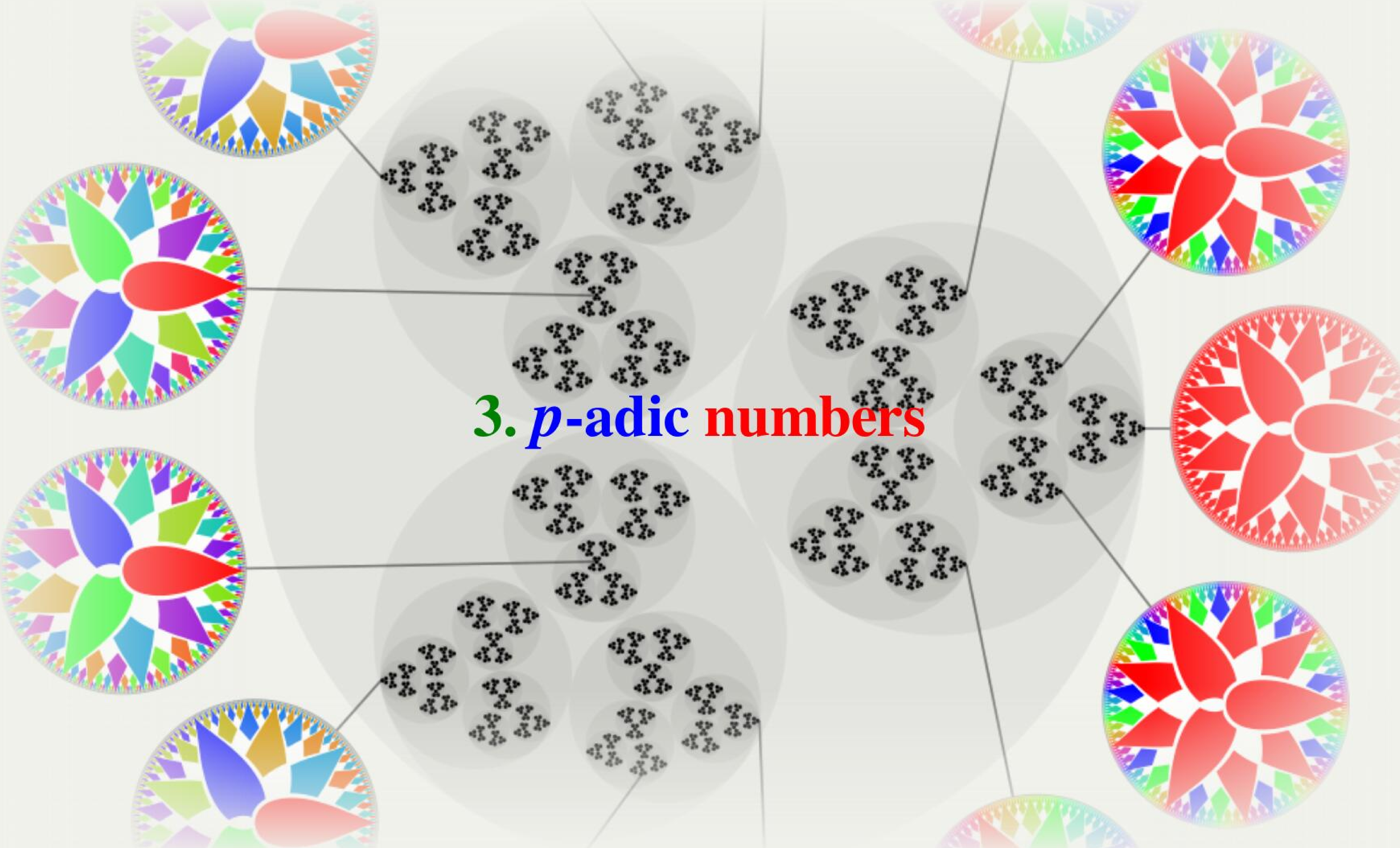
$$\langle\langle 12\rangle, \langle 23\rangle\rangle_{R_5} = \langle|2\rangle\rangle_{R_5} \cap \langle\langle 12\rangle, \langle 23\rangle, \langle 13\rangle, [45]\rangle_{R_5} \cap \langle\langle ij\rangle\, \forall\, i,j\rangle_{R_5}\,,$$

$$\langle\langle 12\rangle, \langle 34\rangle\rangle_{R_5} = \langle\langle 12\rangle, \langle 34\rangle, |1+2|5]\rangle_{R_5} \cap \langle\langle ij\rangle\, \forall i,j\,\rangle_{R_5}$$

with symmetries: 555 reducible varieties in total, 317 irreducible branches.

