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The gradient flow formulation of the electroweak Hamiltonian

Loops and Legs in Quantum Field Theory | April 25 – 30, 2022

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in collaboration with Robert Harlander | April 27, 2022

The effective electroweak Hamiltonian

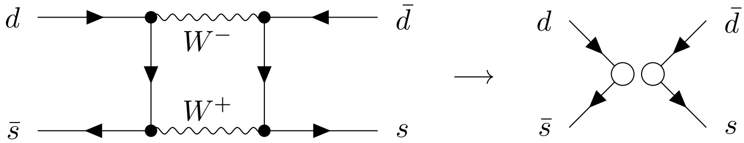
- Observables in flavor physics often computed with effective Hamiltonian of electroweak interactions

$$\mathcal{H}_{\text{eff}} = -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_i C_i \mathcal{O}_i$$

with four-fermion operators like

$$\mathcal{O}^{\Delta S=2} = (\bar{s}\gamma_\mu(1-\gamma_5)d)\,(\bar{s}\gamma_\mu(1-\gamma_5)d)$$

for $K^0 - \bar{K}^0$ mixing:



- Wilson coefficients $C_i(\mu)$ obtained from perturbative matching to Standard Model at $\mu = \mu_W \sim M_W$
- V_{CKM}: relevant entries of the CKM matrix, e.g. $V_{is}^* V_{id} V_{js}^* V_{jd}$ with $i, j = c, t$

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Computing observables

- Flavor observables mostly at low energies

$\Rightarrow$ Use renormalization group equations to evolve down to appropriate scale to avoid large logarithms

- Schematically for Kaon mixing:

$$\langle \bar{K}^0 | \mathcal{H}_{\text{eff}}^{\Delta S=2} | K^0 \rangle \approx C(\mu_W) U(\mu_W, \mu \sim M_K) \langle \bar{K}^0 | \mathcal{O}^{\Delta S=2}(\mu \sim M_K) | K^0 \rangle$$

- Running with $U(\mu_W, \mu)$ determined by anomalous dimension γ of $\mathcal{O}^{\Delta S=2}$
- Matrix element $\langle \bar{K}^0 | \mathcal{O}^{\Delta S=2}(\mu) | K^0 \rangle$ nonperturbative

$\Rightarrow$ Compute on lattice

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Complications

$$\mathcal{H}_{\text{eff}} = -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_i C_i \mathcal{O}_i$$

- While $\mathcal{H}_{\text{eff}}$ is scheme independent, C_i and $\mathcal{O}_i$ are not:
  1. Explicitly depend on renormalization scale μ
  2. Depend on scheme used for γ_5
  3. In dimensional regularization $\mathcal{O}_i$ mix with evanescent operators, which vanish in $D = 4$, but their choice affects the finite pieces in C_i

  $\Rightarrow$ C_i also depend on choice of evanescent operators used for perturbative calculation of C_i

$\Rightarrow$ Scheme matching between lattice and perturbative results is a source of uncertainty

The gradient flow

- Origin in lattice QCD
- Introduce parameter *flow time* $t \geq 0$ [Lüscher 2010]
- *Flowed fields* in $D+1$ dimensions obey differential *flow equations* like

$$\partial_t \Phi(t,x) = D_x \Phi(t,x) \quad \text{with} \quad \Phi(t,x)|_{t=0} = \phi(x)$$

- Flow equation similar to the heat equation (thermodynamics)

$$\partial_t u(t,\vec{x}) = \alpha \Delta u(t,\vec{x}) \quad \text{with} \quad \Delta = \sum_i \partial_{x_i}^2$$

- Fields at positive flow time smeared out with smearing radius $\sqrt{8t}$

$\Rightarrow$ Intuition: Regulates divergencies

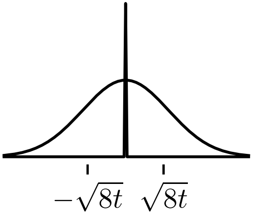

Figure: Sketch of smearing.

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Gluon flow equation [Lüscher 2010]

$$\partial_t B_\mu^a = \mathcal{D}_\nu^{ab} G_{\nu\mu}^b \quad \text{with} \quad B_\mu^a(t,x)\big|_{t=0} = A_\mu^a(x)$$

- Covariant derivative:

$$\mathcal{D}_\mu^{ab} = \delta^{ab}\partial_\mu - f^{abc} B_\mu^c$$

- Field-strength tensor:

$$G_{\mu\nu}^a = \partial_\mu B_\nu^a - \partial_\nu B_\mu^a + f^{abc} B_\mu^b B_\nu^c$$

- $B_\mu^a(t,x)$ and $A_\mu^a(x)$: flowed and fundamental gluon fields
- T^a and f^{abc}: $\mathrm{SU}(N_\mathrm{c})$ generators and structure constants

Quark flow equation [Lüscher 2013]

$$\partial_t \chi = \Delta \chi \quad \text{with} \quad \chi(t,x)|_{t=0} = \psi(x)\,,$$
$$\partial_t \bar{\chi} = \bar{\chi} \overleftarrow{\Delta} \quad \text{with} \quad \overline{\chi}(t,x)|_{t=0} = \overline{\psi}(x)$$

$$\Delta = (\partial_\mu + B_\mu^a T^a)(\partial_\mu + B_\mu^b T^b)\,, \qquad \overleftarrow{\Delta} = (\overleftarrow{\partial}_\mu - B_\mu^a T^a)(\overleftarrow{\partial}_\mu - B_\mu^b T^b)$$

- $\chi(t,x)$ and $\overline{\chi}(t,x)$: flowed quark and anti-quark fields
- $\psi(x)$ and $\overline{\psi}(x)$: fundamental quark and anti-quark fields

