Recent progress in intersection theory for Feynman integrals decomposition.

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Introduction

Feynman integrals

$$\mathcal{I} = \int \prod_{j=1}^{\ell} \mathrm{d}^d k_j \, \frac{1}{\mathcal{D}_1^{a_1} \dots \mathcal{D}_n^{a_n}}$$

1. Dimreg $d = 4 - 2\varepsilon$.
2. Number of propagators $n = \ell(\ell + 1)/2 + E\ell$.

Integration by Parts Identities (IBP)

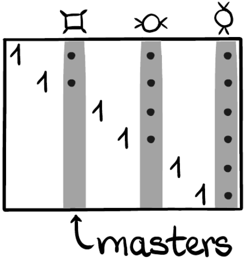
Linear relations among Feynman integrals have a long history and numerous applications:

[Tkachov] [Chetyrkin Tkachov] [Kotikov] [Remiddi] [Laporta] [Gehrmann Remiddi] [Henn] [Papadopoulos] ...

$$0 = \int \mathrm{d}^d k_j \, \frac{\partial}{\partial k_j^\mu} \left(v^\mu \, \frac{1}{\mathcal{D}_1^{a_1} \ldots \mathcal{D}_n^{a_n}} \right) \quad \text{for arbitrary } v^\mu = v^\mu(p,k) \; .$$

Decompose $\mathcal{I}$ in terms of a finite set of master integrals.

- What is the number of master integrals?
- What differential equation do they obey?
- How to compute the IBP coefficients?



Learn about the mathematical structure of pQFT $\Longleftrightarrow$ propose new computational methods.

Intersection theory

Baikov representation

Parametric representation of Feynman integrals:

$$\mathcal{I} = \int \prod_{j=1}^{\ell} \mathrm{d}^d k_j \, \frac{1}{\mathcal{D}_1^{a_1} \dots \mathcal{D}_n^{a_n}} \implies \int_{\mathcal{C}} \mathcal{B}(x)^{\gamma} \, \frac{\mathrm{d}x}{x_1^{a_1} \dots x_n^{a_n}} \, ,$$

where the Baikov polynomial:

$$\mathcal{B}(z) := \det \begin{pmatrix} k_1 \cdot k_1 & k_1 \cdot k_2 & \dots & k_1 \cdot p_E \\ k_1 \cdot k_2 & k_2 \cdot k_2 & \dots & k_2 \cdot p_E \\ \dots & \dots & \dots & \dots \\ p_E \cdot k_1 & p_E \cdot k_2 & \dots & p_E \cdot p_E \end{pmatrix} \Bigg|_{\text{sub } s_{ij} \text{ in terms of } x}$$

Features:

1. $\gamma = (d - \ell - E - 1)/2$
2. $\mathcal{B}(\partial \mathcal{C}) = 0$
3. Unitarity cuts: $\int \mathrm{d}x \mapsto \oint \mathrm{d}x$

Basics of Intersection theory

$$\mathcal{I} = \int_{\mathcal{C}} u(x)\, \varphi(x)$$

Twisted cycle $\int_{\mathcal{C}} u(x)$: contour $\mathcal{C}$ & multivalued twist $u(x)$, s.t. $u(\partial\mathcal{C}) = 0$.
Twisted cocycle: holomorphic n-form $\varphi(x) = \hat{\varphi}(x)\, \mathrm{d}x_1 \wedge \ldots \mathrm{d}x_n$.
Stokes theorem:

$$0 = \int_{\mathcal{C}} \mathrm{d}(u\, \xi) = \int_{\partial\mathcal{C}} u\, \xi = \int_{\mathcal{C}} u\, (\mathrm{d}\xi + \mathrm{d}\log u \wedge \xi) \ .$$

Covariant derivative: $\nabla_\omega := \mathrm{d} + \omega\wedge$ and

$$\omega := \mathrm{d}\log u \ .$$

Linear relation between integrals:

$$\int_{\mathcal{C}} u\, \varphi \equiv \int_{\mathcal{C}} u\, (\varphi + \nabla_\omega \xi) \ .$$

Dual integrals are similar: $\mathcal{I} \mapsto \mathcal{I}^\vee$, $u \mapsto u^{-1}$, $\nabla_\omega \mapsto \nabla_{-\omega}$.