How do we obtain the degree of divergence of the C_i ?

3. *p*-adic numbers



Why p-adic numbers?

- One way to obtain the pole order of the C_i is to evaluate them near the singular varieties.
- But this require a non-trivial notion of size, which finite fields lack because:

$$|a = 0|_{\mathbb{F}_p} = 0 \quad \text{or} \quad |a \neq 0|_{\mathbb{F}_p} = 1\,.$$

- This is known as the *trivial absolute value*.
- It means we are either *exactly on* a variety or *away from* a variety.
- We treat the p-adics just as a field, not covered: p-adic calculus/analysis.

The p-adic numbers

- We can define a p-adic number $x \in \mathbb{Q}_p$ in terms of its expansion in powers of a prime p

$$x = \sum_{i=\nu_p(x)}^{\infty} a_i p^i = a_{\nu_p(x)} p^{\nu_p(x)} + \cdots + a_{-1} p^{-1} + a_0 + a_1 p + a_2 p^2 + \cdots ,$$

  where the a_i are called p-adic digits, $0 \leq a_i < p$, and $\nu_p(x)$ is called the *valuation*.

- The subset with $\nu_p(x) = 0$ is the p-adic integers ($\mathbb{Z}_p$), in which case a_0 behaves as if in $\mathbb{F}_p$.

- The p-adic absolute value is:

$$|x|_p = p^{-\nu_p(x)} \quad \Longrightarrow \quad |p|_p < |1|_p < \left|\frac{1}{p}\right|_p$$

p-adic phase-space points near singular varieties

- Aim: find a p-adic phase-space point **near** a singular variety

Step 0: Consider a singular ideal

$$J = \left\langle q_1, \ldots, q_l, \textstyle\sum_i |i\rangle[i| \right\rangle_{S_n} .$$

Step 1: Find a point $(\eta, \tilde{\eta}) \in V(J)$ in a finite field, i.e. in $\mathbb{F}_p^{4n}$.

Step 2: Lift the $\mathbb{F}_p^{4n}$ solution to $(\eta^\varepsilon, \tilde{\eta}^\varepsilon) \in \mathbb{Q}_p^{4n}$, such that

$$q_1 \sim \cdots \sim q_l \sim \mathcal{O}(p)\,, \qquad \textstyle\sum_i |i\rangle[i| \sim \mathcal{O}(p^k)\,,\; k \gg 1\,,$$

where k denotes the working precision (number of p-adic digits used).

- The procedure is described in details in the paper.

4. Ansatz construction

Zariski-Nagata theorem

[Zariski '49; Nagata '62; Eisenbud-Hochster '79]

- The Zariski-Nagata theorem states that if a polynomial vanishes to k-th order on a variety then it belongs to the so-called k-th symbolic power of the associated radical ideal, in our case:

$$\mathcal{N}(\lambda,\tilde{\lambda})|_{(\eta^\varepsilon,\tilde{\eta}^\varepsilon)\text{ near }V(J)} \sim \varepsilon^{k>0} \qquad \Longrightarrow \qquad \mathcal{N} \in J^{\langle k\rangle}$$

Symbolic powers

[Grifo '18, Lemma 1.18]

- Even if J is irreducible, J^k may not be

$$J^k = \underbrace{J\cdot\cdots\cdot J}_{k\text{ times}} = J^{\langle k\rangle} \cap J_1^{\text{em}} \cap \cdots \cap J_m^{\text{em}}$$

"em" stands for "embedded", i.e. $V(J) \supset V(J_i^{\text{em}})$, while $V(J^{\langle k\rangle}) = V(J)$.

