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UV and IR rational terms in two-loop amplitudes

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JHEP 05 (2020) 077 [2001.11388]
JHEP 10 (2020) 016 [2007.03713]
JHEP 01 (2022) 105 [2107.10288]
& in preparation

Loops and Legs - Ettal - April 2022

Motivation: automated numerical approach at two loops

- Drastically increased complexity of multi-leg amplitudes are successfully addressed by the numerical recursion at one-loop level. [OpenLoops, Recola, MadLoop, HELAC]

- Extension of numerical recursion algorithm to a fully generic two-loop amplitude generator is under development. [see Max Zoller's talk]

- Numerical algorithms construct numerators of loop integrands in 4-dim, while the missing $(D-4)$-dim part needs to be reconstructed in D dimensional regularisation.

- **Rational counterterms** reconstruct $(D-4)$-dim part from loop numerators

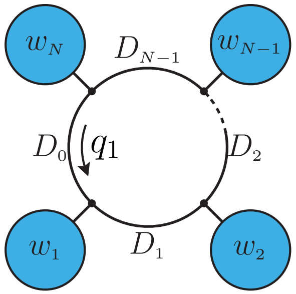
  $\Rightarrow$ one loop rational terms [Ossola, Papadopoulos, Pittau]

  $\Rightarrow$ in this talk:

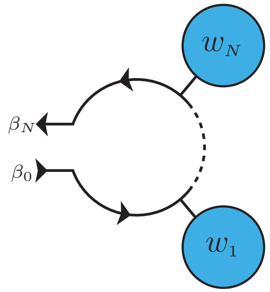
  - Construction and results of two-loop **UV** rational counterterms in renormalisable theories
  - First insights into two-loop **IR** rational terms and their cancellations

One-loop numerical amplitudes: OpenLoops method

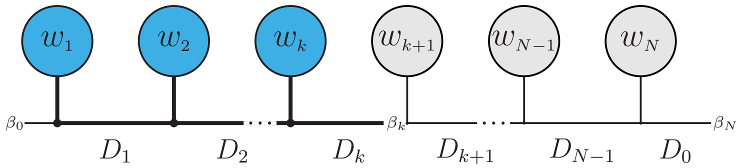
Open loop recursion constructs D-dim regularised amplitudes with 4-dim numerator



$$= \int \mathrm{d}\bar{q}_1 \frac{\bar{\mathcal{N}}(\bar{q}_1)}{D_0 \cdots D_{N-1}} \xrightarrow[\text{in 4-dim}]{\text{cut open}} \left[\mathcal{N}(q_1)\right]^{\beta_N}_{\beta_0} =$$

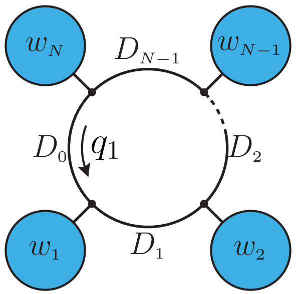


e.g. $\mathcal{N}_k(q_1) =$

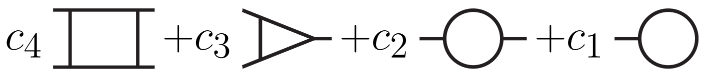


at k-th step of recursion

$\Rightarrow$ **Fully automated, model-flexible, process-independent and highly-efficient** method



$$= \underbrace{\int \mathrm{d}\bar{q}_1 \frac{\mathcal{N}(q_1)}{D_0 \cdots D_{N-1}}}_{} + \underbrace{\textbf{UV rational term}}_{}$$



by reduction to scalar integrals

reconstruct $(D-4)$-part by **UV rational counterterm** insertion and **IR rational terms vanish**

One-loop rational counterterms [Ossola, Papadopoulos, Pittau '08]

Amplitude of an one-loop amputated 1PI diagram γ in $D = 4 - 2\varepsilon$ dimensions

$$\bar{\mathcal{A}}_{1,\gamma} = \underbrace{\mu^{2\varepsilon} \int \mathrm{d}^D \bar{q}_1 \frac{\bar{\mathcal{N}}(\bar{q}_1)}{D_0(\bar{q}_1) \cdots D_{N-1}(\bar{q}_1)}}_{\text{DimReg in } D = 4 - 2\varepsilon} = \sum_{n=-2}^{\infty} I_n \, \varepsilon^n \,, \quad \text{with } D_k(\bar{q}_1) = (\bar{q}_1 + p_k)^2 - m_k^2$$

Rational counterterm (UV) emerges by splitting D-dim numerator into 4-dim and ε-dim parts

$$\bar{\mathcal{N}}(\bar{q}_1) = \mathcal{N}(q_1) + \tilde{\mathcal{N}}(\bar{q}_1) \quad \text{with} \left\{ \begin{array}{lcl} \bar{q}_1 & = & q_1 + \tilde{q}_1 \\ \bar{\gamma}^{\bar{\mu}} & = & \gamma^{\mu} + \tilde{\gamma}^{\tilde{\mu}} \\ \bar{g}^{\bar{\mu}\bar{\nu}} & = & g^{\mu\nu} + \tilde{g}^{\tilde{\mu}\tilde{\nu}} \end{array} \right.$$

$$\Rightarrow \quad \underbrace{\bar{\mathcal{A}}_{1,\gamma}}_{D_{\mathrm{n}}=D} = \underbrace{\mathcal{A}_{1,\gamma}}_{D_{\mathrm{n}}=4} + \delta\mathcal{R}_{1,\gamma} \qquad \text{with numerator dimension } D_{\mathrm{n}} = \{D, 4\}$$

- $\delta\mathcal{R}_{1,\gamma}$ from interplay between ε-dim $\tilde{\mathcal{N}}$ and $\frac{1}{\varepsilon}$ UV poles $\Rightarrow$ extract from UV-divergent part
- No IR rational terms [Bredenstein, Denner, Dittmaier, Pozzorini '08]

Rational counterterms from UV divergence by tadpole expansion

Reduce degree of UV divergence by massive tadpole expansion on denominator

$$\frac{1}{(\bar{q}_1+p_k)^2-m_k^2} \overset{\bar{q}_1\to\infty}{=} \underbrace{\frac{1}{\bar{q}_1^2-M^2}}_{\substack{\text{leading UV tadpole}\\ \mathcal{O}(1/\bar{q}_1^2)}} + \underbrace{\frac{-p_k^2-2\,\bar{q}_1\cdot p_k+m_k^2-M^2}{(\bar{q}_1^2-M^2)^2}}_{\substack{\text{sub-leading UV tadpole}\\ \mathcal{O}(1/\bar{q}_1^3)}} + \mathcal{O}(1/\bar{q}_1^4)$$