De Rham twisted cohomology

Equivalence class of forms: $\langle\varphi| : \varphi \sim \varphi + \nabla_\omega \xi \rightsquigarrow$ left forms.
Twisted cohomology groups:

$$\langle\varphi| \in \mathbb{H}^n_\omega := \{n \text{ forms} \,|\, \nabla_\omega \varphi_n = 0\}/\{\nabla_\omega \varphi_{n-1}\} \ .$$

Equivalence class of dual forms: $|\psi\rangle : \psi \sim \psi + \nabla_{-\omega} \xi \rightsquigarrow$ right forms.
Dual twisted cohomology groups:

$$|\psi\rangle \in \mathbb{H}^n_{-\omega} := \{n \text{ forms} \,|\, \nabla_{-\omega} \psi_n = 0\}/\{\nabla_{-\omega} \psi_{n-1}\} \ .$$

Features of twisted cohomology

Relates various fields of mathematics to Feynman Integrals.

Many ways to count masters r: [Smirnov, Petukhov]

1. Laporta algorithm. [Laporta]
2. Number of critical points $\mathrm{d}\log u = \omega = 0$. [Lee, Pomeransky]
3. Number of independent contours. [Bosma, Sogaard, Zhang] [Primo, Tancredi]
4. Number of independent forms $= \dim(\mathbb{H}^n_{\pm\omega})$. [Mastrolia, Mizera] [FGLMMMM]
5. Holonomic rank of GKZ system (volumes of polytopes). [Henrik's talk] [CGMMMMT]

Now let's turn to IBP coefficients.

Cohomology intersection numbers

Fix contour $\mathcal{C} \implies$ finite dimensional vector space of integrals:

$$\mathcal{I} = \langle \varphi | \mathcal{C}] = \int_{\mathcal{C}} u \, \varphi \, , \quad \mathcal{I}^{\vee} = [\mathcal{C} | \psi \rangle = \int_{\mathcal{C}} u^{-1} \, \psi \, ,$$

$$\langle c_1 \varphi_1 + c_2 \varphi_2 | \mathcal{C}] = c_1 \langle \varphi_1 | \mathcal{C}] + c_2 \langle \varphi_2 | \mathcal{C}] \, .$$

Masters form a basis: $\langle e_\lambda |$ for $\lambda \in \{1, \ldots, r\}$.

Cohomology intersection number = scalar product of integrals:

$$\langle \varphi | \psi \rangle = \frac{1}{(2\pi \mathrm{i})^n} \int \varphi \wedge \psi \, .$$

Is a rational function of parameters.

How to use it?

Master decomposition formula

Allows to directly decompose $\mathcal{I} = \sum_{\lambda=1}^{r} c_\lambda \, \mathcal{J}_\lambda$ in a basis of masters $\mathcal{J}_\lambda = \langle e_\lambda | \mathcal{C}]$!

Master decomposition formula and identity operator: [Mastrolia Mizera] [FGLMMMM]

$$\langle \varphi | = \sum_{\lambda=1}^{r} c_\lambda \, \langle e_\lambda | \; , \quad \mathbb{I}_{\mathrm{c}} = \sum_{\lambda,\mu} | h_\mu \rangle \left(C^{-1} \right)_{\mu\lambda} \langle e_\lambda | \; .$$

The coefficients (independent of $|h_\mu\rangle$):

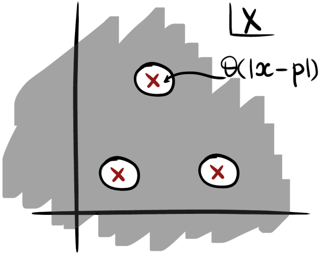
$$c_\lambda = \sum_{\mu=1}^{r} \langle \varphi | h_\mu \rangle \left(C^{-1} \right)_{\mu\lambda} \quad \text{with} \quad C_{\lambda\mu} = \langle e_\lambda | h_\mu \rangle \; .$$

So to compute IBP coefficients we need to learn how to work with $\langle \varphi | \psi \rangle$.