The Ansatz

- Let $\mathcal{V}^{(2)}$ denote the set of codimension-2 irreducible varieties (317 at five-point) and let $\kappa(\mathcal{N}_i, U)$ denote the order of vanishing of $\mathcal{N}_i$ on $U \in \mathcal{V}^{(2)}$.

- We can constrain the common-denominator ansatz as follows

$$\mathcal{N}_i \in \mathcal{M}_{d,\vec{\phi}} \cap \mathfrak{J} \quad \text{with} \quad \mathfrak{J} = \bigcap_{U \in \mathcal{V}^{(2)}} I(U)^{\langle \kappa(\mathcal{N}_i, U) \rangle} .$$

the ansatz is a basis of the vector space $\mathcal{M}_{d,\vec{\phi}} \cap \mathfrak{J}$.

- This can be computed numerically or analytically by polynomial reduction, together with standard linear algebra (null-spaces).

Proof-of-concept results: $0 \to q\bar{q}\gamma\gamma\gamma$ at 2-loop

- Planar two-loop three-photon pentagon-function coefficients:

Remainder	$s_{ij} \cup tr_5$ **vector space**	$\dim(\mathcal{M}_{d,\vec{0}})$	$\langle ij\rangle \cup [ij]$ **vector space**	$\dim(\mathcal{M}_{d,\vec{\phi}})$	$\dim(\mathcal{M}_{d,\vec{\phi}} \cap \mathfrak{J})$	**ratio**
$R^{2,0}_{\gamma^-\gamma^+\gamma^+}$	$\mathcal{M}_{50,\vec{0}}$	41301	$\mathcal{M}_{35,\{3,0,6,-3,-2\}}$	7358	566	73
$R^{2,N_f}_{\gamma^-\gamma^+\gamma^+}$	$\mathcal{M}_{24,\vec{0}}$	2821	$\mathcal{M}_{15,\{-2,-2,0,-3,-3\}}$	378	20	141
$R^{2,0}_{\gamma^+\gamma^+\gamma^+}$	$\mathcal{M}_{32,\vec{0}}$	7905	$\mathcal{M}_{20,\{-2,-4,-2,-2,-2\}}$	1140	18	439
$R^{2,0}_{\gamma^+\gamma^+\gamma^+}$	$\mathcal{M}_{18,\vec{0}}$	1045	$\mathcal{M}_{8,\{1,3,1,1,2\}}$	44	6	174

- Almost two orders of magnitude better than common denominator form.

- Even just allowing for non-zero Little group weights reduces the sampling requirements.

5. Conclusions

Conclusions

- The C_i are more naturally expressed in terms of spinors instead of mandelstams or twistors;
- Algebraic geometry provides us the language to describe singularities of scattering amplitudes;
- p-adic numbers *bridge the gap* between $\mathbb{C}$ and $\mathbb{F}_p$, rescuing a non-trivial absolute value;
- Behaviour on singular varieties gives very constraining information about the analytics;
- The theorem by Zariski and Nagata provides us with a way to interpret the constraints.

Outlook

- Apply this technology directly to numerical evaluations from Caravel [Abreu et al. '20];
- Obtain efficient and stable analytical results for rational coefficients up to high multiplicities.

Unfortunately much had to be left out

- The invariant subring $\mathcal{R}_n \subset R_n$
- Radical vs. non-radical ideals (not really neaded with 5-point massless kinematics)
- Primary decompositions: prime vs. primary ideals
- Computation of symbolic powers
- Construction of common-denominator ansatz and intersection of vector spaces with ideals
- Direction of approach to singular variety (ring extensions)
- Partial fractions from maximal-codimension ideal membership
- High multiplicity results: $\mathcal{A}^{(1)}_{q\bar{q}V(\to\ell\bar{\ell})V'(\to\ell'\bar{\ell}')g}$ with q-loop m dependace [Campbell, DL, Ellis '22]

Thank you!