Isolate UV divergence of amplitudes by expanding all denominators

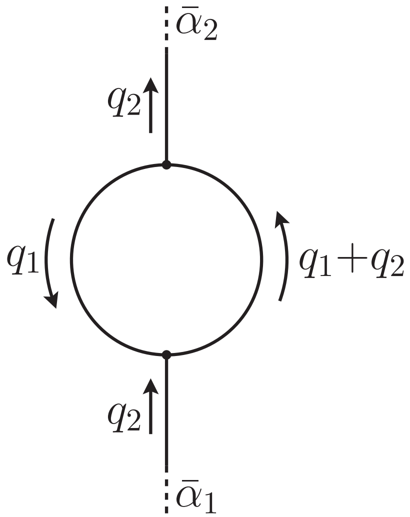
$$\bar{\mathcal{A}}_{1,\gamma} = \underbrace{\text{UV-divergent tadpole integrals}}_{\text{contain numerator } (\mathcal{N}+\tilde{\mathcal{N}})} + \text{UV-finite part}$$

$$\Rightarrow \bar{\mathcal{A}}_{1,\gamma}\big|_{\text{UV-part}} = \underbrace{-\delta Z_{1,\gamma}}_{1/\varepsilon\ \mathrm{MS}\ \text{pole}} + \underbrace{\delta\mathcal{R}_{1,\gamma}}_{\text{finite rational CT}}$$

$\Rightarrow$ $\delta\mathcal{R}_{1,\gamma}$ and $\delta Z_{1,\gamma}$ from same UV part $\Rightarrow$ $\delta\mathcal{R}_{1,\gamma}$ local CT: tree-level Feynman rules (simplest)

One-loop UV subdivergences in two-loop diagram in $D_\mathsf{n} = 4$

- One-loop diagram with 4-dim external momenta $q_2 \Rightarrow \mathcal{A}_{1,\gamma}(q_2)\big|_{\text{UV-part}} = -\delta Z_{1,\gamma}(q_2)$
- One-loop **subdiagram** with D-**dim external loop momenta** $\bar{q}_2 = q_2 + \tilde{q}_2$



$\Rightarrow$ **additional ε-dim $(\tilde{q}_2)^2$ terms** show up, e.g.

$$\frac{1}{(\bar{q}_1 + q_2 + \tilde{q}_2)^2} = \frac{1}{\bar{q}_1^2 - M^2} + \frac{-(q_2 + \tilde{q}_2)^2 - 2\,\bar{q}_1 \cdot (q_2 + \tilde{q}_2) - M^2}{(\bar{q}_1^2 - M^2)^2} + \ldots$$

$\Rightarrow$ yield **extra quadratic pole term** $\delta\tilde{Z}_{1,\gamma}(\tilde{q}_2) \propto \dfrac{\tilde{q}_2^2}{\varepsilon}$ s.t.

$$\mathcal{A}_{1,\gamma}(q_2 + \tilde{q}_2)\big|_{\text{UV-part}} = \underbrace{-\,\delta Z_{1,\gamma}(q_2)}_{\frac{1}{\varepsilon}\text{ MS pole}} \;\; \underbrace{-\;\delta\tilde{Z}_{1,\gamma}(\tilde{q}_2)}_{\text{extra pole of } \mathcal{O}(1)}$$

$\overline{\rm MS}$ renormalisation (subtraction) of one-loop subdiagram

Q.1: How to reconstruct renormalised $D_{\rm n} = D$ one-loop subdiagram from $D_{\rm n} = 4$?

⇒ **Subtract full UV part** (poles and rational CTs) in both $D_{\rm n} = D$ and $D_{\rm n} = 4$ yielding **identical finite remainders** (only differ by $\mathcal{O}(\varepsilon)$)

$$\underbrace{\bar{\mathcal{A}}_{1,\gamma} - \Big[-\delta Z_{1,\gamma} + \delta\mathcal{R}_{1,\gamma} \Big]}_{\text{finite remainder in } D_{\rm n} = D} = \underbrace{\mathcal{A}_{1,\gamma} - \Big[-\delta Z_{1,\gamma} - \delta\tilde{Z}_{1,\gamma} \Big]}_{\text{finite remainder in } D_{\rm n} = 4} + \mathcal{O}(\varepsilon)$$

⇒ **Reconstruct** $\overline{\rm MS}$ renormalised one-loop subdiagram from $D_{\rm n} = 4$

$$\begin{aligned} \mathbf{R}_{\rm UV}\, \bar{\mathcal{A}}_{1,\gamma} &= \Big[\bar{\mathcal{A}}_{1,\gamma} + \delta Z_{1,\gamma} \Big]_{D_{\rm n} = D \text{ renormalisation}} \\ &= \Big[\underbrace{\mathcal{A}_{1,\gamma} + \delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma}}_{D_{\rm n} = 4 \text{ renormalisation}} + \underbrace{\delta\mathcal{R}_{1,\gamma}}_{\substack{\text{rational CT} \\ \text{reconstruction}}} \Big] + \mathcal{O}(\varepsilon) \end{aligned}$$

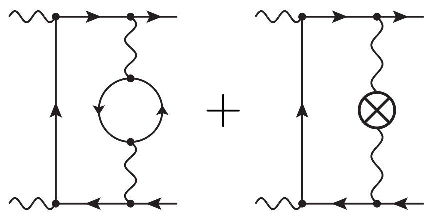
Q.2: How to reconstruct **renormalised** $D_{\rm n} = D$ **two-loop diagram** $(\mathbf{R}\, \bar{\mathcal{A}}_{2,\Gamma})$ from $D_{\rm n} = 4$?

Reconstruct $\mathrm{R}\,\bar{\mathcal{A}}_{2,\Gamma}$ without global divergence

Requirements from multi-loop renormalisation:

- Subdivergence needs to be fully subtracted
- No two-loop rational counterterms for non-global divergent diagrams

$$\Rightarrow \quad \mathbf{R}_{\mathrm{UV}}\,\bar{\mathcal{A}}_{2,\Gamma} = \underbrace{\left(\bar{\mathcal{A}}_{1,\gamma} + \delta Z_{1,\gamma}\right)}_{\text{(a) UV pole subtracted}} \cdot \underbrace{\bar{\mathcal{A}}_{1,\Gamma/\gamma}}_{\text{(b) no divergence}} \quad \leftarrow \quad \text{e.g.}$$



$\delta Z_{1,\gamma}$

$$= \underbrace{\left(\mathcal{A}_{1,\gamma} + \delta Z_{1,\gamma} + \delta \tilde{Z}_{1,\gamma} + \delta \mathcal{R}_{1,\gamma}\right) \cdot \mathcal{A}_{1,\Gamma/\gamma}}_{\text{subtraction in } D_{\mathrm{n}} = 4 \text{ and rational reconstruction}} + \mathcal{O}(\varepsilon)$$

$$= \mathcal{A}_{2,\Gamma} + \left(\delta Z_{1,\gamma} + \delta \tilde{Z}_{1,\gamma} + \delta \mathcal{R}_{1,\gamma}\right) \cdot \mathcal{A}_{1,\Gamma/\gamma} + \mathcal{O}(\varepsilon)$$

$\Rightarrow$ Only globally divergent two-loop diagrams need two-loop rational reconstruction $\delta\mathcal{R}_{2,\Gamma}$

$\Rightarrow$ **Ensure finite number** of $\delta\mathcal{R}_{2,\Gamma}$ terms in any renormalisable theories

Q.3: How to reconstruct $\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma}$ with global divergence?