Univariate intersection numbers $n = 1$

Regulate and compute via residues:

$$\langle\varphi|\psi\rangle = \frac{1}{2\pi \mathrm{i}} \int_X \iota(\varphi) \wedge \psi$$
$$\equiv \frac{1}{2\pi \mathrm{i}} \int_X \left(\varphi - \sum_{p\in\mathcal{P}} \nabla_\omega(\theta_p(x,\bar{x}) f_p)\right) \wedge \psi$$
$$= \sum_{p\in\mathcal{P}} \operatorname*{Res}_{x=p} [f_p\, \psi] \ ,$$



- Integrate over $X = \mathbb{C}\mathrm{P}^1$.
- $\mathcal{P} :=$ {poles of ω}, including $\infty : y = 1/x$ and $y \to 0$.
- Regulate using local potential $\nabla_\omega f_p \equiv (\mathrm{d} + \omega\wedge) f_p = \varphi$ around p .
- Ansatz $f_p = f_{p,\,\min}(x-p)^{\min} + f_{p,\,\min+1}(x-p)^{\min+1} + \ldots + f_{p,\,\max}(x-p)^{\max}$.

Multivariate intersection numbers $n > 1$

Recursive method: apply univariate procedure one variable at a time.
Example: $\varphi(x_1, x_2)$, $\psi(x_1, x_2)$ compute $\langle\varphi|\psi\rangle \rightsquigarrow$ integrate out x_1 first, then x_2 .
Pick $\langle e_\lambda|$ and $|h_\mu\rangle$ bases for x_1-intersections & project φ, ψ onto them:

$$\langle\varphi| = \langle e_\lambda| \wedge \langle\varphi_\lambda| \ , \quad \langle\varphi_\lambda| = \langle\varphi|h_\mu\rangle \left(C^{-1}\right)_{\mu\lambda} \ ,$$
$$|\psi\rangle = |h_\mu\rangle \wedge |\psi_\mu\rangle \ , \quad |\psi_\mu\rangle = \left(C^{-1}\right)_{\mu\lambda} \langle e_\lambda|\psi\rangle \ .$$

Insert the identity operator:

$$\langle\varphi|\psi\rangle = \langle\varphi|h_\mu\rangle \left(C^{-1}\right)_{\mu\lambda} \langle e_\lambda|\psi\rangle = \sum_{p\in\mathcal{P}} \operatorname*{Res}_{x_2=p} \left[f_{p,\lambda}\, C_{\lambda\mu}\, \psi_\mu\right] \ .$$

Requires a local vector potential:

$$\partial_2 f_{p,\lambda} + f_{p,\mu} P_{\mu\lambda} = \varphi_\lambda \quad \text{around pole } p \ , \quad P_{\lambda\nu} := \langle(\partial_2 + \omega_2) e_\lambda|h_\mu\rangle \left(C^{-1}\right)_{\mu\nu} \ .$$

What do we row reduce here?

Solve for ρ:

$$\begin{cases} [x\,\partial_x + P]\,\vec{f} = \vec{\varphi} \\ \rho = \mathrm{Res}_{x=0}\left[\vec{f}\cdot\vec{\psi}\right] \end{cases}$$

Expand in Laurent series:

$$P = \sum_p x^p P_p = P_0 + x P_1 + \dots$$

$$\vec{\varphi} = x^{-k}\vec{\varphi}_{-k} + \dots\,, \qquad \vec{\psi} = x^{-m}\vec{\varphi}_{-m} + \dots$$

Row reduce to find this element

-1	$\vec{\psi}_1$	$\vec{\psi}_0$	$\vec{\psi}_{-1}$	$\vec{\psi}_{-2}$	$\vec{\psi}_{-3}$	$\cdot$
$\cdot$	P_0-2	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\vec{\varphi}_{-2}$
$\cdot$	P_1	P_0-1	$\cdot$	$\cdot$	$\cdot$	$\vec{\varphi}_{-1}$
$\cdot$	P_2	P_1	P_0	$\cdot$	$\cdot$	$\vec{\varphi}_0$
$\cdot$	P_3	P_2	P_1	P_0+1	$\cdot$	$\vec{\varphi}_1$
$\cdot$	P_4	P_3	P_2	P_1	P_0+2	$\vec{\varphi}_2$

Nuances:

- Resonances.
- Higher order poles.
- If P is "simple pole" $\rightsquigarrow$ Global residue theorem.