Applications of the gradient flow in lattice QCD

- Inherent smearing for continuum extrapolation [Lüscher 2010]
- New strategies for scale setting [Lüscher 2010; Borsányi et al. 2012; ...]
- Composite operators do not require renormalization [Lüscher, Weisz 2011]

$\Rightarrow$ No scheme matching for operators and observables in different schemes, e.g. lattice and perturbative schemes:

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$\Rightarrow$ No scheme matching for operators and observables in different schemes, e.g. lattice and perturbative schemes:

- Extract parameters like α_s from lattice simulations [Fodor et al. 2012; Fritsch, Ramos 2013; ...]
- *Flowed operator product expansion* [Suzuki 2013; Makino, Suzuki 2014; Monahan, Orginos 2015]:
  - Define the energy-momentum tensor of QCD on the lattice [Suzuki 2013; Makino, Suzuki 2014; Harlander, Kluth, **FL** 2018]
    $\Rightarrow$ Studies of thermodynamics on the lattice [FlowQCD since 2014]
  - (Potential) alternative determination of vacuum polarization functions on the lattice [Harlander, **FL**, Neumann 2020]
  - Apply to electroweak Hamiltonian $\leftarrow$ this talk (see also [Suzuki, Taniguchi, Suzuki, Kanaya 2020])
  - ...

Lagrangian

- Write Lagrangian for the gradient flow as [Lüscher, Weisz 2011; Lüscher 2013]

$$\mathcal{L} = \mathcal{L}_{\text{QCD}} + \mathcal{L}_B + \mathcal{L}_\chi,$$

$$\mathcal{L}_{\text{QCD}} = \frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} + \sum_{f=1}^{n_\mathrm{f}} \bar{\psi}_f (\not{D}^{\mathrm{F}} + m_f) \psi_f + \ldots$$

- Construct flowed Lagrangian using Lagrange multiplier fields $L^a_\mu(t, x)$ and $\lambda_f(t, x)$:

$$\mathcal{L}_B = -2 \int_0^\infty \mathrm{d}t \, \mathrm{Tr} \left[L^a_\mu T^a \left(\partial_t B^b_\mu T^b - \mathcal{D}^{bc}_\nu G^c_{\nu\mu} T^b \right) \right]$$

$$\mathcal{L}_\chi = \sum_{f=1}^{n_\mathrm{f}} \int_0^\infty \mathrm{d}t \left(\bar{\lambda}_f \left(\partial_t - \Delta \right) \chi_f + \bar{\chi}_f \left(\overleftarrow{\partial_t} - \overleftarrow{\Delta} \right) \lambda_f \right)$$

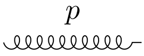
⇒ Flow equations automatically fulfilled

⇒ QCD Feynman rules + gradient-flow Feynman rules (complete list in [Artz, Harlander, **FL**, Neumann, Prausa 2019])

Flowed propagators and flow lines

$$\mathcal{L}_B = -2\int_0^\infty \mathrm{d}t\, \mathrm{Tr}\left[L_\mu^a T^a \left(\partial_t B_\mu^b T^b - \mathcal{D}_\nu^{bc} G_{\nu\mu}^c T^b\right)\right]$$

- Combined Feynman rule for the (flowed) gluon propagator $\left\langle \widetilde{B}_\mu^a(t,p)\widetilde{B}_\nu^b(s,q)\right\rangle$:

s, ν, b 

$t, \mu, a \; = \delta^{ab}\frac{1}{p^2}\delta_{\mu\nu}\, e^{-(t+s)p^2}$

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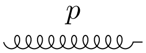
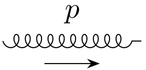


$$s,\nu,b \quad\text{---}\quad t,\mu,a \;=\; \delta^{ab}\frac{1}{p^2}\delta_{\mu\nu}\, e^{-(t+s)p^2}$$

- No squared L_μ^a in $\mathcal{L}_B \Rightarrow$ no propagator
- Instead, there is a mixed propagator $\left\langle \widetilde{B}_\mu^a(t,p)\widetilde{L}_\nu^b(s,q)\right\rangle$ called *flow line*:



$$s,\nu,b \quad\text{---}\quad t,\mu,a \;=\; \delta^{ab}\,\theta(t-s)\delta_{\mu\nu}\, e^{-(t-s)p^2}$$

- Directed towards increasing flow time

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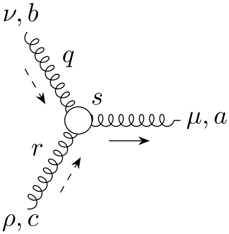
Flow vertices

$$\mathcal{L}_B = -2\int_0^\infty \mathrm{d}t\,\mathrm{Tr}\left[L_\mu^a T^a \left(\partial_t B_\mu^b T^b - \mathcal{D}_\nu^{bc} G_{\nu\mu}^c T^b\right)\right]$$

- Example:



$$= -\mathrm{i}gf^{abc}\int_0^\infty \mathrm{d}s\left(\delta_{\nu\rho}(r-q)_\mu + 2\delta_{\mu\nu}q_\rho - 2\delta_{\mu\rho}r_\nu\right)$$

- Integral restricted by $\theta(t-s)$ from outgoing flow line

Renormalization

- QCD renormalization of QCD parameters like α_s and quark masses
- Flowed gluon fields do not require renormalization [Lüscher 2010; Lüscher, Weisz 2011]
- Flowed quark fields have to be renormalized: $\chi^\mathrm{R} = Z_\chi^{1/2} \chi^\mathrm{B}$ [Lüscher 2013]