Master formula for automated $\mathrm{R}\,\bar{\mathcal{A}}_2$ calculations [Pozzorini, HZ, Zoller '20]

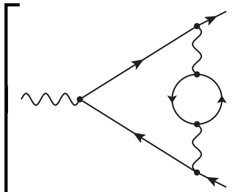
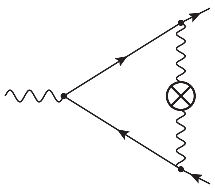
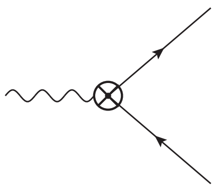
General reconstruction of $\mathbf{R}\,\bar{\mathcal{A}}_2$ in $\overline{\mathrm{MS}}$ scheme from $D_\mathsf{n}=4$ via modified $\mathbf{R}$-operation

$$\mathbf{R}_{\mathrm{UV}}\,\bar{\mathcal{A}}_{2,\Gamma} = \left[\bar{\mathcal{A}}_{2,\Gamma} + \sum_\gamma \delta Z_{1,\gamma}\cdot\bar{\mathcal{A}}_{1,\Gamma/\gamma} + \delta Z_{2,\Gamma}\right]_{D_\mathsf{n}=D \text{ renormalisation}}$$

$$= \left[\mathcal{A}_{2,\Gamma} + \sum_\gamma \underbrace{\left(\delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\right)}_{\substack{\text{sub-divergence subtraction}\\ +\text{ rational reconstruction}}}\cdot\mathcal{A}_{1,\Gamma/\gamma} + \underbrace{\left(\delta Z_{2,\Gamma} + \delta\mathcal{R}_{2,\Gamma}\right)}_{\substack{\text{local two-loop divergence}\\ \text{subtraction } + \text{ reconstruction}}}\right]_{D_\mathsf{n}=4} + \mathcal{O}(\varepsilon)$$

$\Rightarrow$ $\delta\mathcal{R}_{2,\Gamma}$ is **process-independent** local couterterm (proof in [2001.11388])

Example: single 1PI QED diagram

$$\mathbf{R}_{\mathrm{UV}}\,\bar{\mathcal{A}}_{2,\Gamma}\Big|_{D_\mathsf{n}=D} = \Bigg[$$  $+$  $\left(\delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\right) +$  $\left(\delta Z_{2,\Gamma} + \delta\mathcal{R}_{2,\Gamma}\right)\Bigg]_{D_\mathsf{n}=4} + \mathcal{O}(\varepsilon)$

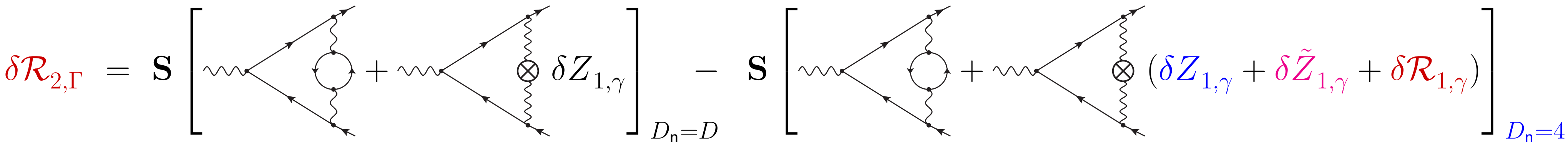
Derivation of two-loop UV rational counterterms

Two-loop rational CTs can be derived **once and for all** by reverting the master formula

$\Rightarrow$ extract $\delta\mathcal{R}_{2,\Gamma}$ from local UV divergences through **two-loop tadpole expansion** S

$$\begin{aligned}\delta\mathcal{R}_{2,\Gamma} \;=\; & \mathbf{S}\left[\bar{\mathcal{A}}_{2,\Gamma} + \sum_{\gamma} \delta Z_{1,\gamma} \cdot \bar{\mathcal{A}}_{1,\Gamma/\gamma}\right]_{D_{\mathsf{n}}=D} \\ & - \mathbf{S}\left[\mathcal{A}_{2,\Gamma} + \sum_{\gamma} \left(\delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\right) \cdot \mathcal{A}_{1,\Gamma/\gamma_i}\right]_{D_{\mathsf{n}}=4}\end{aligned}$$

Example: $\delta\mathcal{R}_{2,\Gamma}$ from single QED diagram



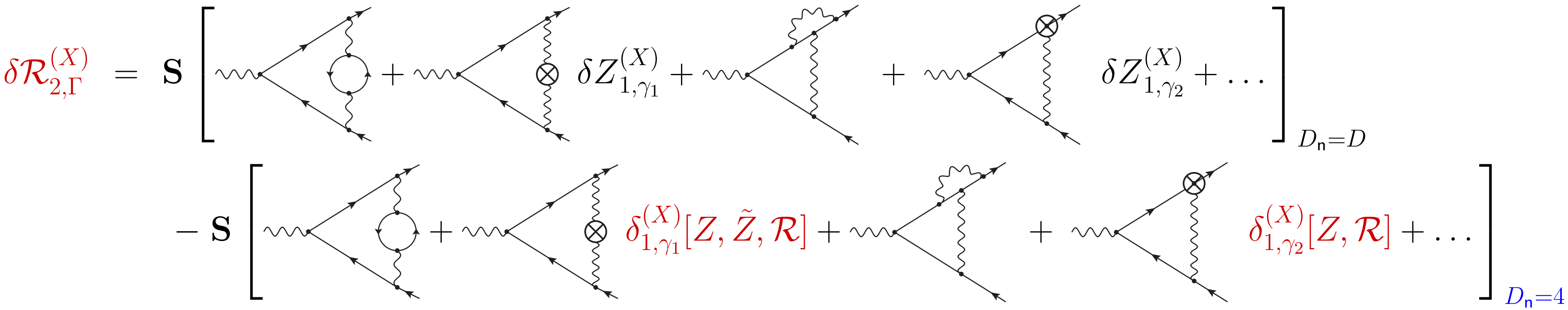
Derivation of UV rational counterterms (beyond $\overline{\mathrm{MS}}$ scheme)

In a scheme X with arbitrary finite multiplicative renormalisations s.t. $\delta Z_{1,\gamma}^{(X)} = \delta Z_{1,\gamma}^{(\mathrm{MS})} +$ finite piece, $\delta\mathcal{R}_{2,\Gamma}^{(X)}$ can be derived at vertex function level

$$\delta\mathcal{R}_{2,\Gamma}^{(X)} = \mathbf{S}\left[\bar{\mathcal{A}}_{2,\Gamma} + \sum_{\gamma} \delta Z_{1,\gamma}^{(X)} \cdot \bar{\mathcal{A}}_{1,\Gamma/\gamma}\right]_{D_{\mathsf{n}}=D}$$