[Weinzierl]

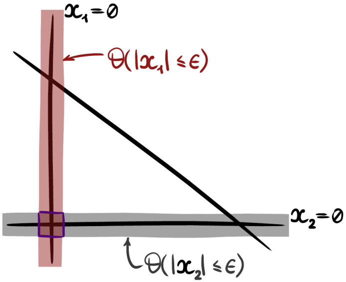
Multivariate DEQ method

From hyperplane arrangement ...

Consider $u(x_1, x_2) = (x_1 x_2(x_1 + x_2 - 1))^\gamma$ with singularities along hyperplanes. [Matsumoto]
Define three potentials:

$$\nabla(\hat{f}_1 \, \mathrm{d}x_2) = \varphi \quad \text{near } x_1 = 0 \, ,$$
$$\nabla(\hat{f}_2 \, \mathrm{d}x_1) = \varphi \quad \text{near } x_2 = 0 \, ,$$
$$\nabla f_{12} = f_1 - f_2 \quad \text{near } x_1 = x_2 = 0 \, .$$

$$\langle \varphi | \psi \rangle = \sum \underset{x_1=0}{\mathrm{Res}} \, \underset{x_2=0}{\mathrm{Res}} \big(f_{12} \, \psi \big) \, .$$



. . . to Multivariate DEQ and Residues (coming soon!)

$$\langle \varphi | \psi \rangle = \int_X \iota(\varphi) \wedge \hat{\psi}\, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \ ,$$

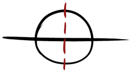
where $X = \mathbb{C}\mathrm{P}^2$ and ι-regulator is:

$$\begin{aligned} \iota(\varphi) &= \varphi - \nabla\big(\theta_1 f_1 + \theta_2(1-\theta_1) f_2 + \theta_2 \mathrm{d}\theta_1 f_{12}\big) \\ &= \ldots - \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2\ f_{12}\ . \end{aligned}$$

The last term $\Longrightarrow$ double residue $\mathrm{Res}\,\mathrm{Res}$.
Define $\nabla_i := \mathrm{d}z_i\, \partial_i + \omega_i\, \mathrm{d}z_i \wedge$.

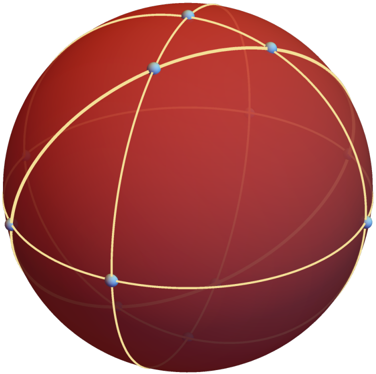
$$\nabla f_{12} \equiv \nabla_1 f_{12} + \nabla_2 f_{12} = f_1 - f_2 \rightsquigarrow \begin{cases} \nabla_2 f_{12} = f_1 \\ \nabla_1 f_{12} = f_2 \end{cases} \rightsquigarrow \boxed{\nabla_1 \nabla_2 f_{12} = \varphi} \Longleftarrow \text{New!}$$

Example: massless sunrise

 $\rightsquigarrow u(x_1, x_2) = \left(-\frac{1}{4s} x_1 x_2 (x_1 + x_2 - s)\right)^{-\varepsilon}$

Singular surfaces:

$$\{x_1 = 0, x_2 = 0, x_1 + x_2 - s = 0\}$$



$\Longrightarrow$ 6 intersection points in $\mathbb{CP}^2$.

Consider $\varphi = \psi = \frac{\mathrm{d}x_1 \mathrm{d}x_2}{x_1 x_2} \rightsquigarrow$ residues:

$$\langle\varphi|\psi\rangle = \frac{-1}{3\varepsilon^2} + \frac{-1}{3\varepsilon^2} + \frac{1}{\varepsilon^2} + 0 + 0 + 0 = \frac{1}{3\varepsilon^2} \; .$$

Features of the multivariate DEQ method (coming soon!)