$\Rightarrow$ χ thus acquire anomalous dimension and are not scheme independent

- "Physical" scheme: Ringed fermions $\mathring{\chi} = \mathring{Z}_\chi^{1/2} \chi^\mathrm{B}$ [Makino, Suzuki 2014]:

$$\mathring{Z}_\chi = -\frac{2N_\mathrm{c}}{(4\pi t)^2 \left.\langle \bar{\chi}^\mathrm{B} \overleftrightarrow{\not{D}} \chi^\mathrm{B} \rangle\right|_{m=0}}$$

$\Rightarrow$ $\mathring{\chi}$ formally independent of renormalization scale μ

- $\mathring{Z}_\chi$ available through NNLO [Artz, Harlander, **FL**, Neumann, Prausa 2019]

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⇒ No operator mixing through renormalization

Flowed operator product expansion

- Small flow-time expansion [Lüscher, Weisz 2011]:

$$\tilde{\mathcal{O}}_i(t,x) = \sum_j \zeta_{ij}(t)\mathcal{O}_j(x) + O(t)$$

- Invert to express operators through flowed operators [Suzuki 2013; Makino, Suzuki 2014; Monahan, Orginos 2015]:

Flowed OPE

$$T = \sum_i C_i \mathcal{O}_i = \sum_{i,j} C_i \zeta_{ij}^{-1}(t)\tilde{\mathcal{O}}_j(t) \equiv \sum_j \tilde{C}_j(t)\tilde{\mathcal{O}}_j(t)$$

- T defined in regular QCD expressed through better behaved flowed operators $\tilde{\mathcal{O}}_j(t)$, which do not require renormalization
- Can relate T in lattice and perturbative schemes without scheme transformation

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Flowed OPE for the electroweak Hamiltonian

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- Write electroweak Hamiltonian as

$$\mathcal{H}_{\text{eff}} = -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_i C_i \mathcal{O}_i = -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_{i,j} C_i \zeta_{ij}^{-1} \tilde{\mathcal{O}}_j$$

- Define flowed operators:

$$\mathcal{O}_1 = -\left(\bar{\psi}_{1,\text{L}}\gamma_\mu T^a \psi_{2,\text{L}}\right)\left(\bar{\psi}_{3,\text{L}}\gamma_\mu T^a \psi_{4,\text{L}}\right) \quad \Rightarrow \quad \tilde{\mathcal{O}}_1 = -\mathring{Z}_\chi^2 \left(\bar{\chi}_{1,\text{L}}\gamma_\mu T^a \chi_{2,\text{L}}\right)\left(\bar{\chi}_{3,\text{L}}\gamma_\mu T^a \chi_{4,\text{L}}\right)$$

$$\mathcal{O}_2 = \left(\bar{\psi}_{1,\text{L}}\gamma_\mu \psi_{2,\text{L}}\right)\left(\bar{\psi}_{3,\text{L}}\gamma_\mu \psi_{4,\text{L}}\right) \quad \Rightarrow \quad \tilde{\mathcal{O}}_2 = \mathring{Z}_\chi^2 \left(\bar{\chi}_{1,\text{L}}\gamma_\mu \chi_{2,\text{L}}\right)\left(\bar{\chi}_{3,\text{L}}\gamma_\mu \chi_{4,\text{L}}\right)$$

- No operator mixing through renormalization for $\tilde{\mathcal{O}}_i$

⇒ Combine <u>without</u> scheme matching between perturbation theory and lattice:
  - C_i known perturbatively through (N)NLO (depending on process)
  - ζ_{ij}^{-1} has to be computed ← this talk (see also [Suzuki, Taniguchi, Suzuki, Kanaya 2020])
  - $\langle \tilde{\mathcal{O}}_j \rangle$ have to be computed on the lattice

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Operator basis

- Operator basis depends on the process under consideration
- We focus on the current-current operators relevant for $|\Delta F| = 2$ processes
- Operator basis not unique even for the same process, but different bases related by basis transformations
- CMM basis [Chetyrkin, Misiak, Münz 1997]:

$$\mathcal{O}_1 = -\left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu T^a \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu T^a \psi_{4,\mathrm{L}}\right),$$
$$\mathcal{O}_2 = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu \psi_{4,\mathrm{L}}\right)$$

  with

$$\psi_{\mathrm{R/L}} = P_\pm \psi = \tfrac{1}{2}(1 \pm \gamma_5)\psi$$

- Advantage of CMM basis: can use anticommuting γ_5

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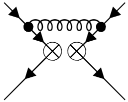
Evanescent operators

$$\mathcal{O}_2 = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu\psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu\psi_{4,\mathrm{L}}\right)$$

- In dimensional regularization, loop corrections produce additional non-reducible γ structures:



$$\Rightarrow \qquad (\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}) \otimes (\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3})$$

- These contributions have to be attributed to *evanescent* operators like

$$E_2^{(1)} = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3}\psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3}\psi_{4,\mathrm{L}}\right) - 16\mathcal{O}_2 \qquad \text{with} \qquad \gamma_{\mu_1\cdots\mu_n} \equiv \gamma_{\mu_1}\cdots\gamma_{\mu_n}$$

- Algebraically, they are of $O(\epsilon)$ and vanish for $D \to 4$
- Nonetheless required to renormalize the physical operators
- Renormalization has to take care of finite pieces from $\frac{1}{\epsilon}$ (poles) $\times\, \epsilon$ (operators)
- Every loop order introduces more evanescent operators

Complete operator basis

- Physical operators:

$$\mathcal{O}_1 = -\left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu T^a \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu T^a \psi_{4,\mathrm{L}}\right),$$
$$\mathcal{O}_2 = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu \psi_{4,\mathrm{L}}\right)$$