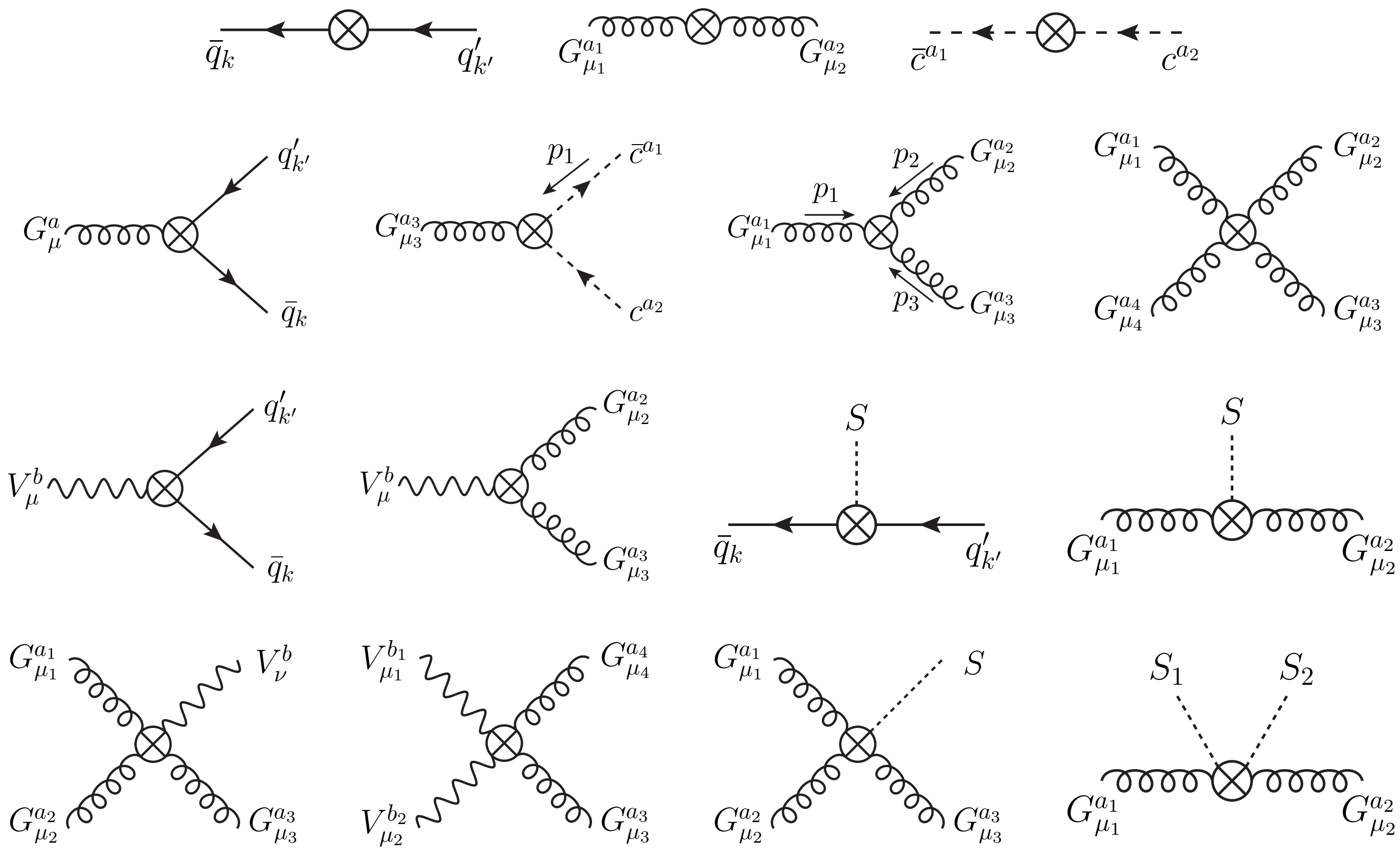
$$- \mathbf{S}\left[\mathcal{A}_{2,\Gamma} + \sum_{\gamma} \underbrace{\left(\delta Z_{1,\gamma}^{(X)} + \delta\tilde{Z}_{1,\gamma}^{(\mathrm{MS})} + \delta\mathcal{R}_{1,\gamma}^{(\mathrm{MS})}\right)}_{\delta_{1,\gamma}^{(X)}[Z,\tilde{Z},\mathcal{R}]} \cdot \mathcal{A}_{1,\Gamma/\gamma_i}\right]_{D_{\mathsf{n}}=4}$$

Example: $\delta\mathcal{R}_{2,\Gamma}^{(X)}$ from QED vertex function



Two-loop QCD rational counterterms in full SM (with massive fermions)

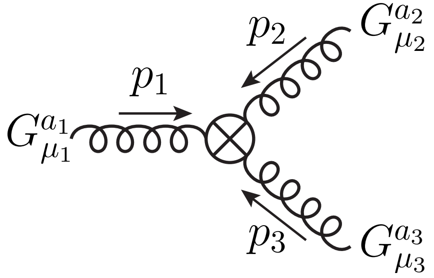
Independent calculations are done with GEFICOM [Chetyrkin, Zoller] (QGRAF [Noguira] →Q2E+EXP [Seidesticker, Harlander, Steinhauser] →FORM [Vermaseren] →MATAD [Steinhauser]), and in-house framework.



where $V = A, Z, W^{\pm}$ and $S = H, \chi, \phi^{\pm}$.

Expressions of two-loop QCD rational counterterms

Example: Triple-gluon function in renormalisation scheme X



$$= g_s\, f^{a_1 a_2 a_3} \left[g^{\mu_1\mu_2}(p_1 - p_2)^{\mu_3} + \text{perm.} \right] \left\{ \sum_{l=1}^{2} \left(\frac{\alpha_s}{4\pi} \right)^l \delta\hat{\mathcal{R}}^{(Y)}_{l,\mathrm{ggg}} \right\},$$

with rational counterterms

$$\delta\hat{\mathcal{R}}^{(Y)}_{1,\mathrm{ggg}} = -\frac{11}{12} C_\mathrm{A} - \frac{4}{3} T_\mathrm{F}\, n_\mathrm{f}\,,$$

$$\delta\hat{\mathcal{R}}^{(Y)}_{2,\mathrm{ggg}} = -\left[\frac{11}{48} C_\mathrm{A}^2 + T_\mathrm{F}\, n_\mathrm{f} \left(\frac{23}{6} C_\mathrm{A} - \frac{8}{3} C_\mathrm{F} \right) \right] \varepsilon^{-1} + T_\mathrm{F}\, n_\mathrm{f} \left(\frac{25}{9} C_\mathrm{A} - \frac{119}{36} C_\mathrm{F} \right) + \frac{145}{288} C_\mathrm{A}^2$$
$$\underbrace{- \left(\frac{11}{8} C_\mathrm{A} + 2\, T_\mathrm{F}\, n_\mathrm{f} \right) \delta\hat{\mathcal{Z}}^{(X)}_{1,\alpha_\mathrm{s}} - \left(\frac{13}{4} C_\mathrm{A} + 2\, T_\mathrm{F}\, n_\mathrm{f} \right) \delta\hat{\mathcal{Z}}^{(X)}_{1,G} + \frac{5}{4} C_\mathrm{A}\, \delta\hat{\mathcal{Z}}^{(X)}_{1,\mathrm{gp}} - \frac{C_\mathrm{A}}{24} \delta\hat{\mathcal{Z}}^{(X)}_{1,c} + \frac{4}{3} T_\mathrm{F} \sum_{f\in\mathcal{F}} \delta\hat{\mathcal{Z}}^{(X)}_{1,f}}_{\text{renormalisation scheme dependence} \Rightarrow \text{applicable to different schemes}}\,.$$