Method:

1. Find intersection points of singular surfaces.
2. Solve multivariate DEQ: $\nabla_1 \ldots \nabla_n f = \varphi$.
3. Compute multivariate residue $\mathrm{Res} \ldots \mathrm{Res}(f\psi)$.

Some nuances:

- Can use rational function reconstruction. [Peraro] [Klappert, Klein, Lange] [Klappert, Lange, Maierhöfer, Usovitsch]
- n-variate intersection numbers $\langle\varphi|\psi\rangle$ = integrals over projective space $\mathbb{CP}^n$
  $\Longrightarrow$ many "infinities" (i.e. $x_1, x_2 \to \infty$, x_1/x_2 fixed) $\rightsquigarrow$ toric geometry.
- Requires resolution of singularities $\rightsquigarrow$ sector decomposition.

Secondary equation approach

Secondary equation for the C matrix

A way to decompose $\varphi = \sum_{\lambda=1}^{r} c_\lambda \, e_\lambda$ via $C_{\lambda\mu} = \langle e_\lambda | h_\mu \rangle$. [FGLMMMM] [Weinzierl]

$$\begin{cases} \partial_i \, e_\lambda = (P_i)_{\lambda\nu} \, e_\nu \\ \partial_i \, h_\mu = (P_i^\vee)_{\mu\xi} \, h_\xi \end{cases} \Longrightarrow \boxed{\partial_i \, C = P_i \cdot C + C \cdot (P_i^\vee)^{\mathrm{T}}}$$

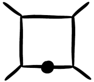
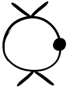
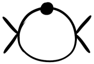
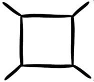
Algorithms for rational solutions of such systems. [IntegrableConnections]
Repeat for auxiliary basis $e^{\mathrm{aux}} := \{e_1, \ldots, e_{r-1}, \varphi\}$: compute C^{aux} using P^{aux} .

$$\begin{pmatrix} e_1 \\ \vdots \\ e_{r-1} \\ \varphi \end{pmatrix} = C^{\mathrm{aux}} \cdot C^{-1} \begin{pmatrix} e_1 \\ \vdots \\ e_{r-1} \\ e_r \end{pmatrix} \quad \rightsquigarrow \quad C^{\mathrm{aux}} \cdot C^{-1} = \left(\begin{array}{ccc|c} & \mathrm{id}_{r-1} & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline c_1 & \cdots & c_{r-1} & c_r \end{array}\right)$$

Compute Pfaffians P and P^{aux} via GKZ formalism and Macaulay matrix.

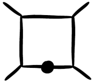
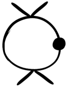
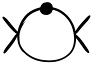
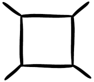
[Henrik's talk] [CGMMMMT] [Risa/Asir]

Secondary equation: box example

Decompose $\varphi =$  into canonical basis: $(e_1, e_2, e_3) = \Big($  ,  ,  $\Big)$

$$P = \begin{pmatrix} -\frac{\epsilon\left(\delta^2(12z+11)+7\delta(z+1)+z+1\right)}{(3\delta+1)z(z+1)} & -\frac{\delta^2\epsilon}{(3\delta+1)(z+1)} & \frac{\delta^2\epsilon(\delta(z+2)+1)}{2(3\delta+1)z(z+1)(\delta\epsilon+1)} \\ \frac{\delta^2\epsilon}{(3\delta+1)z(z+1)} & -\frac{\delta^2\epsilon}{(3\delta+1)(z+1)} & -\frac{\delta^2\epsilon(\delta+2\delta z+z)}{2(3\delta+1)z(z+1)(\delta\epsilon+1)} \\ -\frac{2(2\delta+1)\epsilon(\delta\epsilon+1)}{(3\delta+1)z(z+1)} & \frac{2(2\delta+1)\epsilon(\delta\epsilon+1)}{(3\delta+1)(z+1)} & -\frac{\epsilon\left(\delta^2(5z+7)+\delta(2z+5)+1\right)}{(3\delta+1)z(z+1)} \end{pmatrix}, \quad P^\vee = P\big|_{\varepsilon\to-\varepsilon}$$