- Evanescent operators through NNLO (also from [Chetyrkin, Misiak, Münz 1997]):

$$E_1^{(1)} = -\left(\bar{\psi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} T^a \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} T^a \psi_{4,\mathrm{L}}\right) - 16\mathcal{O}_1,$$
$$E_2^{(1)} = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} \psi_{4,\mathrm{L}}\right) - 16\mathcal{O}_2,$$
$$E_1^{(2)} = -\left(\bar{\psi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} T^a \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} T^a \psi_{4,\mathrm{L}}\right) - 20E_1^{(1)} - 256\mathcal{O}_1,$$
$$E_2^{(2)} = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} \psi_{4,\mathrm{L}}\right) - 20E_2^{(1)} - 256\mathcal{O}_2$$

Flowed operator basis

- Flowed physical operators:

$$\mathcal{O}_1 = -\left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu T^a \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu T^a \psi_{4,\mathrm{L}}\right) \quad \Rightarrow \quad \tilde{\mathcal{O}}_1 = -\mathring{Z}_\chi^2 \left(\bar{\chi}_{1,\mathrm{L}}\gamma_\mu T^a \chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_\mu T^a \chi_{4,\mathrm{L}}\right)$$

$$\mathcal{O}_2 = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu \psi_{4,\mathrm{L}}\right) \quad \Rightarrow \quad \tilde{\mathcal{O}}_2 = \mathring{Z}_\chi^2 \left(\bar{\chi}_{1,\mathrm{L}}\gamma_\mu \chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_\mu \chi_{4,\mathrm{L}}\right)$$

- Flowed evanescent operators:

$$\begin{aligned}
\tilde{E}_1^{(1)} &= -\mathring{Z}_\chi^2 \left(\bar{\chi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} T^a \chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} T^a \chi_{4,\mathrm{L}}\right) - 16\tilde{\mathcal{O}}_1\,,\\
\tilde{E}_2^{(1)} &= \mathring{Z}_\chi^2 \left(\bar{\chi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} \chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} \chi_{4,\mathrm{L}}\right) - 16\tilde{\mathcal{O}}_2\,,\\
\tilde{E}_1^{(2)} &= -\mathring{Z}_\chi^2 \left(\bar{\chi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} T^a \chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} T^a \chi_{4,\mathrm{L}}\right) - 20\tilde{E}_1^{(1)} - 256\tilde{\mathcal{O}}_1\,,\\
\tilde{E}_2^{(2)} &= \mathring{Z}_\chi^2 \left(\bar{\chi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} \chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} \chi_{4,\mathrm{L}}\right) - 20\tilde{E}_2^{(1)} - 256\tilde{\mathcal{O}}_2
\end{aligned}$$

- Note: Since flowed operators do not have to be renormalized, the flowed evanescent operators actually vanish and could be dropped
- Keeping them allows us to check our results

Matching matrix and renormalization

- Small-flow-time expansion for operators of electroweak Hamiltonian:

$$\begin{pmatrix} \tilde{\mathcal{O}}(t) \\ \tilde{E}(t) \end{pmatrix} \asymp \zeta^{\mathrm{B}}(t) \begin{pmatrix} \mathcal{O} \\ E \end{pmatrix}$$

$$\text{with} \qquad \mathcal{O} = (\mathcal{O}_1, \mathcal{O}_2)^{\mathrm{T}}, \qquad E = (E_1^{(1)}, E_2^{(1)}, E_1^{(2)}, E_2^{(2)})^{\mathrm{T}}$$

- Since regular operators are divergent, $\zeta^{\mathrm{B}}(t)$ is divergent as well
- Regular operators renormalized through

$$\begin{pmatrix} \mathcal{O} \\ E \end{pmatrix}^{\mathrm{R}} = Z \begin{pmatrix} \mathcal{O} \\ E \end{pmatrix} \equiv \begin{pmatrix} Z_{\mathrm{PP}} & Z_{\mathrm{PE}} \\ Z_{\mathrm{EP}} & Z_{\mathrm{EE}} \end{pmatrix} \begin{pmatrix} \mathcal{O} \\ E \end{pmatrix}$$

- Renormalized $\zeta(t)$:

$$\begin{pmatrix} \tilde{\mathcal{O}}(t) \\ \tilde{E}(t) \end{pmatrix} \asymp \zeta^{\mathrm{B}}(t) Z^{-1} \begin{pmatrix} \mathcal{O} \\ E \end{pmatrix}^{\mathrm{R}} \equiv \zeta(t) \begin{pmatrix} \mathcal{O} \\ E \end{pmatrix}^{\mathrm{R}} \equiv \begin{pmatrix} \zeta_{\mathrm{PP}}(t) & \zeta_{\mathrm{PE}}(t) \\ \zeta_{\mathrm{EP}}(t) & \zeta_{\mathrm{EE}}(t) \end{pmatrix} \begin{pmatrix} \mathcal{O} \\ E \end{pmatrix}^{\mathrm{R}}$$

Method of projectors

- Define projectors [Gorishny, Larin, Tkachov 1983; Gorishny, Larin 1987]

$$P_k[\mathcal{O}_i(x)] \equiv D_k\langle 0|\mathcal{O}_i(x)|k\rangle = \delta_{ik} + O(\alpha_s)$$

- Apply to small flow-time expansion:

$$P_k[\tilde{\mathcal{O}}_i(t,x)] = \sum_j \zeta_{ij}^{\mathrm{B}}(t) P_k[\mathcal{O}_j(x)]$$

- $\zeta_{ij}^{\mathrm{B}}(t)$ only depend on t

$\Rightarrow$ Set all other scales to zero

$\Rightarrow$ No perturbative corrections to $P_k[\mathcal{O}_j(x)]$, because all loop integrals are scaleless