- Full results of UV rational terms at $\mathcal{O}(\alpha_s^2)$ in the full SM with massive quarks in [2007.03713] and [2107.10288]
- Broken phase results are derived from symmetric phase via vev-expansion techniques [2107.10288]

Two-loop IR rational terms in massive QED (preliminary)

- **idea:** IR rational terms cancel in IR-subtracted two-loop amplitudes
- We consider massive QED and assume amplitudes $\bar{\mathcal{A}}_{2,\Gamma}$ are UV-subtracted.
- We want to demonstrate that

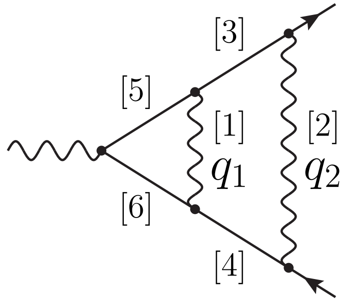
$$\mathbf{R}_{\mathrm{IR}}\,\bar{\mathcal{A}}_{2,\Gamma} = \left[\bar{\mathcal{A}}_{2,\Gamma} - \mathbf{I}^{(1)}_{\mathrm{IR}}\,\bar{\mathcal{A}}_{1,\Gamma} - \mathbf{I}^{(2)}_{\mathrm{IR}}\,\bar{\mathcal{A}}_{0,\Gamma}\right]_{D_{\mathsf{n}}=D}$$

$$= \left[\mathcal{A}_{2,\Gamma} - \mathbf{I}^{(1)}_{\mathrm{IR}}\,\mathcal{A}_{1,\Gamma} - \mathbf{I}^{(2)}_{\mathrm{IR}}\,\mathcal{A}_{0,\Gamma} + \Delta\mathbf{I}^{(2)}_{\mathrm{IR}}\,\mathcal{A}_{0,\Gamma}\right]_{D_{\mathsf{n}}=4} + \mathcal{O}(\varepsilon)$$

where $\mathbf{I}^{(1)}_{\mathrm{IR}}$ and $\mathbf{I}^{(2)}_{\mathrm{IR}}$ are universal IR subtraction operators, and $\Delta\mathbf{I}^{(2)}_{\mathrm{IR}}$ is a minimal modification of $\mathbf{I}^{(2)}_{\mathrm{IR}}$ that reabsorbs IR rational terms

Example 1: soft divergences in massive QED diagram

Consider an example with soft divergences in the diagram (UV renormalised) in $D_\mathsf{n} = D$



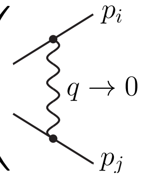
$$= \int \mathrm{d}\bar{q}_1 \mathrm{d}\bar{q}_2 \, \frac{\bar{\mathcal{N}}_{234}(\bar{q}_2)\, \bar{\mathcal{N}}_{156}(\bar{q}_1, \bar{q}_2)}{\mathcal{D}_{234}(\bar{q}_2)\, \mathcal{D}_{156}(\bar{q}_1, \bar{q}_2)} = \bar{\mathcal{A}}_{2,\Gamma}$$

IR divergences can be isolated through integrand-level approximation of single- and double-soft limits

$$\bar{\mathcal{A}}_{2,\Gamma} = \underbrace{t_{S_1 S_2}\, \bar{\mathcal{A}}_{2,\Gamma}}_{\text{double-soft div.}} + \underbrace{\left(t_{S_2}\, \bar{\mathcal{A}}_{2,\Gamma} - t_{S_2} t_{S_1 S_2} \bar{\mathcal{A}}_{2,\Gamma}\right)}_{\text{remnant single-soft div.}} + \bar{\mathcal{A}}_{2,\Gamma}^{\text{fin}}$$

$$= \underbrace{\int_{\mathrm{d}\bar{q}_2} \frac{\bar{\mathcal{N}}_{234}(0)}{\mathcal{D}_{234}(\bar{q}_2)} \int_{\mathrm{d}\bar{q}_1} \frac{\bar{\mathcal{N}}_{156}(\bar{q}_1, 0)}{\mathcal{D}_{156}(\bar{q}_1, 0)}}_{= \left(\mathbf{I}_{\text{IR}}^{(1)} + \mathbf{I}_{\text{fin}}^{(1)}\right) \bar{\mathcal{A}}_{1,\Gamma}} + \underbrace{\int_{\mathrm{d}\bar{q}_1 \mathrm{d}\bar{q}_2} \frac{\bar{\mathcal{N}}_{234}(0)}{\mathcal{D}_{234}(\bar{q}_2)} \left(\frac{\bar{\mathcal{N}}_{156}(0,0)}{\mathcal{D}_{156}(\bar{q}_1, \bar{q}_2)} - \frac{\bar{\mathcal{N}}_{156}(0,0)}{\mathcal{D}_{156}(\bar{q}_1, 0)} \right)}_{= \left(\mathbf{I}_{\text{IR}}^{(2)} - \mathbf{I}_{\text{IR}}^{(1)} \mathbf{I}_{\text{fin}}^{(1)} + \mathbf{I}_{\text{fin}}^{(2)}\right) \mathcal{A}_{0,\Gamma}} + \bar{\mathcal{A}}_{2,\Gamma}^{\text{fin}}$$

Example 1: cancellation of IR rational terms

Eikonal approximations of soft photon exchanges $\Bigg($ 

$\propto p_i \cdot p_j \Bigg)$ yield 4-dim scalar product of external momenta, hence $D_{\mathsf{n}} = 4$ projections are allowed for $\bar{\mathcal{N}}_{234}(0)$ and $\bar{\mathcal{N}}_{156}(0,0)$ such that

$$\bar{\mathcal{A}}_{2,\Gamma} = \left(\mathbf{I}^{(1)}_{\mathrm{IR}} + \mathbf{I}^{(1)}_{\mathrm{fin}}\right) \bar{\mathcal{A}}_{1,\Gamma} + \left(\mathbf{I}^{(2)}_{\mathrm{IR}} - \mathbf{I}^{(1)}_{\mathrm{IR}}\mathbf{I}^{(1)}_{\mathrm{fin}} + \mathbf{I}^{(2)}_{\mathrm{fin}}\right) \mathcal{A}_{0,\Gamma} + \bar{\mathcal{A}}^{\mathrm{fin}}_{2,\Gamma} \Big|_{D_{\mathsf{n}}=D}$$

$$\mathcal{A}_{2,\Gamma} = \left(\mathbf{I}^{(1)}_{\mathrm{IR}} + \mathbf{I}^{(1)}_{\mathrm{fin}}\right) \mathcal{A}_{1,\Gamma} + \left(\mathbf{I}^{(2)}_{\mathrm{IR}} - \mathbf{I}^{(1)}_{\mathrm{IR}}\mathbf{I}^{(1)}_{\mathrm{fin}} + \mathbf{I}^{(2)}_{\mathrm{fin}}\right) \mathcal{A}_{0,\Gamma} + \mathcal{A}^{\mathrm{fin}}_{2,\Gamma} \Big|_{D_{\mathsf{n}}=4}$$