This presentation was powered by:

jupyter, RISE, Reveal, Mathjax, Singular

syngular, lips, numpy, matplotlib, sympy

Backup Slides

The QCD NMHV tree from the example

- The tree used as an example, written in 3 different ways (1^{st} is BCFW)

$$
\begin{aligned}
\mathcal{A}^{(0)}_{q^+g^+g^+\bar{q}^-g^-g^-} &= \frac{\langle 6|1+2|3]^2}{\langle 12\rangle[45]\langle 2|1+6|5]s_{345}} - \frac{\langle 4|2+3|1]^2}{[16]\langle 23\rangle\langle 34\rangle[56]\langle 2|1+6|5]} \\
&= -\frac{[12]\langle 45\rangle\langle 6|1+2|3]}{\langle 12\rangle[16]\langle 34\rangle[45]s_{345}} - \frac{\langle 4|2+3|1]s_{123}}{\langle 12\rangle[16]\langle 23\rangle\langle 34\rangle[45][56]} \\
&= -\frac{\langle 6|1+2|3]s_{123}}{\langle 12\rangle\langle 23\rangle[45][56]s_{345}} - \frac{[12]\langle 45\rangle\langle 4|2+3|1]}{[16]\langle 23\rangle\langle 34\rangle[56]s_{345}}
\end{aligned}
$$

Varieties of ideals, ideals of varieties

- A variety U associated to an ideal J of a ring R and denoted as $U = V(J)$ is the set of points at which the generators $p_1, \ldots, p_k$ of J evaluate to zero

$$V(\langle p_1, \ldots, p_k \rangle_R) = \left\{ (x_1, \ldots, x_n) \in \mathbb{F}^n \, : \, p_i(x_1, \ldots, x_n) = 0 \text{ for } 1 \leq i \leq k \right\}.$$

- An ideal J of a ring R associated to a variety U and denoted as $J = I(U)$ is the subset of polynomials $\{p_i\} \subseteq R$ which evaluate to zero on U

$$I(U) = \left\{ p_i(x_1, \ldots, x_n) \in R \, : \, p_i(x_1, \ldots, x_n) = 0 \text{ for } (x_1, \ldots, x_n) \in \mathbb{F}^n \right\}.$$

The field of fractions

- The coefficients $c_i(\lambda, \tilde{\lambda})$ belong to the field of fractions of R_n

$$c_i(\lambda, \tilde{\lambda}) \in FF(R_n)$$

- This is non-trivial and there are subtleties at low multiplicities [Campbell, DL, Ellis '22]

1. $FF(R_3)$ is actually not a field, because R_3 is not an Integral Domain
2. $FF(R_4)$ is a field, but it is not a Unique Factorization Domain

Minimal decompositions of varieties

- A variety is irreducible if

$$U = U_1 \cup U_2 \quad \Rightarrow \quad U_1 = U \quad \text{or} \quad U_2 = U\,.$$

- All varieties admit a *minimal decomposition*

$$U = \bigcup_{k=1}^{n_B(U)} U_k \quad \text{s.t.} \quad U_i \nsubseteq U_j \;\forall\, i \neq j \text{ and } n_B(U_k) = 1 \;\forall k\,.$$

where we call U_k branches, and $n_B(U)$ denotes the number of branches.

Radical, primary and prime ideals

$$J \text{ is radical if } a^k \in J \;\; \Rightarrow \;\; a \in J\,;$$

$$Q \text{ is primary if } ab \in Q \;\; \Rightarrow \;\; a \in Q \text{ or } b^k \in Q\,,\;\; k \in \mathbb{Z}_{\geq 0}\,;$$

$$P \text{ is prime if } P \text{ is radical and primary.}$$

Primary decompositions

$$J = \bigcap_{i=1}^{n_Q(J)} Q_i\,, \quad \text{with} \;\; Q_i \;\; \text{primary,} \; \sqrt{Q_i} \text{ distinct}\,,$$

$$\text{and } Q_m \not\supseteq \textstyle\bigcap_{l \neq m} Q_l\,.$$

$P_i = \sqrt{Q_i}$ are prime ideals. In general one has: $n_Q(J) \geq n_U(V(J))$.