“Master formula”

$$\zeta_{ij}^{\mathrm{B}}(t) = P_j[\tilde{\mathcal{O}}_i(t,x)]\Big|_{p=m=0}$$

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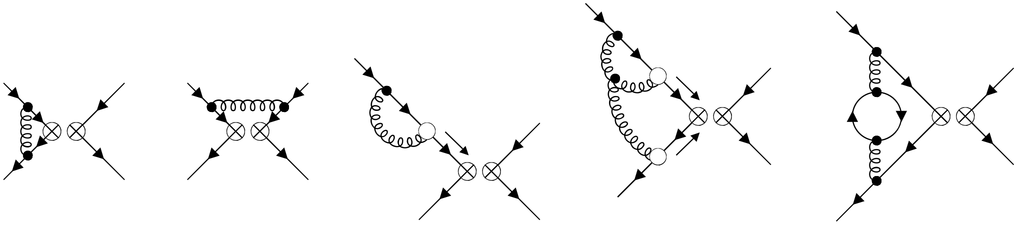
Projectors and example diagrams

- Projectors for $\mathcal{O}_1$ and $\mathcal{O}_2$ (schematically):

$$P_1[\mathcal{O}] = -\frac{1}{16 T_{\rm R}^2 N_{\rm A}} \mathrm{Tr}_{\rm line\,1}\mathrm{Tr}_{\rm line\,2} \, \langle 0| \left(\psi_{4,\rm L} T^b \gamma_\nu \bar{\psi}_{3,\rm L}\right)\left(\psi_{2,\rm L} T^b \gamma_\nu \bar{\psi}_{1,\rm L}\right) \mathcal{O} |0\rangle \Big|_{p=m=0} ,$$

$$P_2[\mathcal{O}] = \frac{1}{16 N_{\rm c}^2} \mathrm{Tr}_{\rm line\,1}\mathrm{Tr}_{\rm line\,2} \, \langle 0| \left(\psi_{4,\rm L} \gamma_\nu \bar{\psi}_{3,\rm L}\right)\left(\psi_{2,\rm L} \gamma_\nu \bar{\psi}_{1,\rm L}\right) \mathcal{O} |0\rangle \Big|_{p=m=0}$$

- Sample diagrams:



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Automatized calculation

- `qgraf` [Nogueira 1991]: Generate Feynman diagrams
- `q2e` and `exp` [Harlander, Seidensticker, Steinhauser 1998; Seidensticker 1999]: Assign diagrams to topologies and prepare `FORM` code
- `FORM` [Vermaseren 2000; Kuipers, Ueda, Vermaseren, Vollinga 2013]: Insert Feynman rules, perform tensor reduction, Dirac traces, color algebra, and expansions
- Generate system of equations employing integration-by-parts-like relations [Tkachov 1981; Chetyrkin, Tkachov 1981] with in-house `Mathematica` code
- `Kira` [Maierhöfer, Usovitsch, Uwer 2017; Klappert, **FL**, Maierhöfer, Usovitsch 2020] $\oplus$ `FireFly` [Klappert, **FL** 2019; Klappert, Klein, **FL** 2020]: Solve system to express all integrals through master integrals
- Master integrals available from [Harlander, Kluth, **FL** 2018]

Results in CMM basis

- Physical matching matrix $(\zeta^{-1})_{\mathrm{PP}}$:

$$(\zeta^{-1})_{11}(t) = 1 + a_s\left(4.212 + \frac{1}{2}L_{\mu t}\right) + a_s^2\left[22.72 - 0.7218\,n_f + L_{\mu t}\,(16.45 - 0.7576\,n_f) + L_{\mu t}^2\left(\frac{17}{16} - \frac{1}{24}\,n_f\right)\right],$$

$$(\zeta^{-1})_{12}(t) = a_s\left(-\frac{5}{6} - \frac{1}{3}L_{\mu t}\right) + a_s^2\left[-4.531 + 0.1576\,n_f + L_{\mu t}\left(-3.133 + \frac{5}{54}\,n_f\right) + L_{\mu t}^2\left(-\frac{13}{24} + \frac{1}{36}n_f\right)\right],$$

$$(\zeta^{-1})_{21}(t) = a_s\left(-\frac{15}{4} - \frac{3}{2}\,L_{\mu t}\right) + a_s^2\left[-23.20 + 0.7091\,n_f + L_{\mu t}\left(-15.22 + \frac{5}{12}\,n_f\right) + L_{\mu t}^2\left(-\frac{39}{16} + \frac{1}{8}\,n_f\right)\right],$$

$$(\zeta^{-1})_{22}(t) = 1 + a_s\,3.712 + a_s^2\left[19.47 - 0.4334\,n_f + L_{\mu t}\,(11.75 - 0.6187\,n_f) + \frac{1}{4}\,L_{\mu t}^2\right]$$

- $a_s = \alpha_s(\mu)/\pi$ renormalized in $\overline{\mathrm{MS}}$ scheme and $L_{\mu t} = \ln 2\mu^2 t + \gamma_E$
- Set $N_c = 3$, $T_R = \frac{1}{2}$, and transcendental coefficients replaced by floating-point numbers

Checks

$$\zeta^{-1} = Z(\zeta^{\mathrm{B}})^{-1} = \begin{pmatrix} (\zeta^{-1})_{\mathsf{PP}} & (\zeta^{-1})_{\mathsf{PE}} \\ (\zeta^{-1})_{\mathsf{EP}} & (\zeta^{-1})_{\mathsf{EE}} \end{pmatrix}$$

- Finite after α_{s} + field renormalization and with Z from [Chetyrkin, Misiak, Münz 1997; Gambino, Gorbahn, Haisch 2003; Gorbahn, Haisch 2004]
- $(\zeta^{-1})_{\mathsf{EP}} = O(\epsilon)$
- Independent of QCD gauge parameter
- Non-trivial basis transformation to non-mixing basis of [Buras, Gorbahn, Haisch, Nierste 2006] leads to diagonal ζ^{-1}