$\Rightarrow$ IR rational terms in $\mathbf{I}^{(1)}_{\mathrm{IR}} \bar{\mathcal{A}}_{1,\Gamma}$ cancel in the standard IR-subtracted amplitude, since

$$\left[\bar{\mathcal{A}}_{2,\Gamma} - \mathbf{I}^{(1)}_{\mathrm{IR}} \bar{\mathcal{A}}_{1,\Gamma} - \mathbf{I}^{(2)}_{\mathrm{IR}} \bar{\mathcal{A}}_{0,\Gamma}\right]_{D_{\mathsf{n}}=D} - \left[\mathcal{A}_{2,\Gamma} - \mathbf{I}^{(1)}_{\mathrm{IR}} \mathcal{A}_{1,\Gamma} - \mathbf{I}^{(2)}_{\mathrm{IR}} \bar{\mathcal{A}}_{0,\Gamma}\right]_{D_{\mathsf{n}}=4}$$

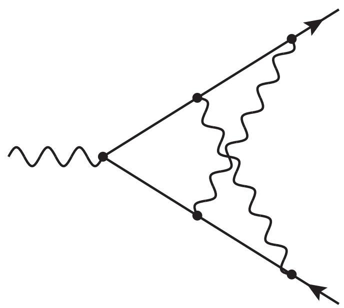
$$= \mathbf{I}^{(1)}_{\mathrm{fin}} \underbrace{\left(\bar{\mathcal{A}}_{1,\Gamma} - \mathcal{A}_{1,\Gamma}\right)}_{\mathcal{O}(\varepsilon)} + \underbrace{\left(\bar{\mathcal{A}}^{\mathrm{fin}}_{2,\Gamma} - \mathcal{A}^{\mathrm{fin}}_{2,\Gamma}\right)}_{\mathcal{O}(\varepsilon)} = \mathcal{O}(\varepsilon)$$

where we used one-loop decomposition $\bar{\mathcal{A}}_{1,\Gamma} = \left(\mathbf{I}^{(1)}_{\mathrm{IR}} + \mathbf{I}^{(1)}_{\mathrm{fin}}\right)\mathcal{A}_{0,\Gamma} + \bar{\mathcal{A}}^{\mathrm{fin}}_{1,\Gamma}$

$\Rightarrow$ $\mathbf{R}_{\mathrm{IR}}\, \bar{\mathcal{A}}_{2,\Gamma}$ is free from IR rational terms.

Example 2: "non-factorised" soft divergences

Example diagram of non-factorisable soft divergences with two crossed photons:



$$= \bar{\mathcal{A}}_{2,\Gamma}$$

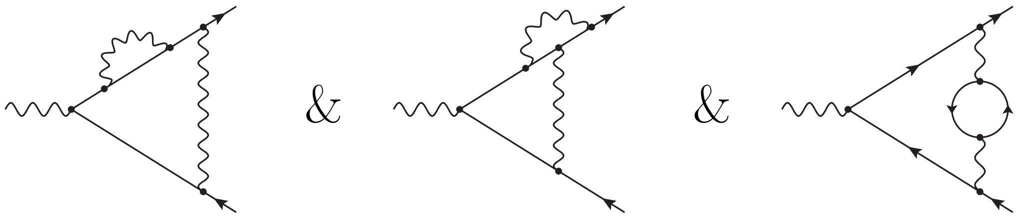
It contains only IR divergence from double-soft photon exchanges such that

$$\bar{\mathcal{A}}_{2,\Gamma}\Big|_{\text{IR-part}} \;=\; t_{S_1 S_2}\bar{\mathcal{A}}_{2,\Gamma} \;=\; \left(\mathbf{I}^{(2)}_{\mathrm{IR}} + \mathbf{I}^{(2)}_{\mathrm{fin}}\right)\mathcal{A}_{0,\Gamma}$$

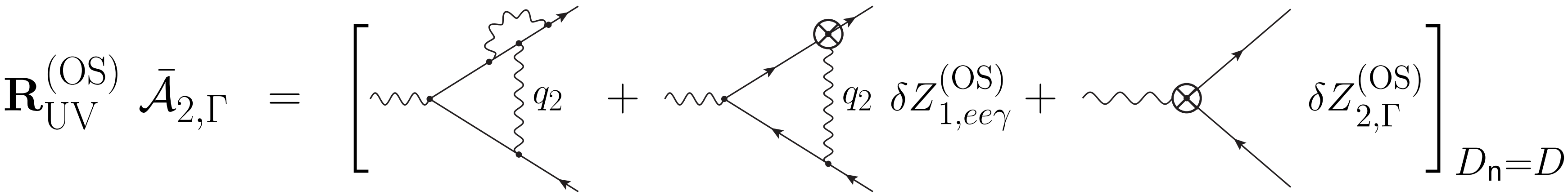
Hence $\mathbf{R}_{\mathrm{IR}}\,\bar{\mathcal{A}}_{2,\Gamma}$ is again free from IR rational terms. (same Eikonal approximation mechanism)

Example 3: IR divergences in On-Shell (OS) renormalisation

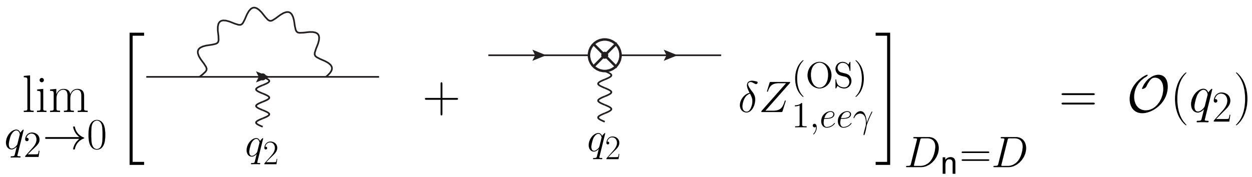
Non-trivial terms arise from IR divergences, which are subtracted by renormalisation CTs but not by $\mathbf{I}^{(l)}$-operators.