Examples of fully fledged primary decompositions

In 6-point kinematics (R_6):

$$\Big\langle \langle 12\rangle, \Delta_{12|34|56} \Big\rangle = \Big\langle \langle 12\rangle, (s_{56} - s_{34})^2 \Big\rangle \,,$$

$$\sqrt{\Big\langle \langle 12\rangle, \Delta_{12|34|56} \Big\rangle} = \Big\langle \langle 12\rangle, (s_{56} - s_{34}) \Big\rangle \,.$$

In 7-point kinematics (R_7):

$$\Big\langle \langle 7|\Gamma_{12}|7], \langle 7|\Gamma_{34|56}|7\rangle \Big\rangle = \Big\langle \langle 12\rangle, \langle 17\rangle, \langle 27\rangle \Big\rangle \;\cap\; \Big\langle [12], \Gamma_{12}|7\rangle \Big\rangle \;\cap\; \Big\langle \langle 7|\Gamma_{12}|7], |7\rangle\langle 7| \Big\rangle$$
$$\cap\; \Big\langle \langle 7|\Gamma_{34}|7], \langle 7|\Gamma_{56}|7], \langle 7|\Gamma_{34|56}|7\rangle, [7|\Gamma_{34|56}|7] \Big\rangle$$

$$\sqrt{\Big\langle \langle 7|\Gamma_{12}|7], |7\rangle\langle 7| \Big\rangle} = \Big\langle |7\rangle \Big\rangle \,.$$

Absolute values on the rationals

- An absolute value is a map: $\mathbb{F} \to \mathbb{R}_{\geq 0}$, satisfying:

  1. non-negativity; 2. positive-definiteness; 3. multiplicativity; 4. the triangle inequality.

- Ostrowski's theorem: there exist only 3 possible absolute values on $\mathbb{Q}$:

  0. The trivial absolute value $|x|_0 = \{0 \text{ if } x \text{ is } 0 \text{ else } 1\}$,
  1. The usual absolute value $|x|_\infty = \{x \text{ if } x \geq 0 \text{ else } -x\}$,
  2. The p-adic absolute value $|x|_p = p^{-\nu_p(x)}$.

The least common denominator form

- A single p-adic evaluation near each codimension-one variety yields the pole orders

$$q_{ij} = \nu_p(\mathcal{C}_i(\eta^\epsilon, \tilde{\eta}^\epsilon))\,, \quad \text{where} \quad (\eta, \tilde{\eta}) \in V(\langle \mathcal{D}_j \rangle)$$

starting from R_5.

- This gives us the common-denominator form ($\mathcal{N}_i$ unknown)

$$\mathcal{C}_i = \frac{\mathcal{N}_i}{\prod_j \mathcal{D}_j^{q_{ij}}}\,,$$

where q_{ij} may be negative (common numerator factors). I am assuming pure $\mathcal{F}_i$.

The least common denominator ansatz

- We have already encountered the vector space

$$\mathcal{M}_{d,\vec{\phi}} = \Big\{ \langle ij\rangle^{\alpha_{ij}} [ij]^{\beta_{ij}} \;:\; j > i\,,\; \sum_{ij} \alpha_{ij} + \beta_{ij} = d\,, \\ \textstyle\sum_{ij} \alpha_{ij}(\delta_{ik} + \delta_{jk}) - \beta_{ij}(\delta_{ik} + \delta_{jk}) = \phi_k \Big\}$$

- Let $[x]$ denote the mass-dimension of x and $\{x\}$ its Little-group weights
- The least-common-denominator ansatz is (assuming pure $\mathcal{F}_i$)

$$\mathcal{C}_i = \frac{\mathcal{N}_i}{\prod_j \mathcal{D}_j^{q_{ij}}} \quad \Longrightarrow \quad \mathcal{N}_i \in \mathcal{M}_{[\mathcal{A}]+\left[\prod_j \mathcal{D}_j^{q_{ij}}\right],\{\mathcal{A}\}+\left\{\prod_j \mathcal{D}_j^{q_{ij}}\right\}}$$

Hilbert's Nullstellensatz

- A perhaps better-known version of the Zariski-Nagata theorem is Hilbert's Nullstellensatz:

$$\mathcal{N}(\lambda, \tilde{\lambda})|_{\varepsilon \text{ away from } V(J)} \sim \varepsilon^{k>0} \quad \Longrightarrow \quad \mathcal{N} \in \sqrt{J}$$

this is, however, less powerful as it does not use information about the degree of vanishing.