Application to flavor physics

$$\mathcal{H}_{\text{eff}} = -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_i C_i \mathcal{O}_i = -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_{i,j} C_i \zeta_{ij}^{-1} \tilde{\mathcal{O}}_j \equiv -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_i \tilde{C}_i \tilde{\mathcal{O}}_i$$

- Flowed Wilson coefficients $\tilde{C}_i$ and flowed operators $\tilde{\mathcal{O}}_j$ separately completely scheme independent:
  - Do not explicitly depend on renormalization scale μ
  - Do not depend on scheme used for γ_5
  - Do not depend on choice of evanescent operators

$\Rightarrow$ $\tilde{C}_i$ and $\langle \tilde{\mathcal{O}}_j \rangle$ can be computed in different schemes, e.g. perturbatively and on the lattice

- Perturbative ingredients C_i and ζ_{ij}^{-1} have to be computed in the same scheme, but this is no major problem

Status

$$\mathcal{H}_{\text{eff}} = -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_{i,j} C_i \zeta_{ij}^{-1} \tilde{\mathcal{O}}_j :$$

- $|\Delta F| = 2$:
  - C_i: NLO with two contributions for $|\Delta S| = 2$ known through NNLO [Brod, Gorbahn 2010 + 2012]
  - ζ_{ij}^{-1}: NNLO [Harlander, Lange 2022]
- $|\Delta F| = 1$:
  - C_i: NNLO [Bobeth, Misiak, Urban 2000; Gorbahn, Haisch 2004]
  - ζ_{ij}^{-1}: NNLO, but without penguins yet [Harlander, Lange 2022]
- Evolution to small μ / large t:
  - RGE: NNLO [Gorbahn, Haisch 2004]
  - t-evolution: NNLO, but without penguins yet [Harlander, Lange 2022]
- $\langle \tilde{\mathcal{O}}_j \rangle$ not computed yet

Conclusion and outlook

- Gradient flow useful in lattice QCD (scale setting, smearing, non-renormalization)
- Cross-fertilization between lattice and perturbative QCD
- Apply flowed operator product expansion to the electroweak Hamiltonian to avoid scheme matching between perturbative Wilson coefficients and lattice determinations of matrix elements
- Computed the matching matrix between regular and flowed current-current operators through NNLO
  - Non-trivial comparison with NLO result of [Suzuki, Taniguchi, Suzuki, Kanaya 2020] (different basis and different scheme for γ_5) should be done
  - Extension to penguin operators for $|\Delta F| = 1$ planned
- Lattice calculations of matrix elements still in exploratory stages (as far as I know)
- Comparison to traditional approaches with schemes like RI-SMOM to be studied once matrix elements available

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Solving the flow equations

- Split the flow equation into a linear part and a remainder [Lüscher 2010]

$$\partial_t B_\mu^a = \partial_\nu \partial_\nu B_\mu^a + R_\mu^a \quad \text{with} \quad B_\mu^a(t,x)\big|_{t=0} = A_\mu^a(x)$$

- Solved by

$$B_\mu^a(t,x) = \int_y K_{\mu\nu}(t, x-y) A_\nu^a(y) + \int_y \int_0^t \mathrm{d}s\, K_{\mu\nu}(t-s, x-y) R_\nu^a(s,y)$$

with the integration kernel

$$K_{\mu\nu}(t,x) = \int_p e^{\mathrm{i}p\cdot x} \delta_{\mu\nu} e^{-tp^2} \equiv \int_p e^{\mathrm{i}p\cdot x} \widetilde{K}_{\mu\nu}(t,p)$$

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Propagators

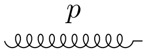
$$B^a_\mu(t,x) = \int_y K_{\mu\nu}(t,x-y)A^a_\nu(y) + \int_y \int_0^t \mathrm{d}s\, K_{\mu\nu}(t-s,x-y)R^a_\nu(s,y)$$

$$K_{\mu\nu}(t,x) = \int_p e^{\mathrm{i}p\cdot x}\delta_{\mu\nu}e^{-tp^2} \equiv \int_p e^{\mathrm{i}p\cdot x}\widetilde{K}_{\mu\nu}(t,p)$$

- Flowed gluon propagator contains the fundamental gluon propagator:

$$\left\langle \widetilde{B}^a_\mu(t,p)\widetilde{B}^b_\nu(s,q)\right\rangle\Big|_{\mathrm{LO}} = \widetilde{K}_{\mu\rho}(t,p)\widetilde{K}_{\nu\sigma}(s,q)\left\langle \widetilde{A}^a_\rho(p)\widetilde{A}^b_\sigma(q)\right\rangle$$

$\Rightarrow$ Can express both by the same Feynman rule

s,ν,b  t,μ,a $= \delta^{ab}\dfrac{1}{p^2}\delta_{\mu\nu}\, e^{-(t+s)p^2}$

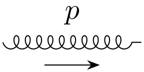
Flow lines

- Flowed gluon Lagrangian:

$$\mathcal{L}_B = -2 \int_0^\infty \mathrm{d}t \, \mathrm{Tr} \left[L_\mu^a T^a \left(\partial_t B_\mu^b T^b - \mathcal{D}_\nu^{bc} G_{\nu\mu}^c T^b \right) \right]$$

$\Rightarrow$ No squared L_μ^a in $\mathcal{L}_B \Rightarrow$ no propagator

- Instead, there is a mixed propagator $\left\langle \widetilde{B}_\mu^a(t,p) \widetilde{L}_\nu^b(s,q) \right\rangle$ called *flow line*:

s, ν, b 

t, μ, a $= \delta^{ab} \, \theta(t-s) \delta_{\mu\nu} \, e^{-(t-s)p^2}$

- Directed towards increasing flow time

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Renormalization (II)