Here we adopt OS renormalisation and discuss the diagram with vertex correction (middle)

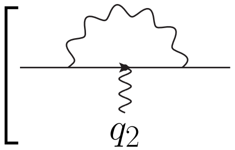


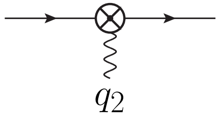
The OS renormalisation (in Thompson limit) removes the single-soft IR divergence at $q_2 \to 0$ by

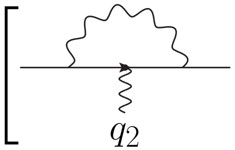


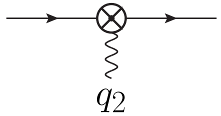
Example 3: IR rational terms in OS scheme

The IR cancellation in soft limit is destroyed in $D_{\mathsf{n}} = 4$

$\lim_{q_2 \to 0} \Big[$ 

$+$ 

$\delta Z^{(\mathrm{OS})}_{1,ee\gamma} \Big]_{D_{\mathsf{n}}=D} - \Big[$ 

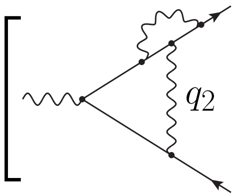
$+$ 

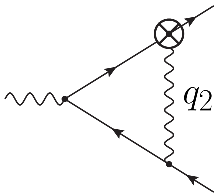
$\left(\delta Z^{(\mathrm{OS})}_{1,ee\gamma} + \delta \mathcal{R}^{(\mathrm{OS})}_{1,ee\gamma} \right) \Big]_{D_{\mathsf{n}}=4}$

$$= \Delta B_\mu(\varepsilon) + \mathcal{O}(q_2)$$

where $\Delta B_\mu(\varepsilon) = \mathcal{O}(\varepsilon)$ survives at $q_2 \to 0$

$\Rightarrow$ The combination of $\Delta B_\mu(\varepsilon)$ with soft IR singularity yields

$\mathbf{R}^{(\mathrm{OS})}_{\mathrm{UV}}\, \bar{\mathcal{A}}_{2,\Gamma} = \Big[$ 

$+$ 

$\left(\delta Z^{(\mathrm{OS})}_{1,ee\gamma} + \delta \mathcal{R}^{(\mathrm{OS})}_{1,ee\gamma} \right) \Big]_{D_{\mathsf{n}}=4} + \left(\mathbf{I}^{(1)}_{\mathrm{IR}}\, \Delta B(\varepsilon) + \delta Z^{(\mathrm{OS})}_{2,\Gamma} \right) \mathcal{A}_{0,\Gamma}$

The factorisation $\mathbf{I}^{(1)}_{\mathrm{IR}}\, \Delta B(\varepsilon)\, \mathcal{A}_{0,\Gamma}$ is achieved by applying Dirac equation (again Eikonal mechanism), and it can be further absorbed into a minimal modification of $\mathbf{I}^{(2)}_{\mathrm{IR}}$ operator in $D_{\mathsf{n}} = 4$

$$\mathbf{I}^{(1)}_{\mathrm{IR}}\, \Delta B(\varepsilon)\, \mathcal{A}_{0,\Gamma} := \Delta \mathbf{I}^{(2)}_{\mathrm{IR}}\, \mathcal{A}_{0,\Gamma} \tag{1}$$

as a **process-independent IR-subtraction operator**

Summary and Outlook

- Renormalised two-loop amplitudes can be constructed via diagrams with 4-dim numerator + UV rational counterterms in a generic renormalisation scheme

$$\mathbf{R}_{\mathrm{UV}}\,\bar{\mathcal{A}}_{2,\Gamma} \;=\; \mathcal{A}_{2,\Gamma} + \sum_{\gamma}\left(\delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\right)\cdot\mathcal{A}_{1,\Gamma/\gamma} + \delta Z_{2,\Gamma} + \delta\mathcal{R}_{2,\Gamma}$$

- We have presented a general method to compute $\delta\mathcal{R}_{2,\Gamma}$, and calculated full QCD $\delta\mathcal{R}_{2,\Gamma}$ in the SM with arbitrary fermion masses

- We have shown the first preliminary evidence (in massive QED) that the IR rational terms cancel in the modified IR-subtracted amplitudes

$$\mathbf{R}_{\mathrm{IR}}\,\bar{\mathcal{A}}_{2,\Gamma} \;=\; \mathcal{A}_{2,\Gamma} - \mathbf{I}^{(1)}_{\mathrm{IR}}\,\mathcal{A}_{1,\Gamma} \;-\; \mathbf{I}^{(2)}_{\mathrm{IR}}\,\mathcal{A}_{0,\Gamma} + \Delta\mathbf{I}^{(2)}_{\mathrm{IR}}\,\mathcal{A}_{0,\Gamma}$$

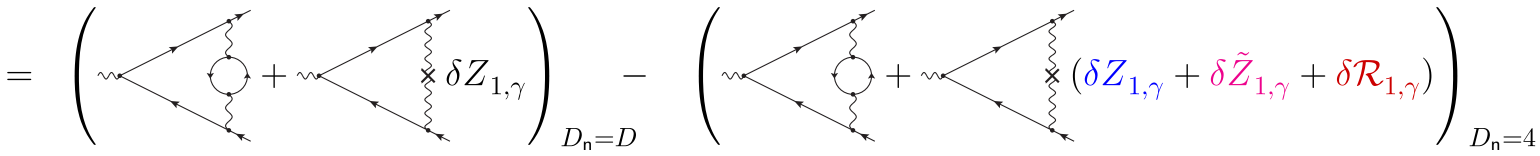
- Next step: include collinear singularity, and complete analysis of IR rational terms in QCD amplitudes

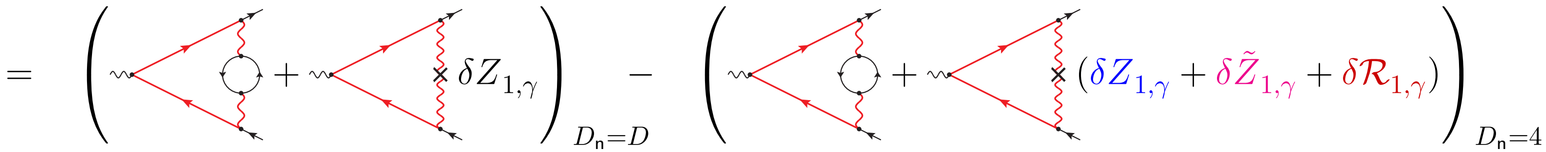
  $\Rightarrow$ open the door to numerical automated calculations of various non-trivial processes