$$C = \begin{pmatrix} -\frac{(2\delta+1)(4\delta+1)}{\delta} & \delta & -2(\delta\epsilon-1) \\ \delta & -\frac{(2\delta+1)(4\delta+1)}{\delta} & -2(\delta\epsilon-1) \\ 2(\delta\epsilon+1) & 2(\delta\epsilon+1) & -\frac{4\left(10\delta^2+6\delta+1\right)(\delta\epsilon-1)(\delta\epsilon+1)}{\delta^3} \end{pmatrix}$$

 $= -\frac{2\varepsilon(2\varepsilon+1)}{z(\varepsilon+1)} \cdot$  $+ 0 \cdot$  $+ (2\varepsilon+1) \cdot$  , $\quad z = t/s$.

Conclusions

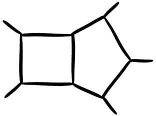
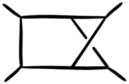
1. Twisted cohomology unites areas of math related to Feynman integrals.
2. Intersection number acts as scalar product $\rightsquigarrow$ direct projection onto basis of masters.
3. Multivariate DEQ method:
   - Localizes on intersection points of singular surfaces $\rightsquigarrow$ toric geometry.
   - Resolution of singularities (sector decomposition).
4. Secondary equation: get IBP coefficients from rational solutions to DEQ.

   [CGMMMMT] [IntegrableConnections] [DiffTools] [Sigma]

5. Employ rational function reconstruction (FiniteFlow, FireFly & Kira).

   [Peraro] [Klappert Klein, Lange] [Klappert, Lange Maierhöfer, Usovitsch]

Outlook:

- Automate resolution of singularities. [pySecDec] [desing]
- Practical issues: canonical basis, more loops and legs. [Chen, Jiang Xu, Yang] [Chen, Jiang Ma, Xu, Yang]
- Recursive method:   coming soon!

Bonus slides

Riemann twisted period relations (a slide for David)

Quadratic relations between Feynman Integrals. [Broadhurst, Roberts] [Zhou] [Broadhurst, Mellit]

Two identity operators: cohomological [Cho, Matsumoto] [Mastrolia, Mizera] [FGLMMMM]

$$\mathbb{I}_{\rm c} = \sum_{\lambda,\mu} |h_\mu\rangle \left(C^{-1}\right)_{\mu\lambda} \langle e_\lambda| \ , \quad C_{\lambda\mu} = \langle e_\lambda | h_\mu \rangle \ ,$$

($C_{\lambda\mu}$ from intersections or Macaulay matrix) and homological

$$\mathbb{I}_{\rm h} = \sum_{\lambda,\mu} |\mathcal{C}_\mu] \left(H^{-1}\right)_{\mu\lambda} [\mathcal{D}_\lambda| \ , \quad H_{\lambda\mu} = [\mathcal{D}_\lambda | \mathcal{C}_\mu] \ .$$

Twisted Riemann Period Relations:

$$\langle \varphi, \psi \rangle = \langle \varphi | \mathbb{I}_{\rm h} | \psi \rangle = \sum_{\lambda,\mu} \underbrace{\langle \varphi | \mathcal{C}_\mu]}_{\text{Feyn. Int.}} \left(H^{-1}\right)_{\mu\lambda} \underbrace{[\mathcal{D}_\lambda | \psi\rangle}_{\text{Feyn. Int.}} \ .$$

Can continue inserting $\mathbb{I}_{\rm h}$ and $\mathbb{I}_{\rm c}$ $\rightsquigarrow$ higher order identities, coaction? [Abreau, Britto, Duhr, Gradi]

Beyond hyperplanes: double box on the maximal cut

$$u(x_1, x_2) = \left(x_1 x_2 (st + s(x_1 + x_2) + x_1 x_2)\right)^{\gamma} \ .$$

Two of the intersection points degenerate $\rightsquigarrow$ resolution of singularities is needed.