- Renormalization matrix Z includes finite renormalization:

$$Z_{ij} = \delta_{ij} + \sum_{k=1}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^k Z_{ij}^{(k)} \quad \text{with} \quad Z_{ij}^{(k)} = \sum_{l=0}^{k} \frac{1}{\epsilon^l} Z_{ij}^{(k,l)}$$

- Related to anomalous dimension of the operators and Wilson coefficients:

$$\mu \frac{\mathrm{d}\mathcal{O}_i(\mu)}{\mathrm{d}\mu} \equiv \gamma_{ij}\mathcal{O}_j(\mu) \quad \text{and} \quad \mu \frac{\mathrm{d}C_i(\mu)}{\mathrm{d}\mu} \equiv \gamma_{ji}C_j(\mu) \quad \Rightarrow \quad \gamma_{ij} = 2\alpha_s\beta_\epsilon Z_{ik}\frac{\partial Z_{kj}^{-1}}{\partial \alpha_s}$$

- Block form [Buras, Weisz 1990; Dugan, Grinstein 1991; Herrlich, Nierste 1995]:

$$\gamma^{(k)} = \begin{pmatrix} \gamma_{\mathrm{PP}}^{(k)} & \gamma_{\mathrm{PE}}^{(k)} \\ 0 & \gamma_{\mathrm{EE}}^{(k)} \end{pmatrix} \quad \text{and} \quad Z^{(k,0)} = \begin{pmatrix} 0 & 0 \\ Z_{\mathrm{EP}}^{(k,0)} & 0 \end{pmatrix}$$

- This ensures that matrix elements of renormalized evanescent operators vanish:

$$\langle E^{\mathrm{R}} \rangle = Z_{\mathrm{EP}}\langle \mathcal{O} \rangle + Z_{\mathrm{EE}}\langle E \rangle \stackrel{!}{=} O(\epsilon)$$

Feynman rules and projectors (I)

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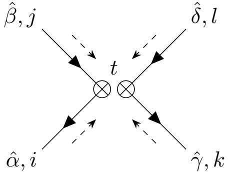
- Feynman rules for operators:

$$\tilde{\mathcal{O}}_2 = \left(\bar{\chi}_{1,\mathrm{L}}\gamma_\mu\chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_\mu\chi_{4,\mathrm{L}}\right)$$

$\Rightarrow$



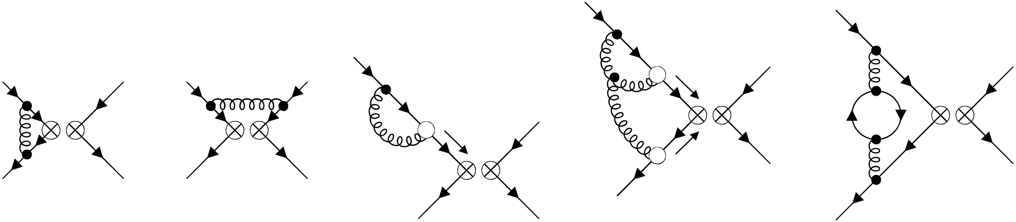
$$= \quad \delta_{ij}\delta_{kl}\{P_+\gamma_\mu P_-\}_{\hat{\alpha}\hat{\beta}}\{P_+\gamma_\mu P_-\}_{\hat{\gamma}\hat{\delta}}$$

- Feynman rule for regular operator $\mathcal{O}_2$ same except for $t = 0$ (no flow lines allowed)
- Suitable projector (schematically):

$$\hat{P}_2[\mathcal{O}] = \frac{1}{16N_\mathrm{c}^2}\,\mathrm{Tr}_{\mathrm{line}\,1}\mathrm{Tr}_{\mathrm{line}\,2}\,\langle 0|\left(\psi_{4,\mathrm{L}}\gamma_\nu\bar{\psi}_{3,\mathrm{L}}\right)\left(\psi_{2,\mathrm{L}}\gamma_\nu\bar{\psi}_{1,\mathrm{L}}\right)\mathcal{O}|0\rangle\Big|_{p=m=0}$$

Feynman rules and projectors (II)

- Sample diagrams:



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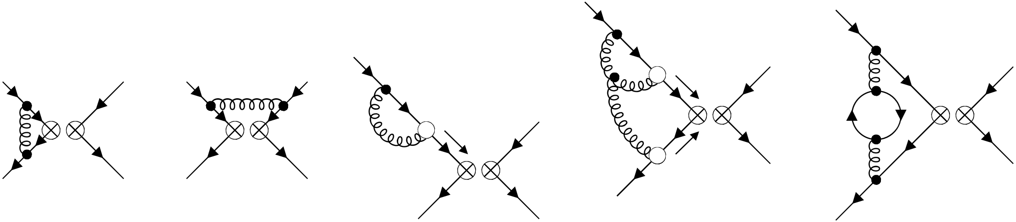
Feynman rules and projectors (II)

- Sample diagrams:



- Projectors not diagonal yet: $\hat{P}_1[E_1^{(1)}] \neq 0$, $\hat{P}_2[E_2^{(1)}] \neq 0, \ldots$

$\Rightarrow$ Construct linear combinations $P_i[\mathcal{O}] = \sum_j C_{ij} P_j[\mathcal{O}]$ such that $P_i[\mathcal{O}_i] = \delta_{ij}$

- Linear combinations have to be adapted when additional evanescent operators are included, i.e. going to NNNLO

Treatment of γ_5 (I)

- In dimensional regularization,

$$\{\gamma_\mu, \gamma_5\} = 0$$

is incompatible with the trace requirement

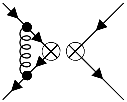
$$\mathrm{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_5) \neq 0 \xrightarrow[D \to 4]{} 4\mathrm{i}\epsilon_{\mu\nu\rho\sigma}$$