B.1. Example: two-loop tadpole expansion

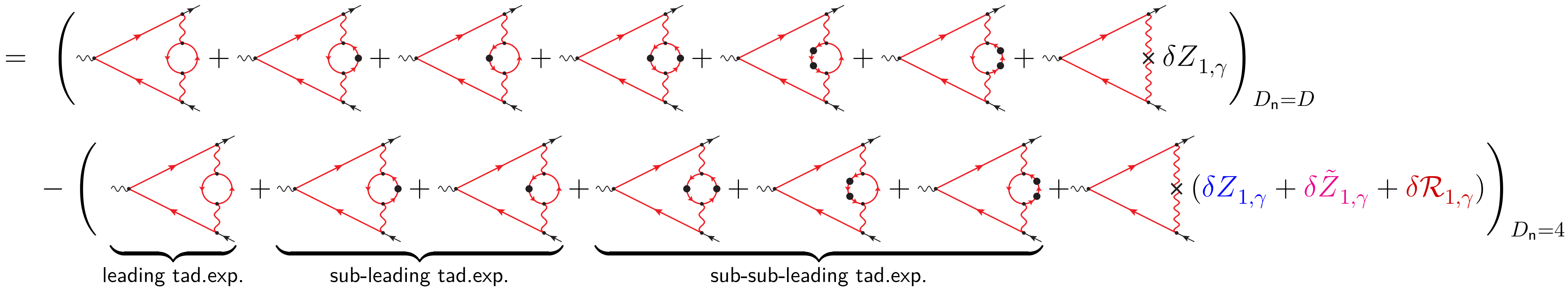
Expand chain 1 (outer loop) → expand chains 2 and 3 (photon self-energy subdiagram) w.r.t. external momenta

$\delta\mathcal{R}_2$
of single
diagram





$+\ \mathcal{O}(\varepsilon)$ terms (truncated diagrams without global divergence)

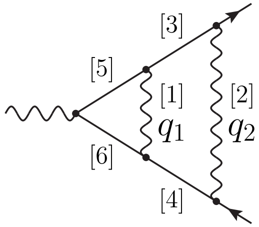


$+\ \mathcal{O}(\varepsilon)$ terms (truncated diagrams without global divergence or subdivergence)

where red lines denote $\left\{ \frac{1}{\bar{q}_1^2 - M^2}, \frac{1}{\bar{q}_2^2 - M^2}, \frac{1}{(\bar{q}_1^2 + \bar{q}_2^2) - M^2} \right\}$ tadpole denominators, black lines denote original denominators, and dots denote numerator polynomials resulted from tadpole expansions

Example 1: Alternative parametrisation of IR divergences

For previous example



$$= \int \mathrm{d}\bar{q}_1 \mathrm{d}\bar{q}_2 \, \frac{\bar{\mathcal{N}}_{234}(\bar{q}_2)\, \bar{\mathcal{N}}_{156}(\bar{q}_1,\bar{q}_2)}{\mathcal{D}_{234}(\bar{q}_2)\, \mathcal{D}_{156}(\bar{q}_1,\bar{q}_2)} = \bar{\mathcal{A}}_{2,\Gamma}$$

we can parametrise the IR divergence in a different way

$$\begin{aligned}
\bar{\mathcal{A}}_{2,\Gamma}\big|_{\text{IR-part}} &= \underbrace{t_{S_1S_2}\,\bar{\mathcal{A}}_{2,\Gamma}}_{\text{double-soft div.}} + \underbrace{\left(t_{S_2}\,\bar{\mathcal{A}}_{2,\Gamma} - t_{S_2}t_{S_1S_2}\bar{\mathcal{A}}_{2,\Gamma}\right)}_{\text{nested } S_1S_2 \text{ div. subtracted from } S_2 \text{ region}} \\
&= \underbrace{\int_{\mathrm{d}\bar{q}_1\mathrm{d}\bar{q}_2} \frac{\bar{\mathcal{N}}_{234}(0)}{\mathcal{D}_{234}(\bar{q}_2)} \left(\frac{\bar{\mathcal{N}}_{156}(0,0)}{\mathcal{D}_{156}(\bar{q}_1,\bar{q}_2)} - \frac{\bar{\mathcal{N}}_{156}(0,0)}{\mathcal{D}_{156}(\bar{q}_1,0)}\right)}_{=\left(\mathbf{I}^{(2)}_{\text{IR,Eik}}+\mathbf{I}^{(2)}_{\text{fin}}\right)\bar{\mathcal{A}}_{0,\Gamma} - \left(\mathbf{I}^{(1)}_{\text{IR}}+\mathbf{I}^{(1)}_{\text{fin}}\right)^2\bar{\mathcal{A}}_{0,\Gamma}} + \underbrace{\int_{\mathrm{d}\bar{q}_2} \frac{\bar{\mathcal{N}}_{234}(0)}{\mathcal{D}_{234}(\bar{q}_2)} \int_{\mathrm{d}\bar{q}_1} \frac{\bar{\mathcal{N}}_{156}(\bar{q}_1,0)}{\mathcal{D}_{156}(\bar{q}_1,0)}}_{=\left(\mathbf{I}^{(1)}_{\text{IR}}+\mathbf{I}^{(1)}_{\text{fin}}\right)\bar{\mathcal{A}}_{1,\Gamma}} \\
&= \left(\mathbf{I}^{(2)}_{\text{IR,Eik}} + \mathbf{I}^{(2)}_{\text{fin}}\right)\bar{\mathcal{A}}_{0,\Gamma} - \left(\mathbf{I}^{(1)}_{\text{IR}} + \mathbf{I}^{(1)}_{\text{fin}}\right)^2 \bar{\mathcal{A}}_{0,\Gamma} + \left(\mathbf{I}^{(1)}_{\text{IR}} + \mathbf{I}^{(1)}_{\text{fin}}\right)\left((\mathbf{I}^{(1)}_{\text{IR}} + \mathbf{I}^{(1)}_{\text{fin}})\bar{\mathcal{A}}_{0,\Gamma} + \bar{\mathcal{A}}^{\text{fin}}_{1,\Gamma}\right) \\
&= \mathbf{I}^{(2)}_{\text{IR,Eik}}\,\bar{\mathcal{A}}_{0,\Gamma} - \left(\mathbf{I}^{(1)}_{\text{IR}}\right)^2 \bar{\mathcal{A}}_{0,\Gamma} + 2\mathbf{I}^{(1)}_{\text{IR}}\mathbf{I}^{(1)}_{\text{fin}}\bar{\mathcal{A}}_{0,\Gamma} + \mathbf{I}^{(1)}_{\text{fin}}\,\mathbf{I}^{(1)}_{\text{IR}}\bar{\mathcal{A}}_{0,\Gamma} + \mathbf{I}^{(1)}_{\text{IR}}\bar{\mathcal{A}}_{1,\Gamma} + \mathcal{O}(\varepsilon^0) \\
&= \underbrace{\left(\mathbf{I}^{(2)}_{\text{IR,Eik}} - \left(\mathbf{I}^{(1)}_{\text{IR}}\right)^2 - \mathbf{I}^{(1)}_{\text{IR}}\mathbf{I}^{(1)}_{\text{fin}}\right)\bar{\mathcal{A}}_{0,\Gamma}}_{\mathbf{I}^{(2)}_{\text{IR}}\,\bar{\mathcal{A}}_{0,\Gamma}} + \mathbf{I}^{(1)}_{\text{IR}}\bar{\mathcal{A}}_{1,\Gamma} + \mathcal{O}(\varepsilon^0)
\end{aligned}$$

and the neglected $\mathcal{O}(\varepsilon^0)$ part is free from IR rational terms.