- Different prescriptions for γ_5 (NDR, 't Hooft-Veltmann, DREG) lead to different results for scheme-dependent quantities like Wilson coefficients

Treatment of γ_5 (II)

$$P_2[\mathcal{O}] = \frac{1}{16N_c^2}\,\mathrm{Tr}_{\text{line 1}}\mathrm{Tr}_{\text{line 2}}\,\langle 0|\left(\psi_{4,\mathrm{L}}\gamma_\nu\bar{\psi}_{3,\mathrm{L}}\right)\left(\psi_{2,\mathrm{L}}\gamma_\nu\bar{\psi}_{1,\mathrm{L}}\right)\mathcal{O}|0\rangle\Big|_{p=m=0}$$

$$\mathcal{O}_2 = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu\psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu\psi_{4,\mathrm{L}}\right)$$



- The quarks in our operators cannot annihilate due to different flavors

⇒ No γ_5 in traces produced by loop corrections

- Define external quarks in projectors to be left-handed, anticommute γ_5 from operator, and use $P_L^2 = P_L = \frac{1}{2}(1-\gamma_5)$

⇒ No traces with γ_5, simply use naively anticommuting γ_5

- Note: CMM basis avoids γ_5 in traces also for penguin operators ($|\Delta F| = 1$) [Chetyrkin, Misiak, Münz 1997]

Basis transformations

- Different operator bases related by

$$\vec{\mathcal{O}}' = R(\vec{\mathcal{O}} + W\vec{E}) \qquad \text{and} \qquad \vec{E}' = M(\epsilon U\vec{\mathcal{O}} + [1 + \epsilon V]\vec{E})$$

- Not sufficient to simply rotate the physical submatrix with R: $\zeta'_{\mathrm{PP}} \neq R\zeta_{\mathrm{PP}}R^{-1}$
- 1. possibility:
  - Transform whole ζ^{B}
  - Perform renormalization in the same way as before with a different Z
- 2. possibility:
  - Rotate renormalized ζ_{PP}: $\zeta'_{\mathrm{PP}} = R\zeta_{\mathrm{PP}}R^{-1}$
  - But: basis transformation also changes the scheme of Z

$\Rightarrow$ Restore the scheme by an additional finite renormalization [Chetyrkin, Misiak, Münz 1997; Gambino, Gorbahn, Haisch 2003; Gorbahn, Haisch 2004]

Transformation to non-mixing basis

- Physical operators:

$$\mathcal{O}_\pm = \frac{1}{2}\left[\left(\bar\psi_1^\alpha \gamma_\mu^{\mathrm{L}} \psi_2^\alpha\right)\left(\bar\psi_3^\beta \gamma_\mu^{\mathrm{L}} \psi_4^\beta\right) \pm \left(\bar\psi_1^\alpha \gamma_\mu^{\mathrm{L}} \psi_2^\beta\right)\left(\bar\psi_3^\beta \gamma_\mu^{\mathrm{L}} \psi_4^\alpha\right)\right]$$

- Evanescent operators and transformation matrices through NNLO defined in [Buras, Gorbahn, Haisch, Nierste 2006]
- Anomalous dimension is diagonal, i.e. the operators do not mix under RGE running
- We did the transformation in both ways and find agreement as well as a diagonal form:

$$\zeta_{++}^{-1} = 1 + a_{\mathrm{s}}\left(2.796 - \frac{1}{2}L_{\mu t}\right) + a_{\mathrm{s}}^2\left[14.15 - 0.1739\, n_{\mathrm{f}} + L_{\mu t}\left(6.509 - 0.4798\, n_{\mathrm{f}}\right) + L_{\mu t}^2\left(-\frac{9}{16} + \frac{1}{24}\, n_{\mathrm{f}}\right)\right],$$

$$\zeta_{--}^{-1} = 1 + a_{\mathrm{s}}\left(5.546 + L_{\mu t}\right) + a_{\mathrm{s}}^2\left[32.01 - 0.9524\, n_{\mathrm{f}} + L_{\mu t}(21.23 - 0.8965\, n_{\mathrm{f}}) + L_{\mu t}^2\left(\frac{15}{8} - \frac{1}{12}\, n_{\mathrm{f}}\right)\right]$$

Evolving $\tilde{C}_i$

$$\mathcal{H}_{\text{eff}} = -\left(\frac{4G_F}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_{i,j} C_i \zeta_{ij}^{-1} \tilde{\mathcal{O}}_j \equiv -\left(\frac{4G_F}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_i \tilde{C}_i \tilde{\mathcal{O}}_i$$

- SM matching done at $\mu_W \sim M_W$, lattice calculation done at small $\mu \sim \sqrt{1/t}$
- Avoid large logarithms by either:

1
- Evolve regular Wilson coefficients C_i down to $\mu \sim \sqrt{1/t}$ with the known RGE:

$$C_i(\mu) = \sum_j C_j(\mu_W) U_{ji}(\mu_W, \mu)$$

2
- Construct flowed Wilson coefficents $\tilde{C}_i$ at $\mu \sim \sqrt{1/t}$
- Construct flowed Wilson coefficents $\tilde{C}_i$ at $\mu \sim M_W$
- Use the flowed anomalous dimension

$$\tilde{\gamma}(t) = (t\partial_t \zeta(t))\zeta^{-1}(t) \qquad \text{defined through} \qquad t\partial_t \tilde{\mathcal{O}}(t) = \tilde{\gamma}(t)\tilde{\mathcal{O}}(t)$$

to evolve to t large enough for lattice calculation using

$$t\partial_t \tilde{C}_i(t) = -\sum_j \tilde{C}_j(t) \tilde{\gamma}_{ji}(t)$$

- Compatibility of both methods to be studied