From amplitudes to the dynamics of binary black holes

UCLA Mani L. Bhaumik Institute for Theoretical Physics

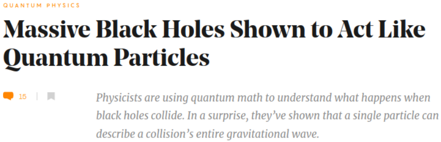
Michael Ruf

Loops and Legs, 27. Apr 2022

based on [2101.07254,2112.10750] w/ Bern, Parra-Martinez, Roiban, Shen, Solon and Zeng

Graviatiational waves at workshop about elementary particle physics...

- GW observation largest collider in the universe (although angular covering is somewhat poor)
- In the end of the day all we care about is high prescision (loops & legs, Feynman diagrams)
- Aim of this talk: reformulate the relativistic two-body as a collider problem (and solve it)

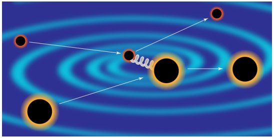

QUANTUM PHYSICS

Massive Black Holes Shown to Act Like Quantum Particles

15

Physicists are using quantum math to understand what happens when black holes collide. In a surprise, they've shown that a single particle can describe a collision's entire gravitational wave.

[quantamagazine.org]

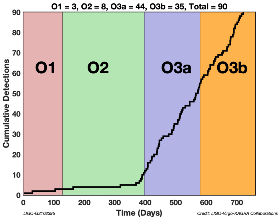


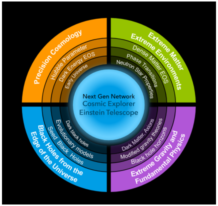
[physics.aps.org]

Related talk by Andreas Maier tomorrow

Gravitational waves

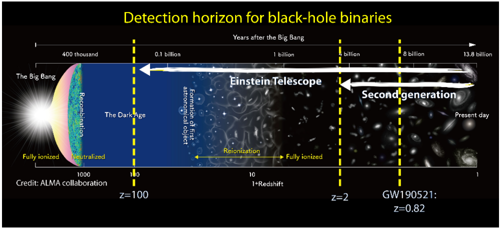
- Many interesting sources (early Universe, supernovae, ...)
- Abundant prospects:
  - Strong-field tests of GR (non-perturbative effects, horizons)
  - Cataloging black hole (BH) binaries (properties, abundance)
  - Equation of state of neutron stars (NS)
  - Multimessenger astronomy
  - ...
- $\mathcal{O}(100)$ mergers events: BH-BH, BH-NS, NS-NS
- This talk: binary black-hole systems without spin

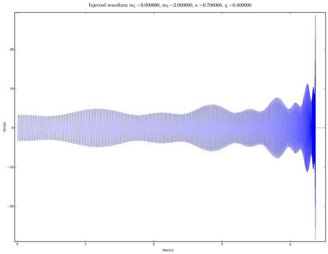




Gravitational waves

- GW astronomy is a precision game:
  - Relative length changes $\Delta\ell/\ell \sim 10^{-20}$
  - Observations of up to $\mathcal{O}(10^3)$ cycles; errors accumulate
  - Features even smaller (angular momentum, structure of constituents, new physics,...)
- Next-gen. experiments: factor 10 improved S/N ratio
- High-precision theory predictions required



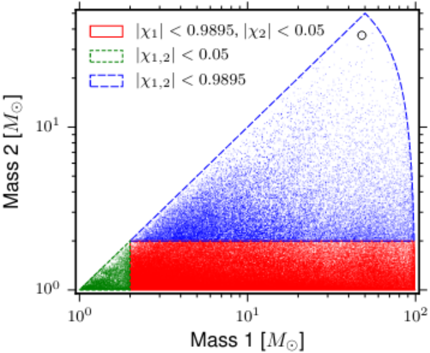


[Diego Fazi, PhD Thesis]

Gravitational waves

- Target observable waveform/strain $h_{\mu\nu}$
- To predict $h_{\mu\nu}$, solve $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$
- Non-linear system, hard
- Numerics challenging; viable since ~ 15 years
- Template database (250k for first detection), supercomputers

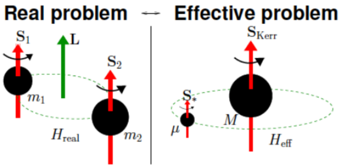
$\rightarrow$ Need analytic input



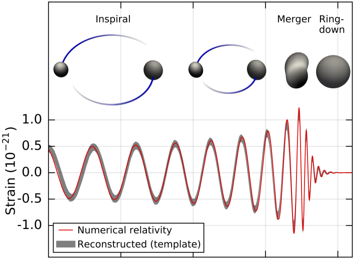
GW150914 templates [Ligo]

Two-body problem

- Inspiral accessible in perturbation theory $r \gg r_S$
- Weak field two-body Hamiltonians
- Adiabatic separation. Radiation through fluxes $\frac{\mathrm{d}J_{ij}}{\mathrm{d}t}, \frac{\mathrm{d}E}{\mathrm{d}t}$
- Effective one-body [Damour, Buonanno ('99)]: Resumming analytic input into geometry + NR tuning



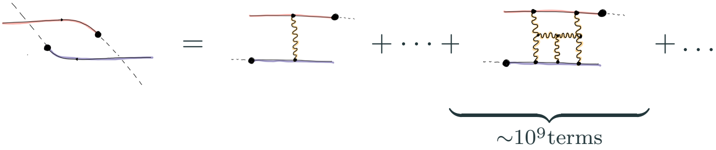
[S. Babak]



[Ligo.org]

Two-body problem

- Perturbation theory is hard (even in classical physics!)
- Perturbation theory is *harder* if gravity is involved: gauge dependence, complicated Feynman rules, power-counting, inherently non-planar,. . .

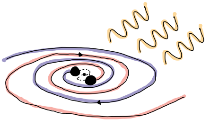


- Very familiar problem in collider phenomenology

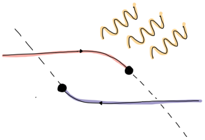
Can we use tools from collider physics for the two-body problem?

Binary dynamics and amplitudes

- From afar, compact objects are point-like [Golberger & Rothstein]
- Analytic continuation above threshold: bound $\implies$ scattering



$\xleftrightarrow{\text{analytic cont.}}$



- S-matrix: all information about *classical* scattering
- Old idea [Iwasaki,'71,...] recently revived [Damour '17]

Binary problem described through $2 \to 2$ scattering amplitudes of massive particles

Binary dynamics and amplitudes

- Exchanged hard problem for an even harder one (c.f. $(\mu/e)^+e^- \to (\mu/e)^+e^-$)
- In the classical limit problem simplifies again

$\to$ apply early and thoroughly!

- Comments:
  - Amplitudes are computed from low-energy effective theory of gravity, UV completion irrelevant
  - Extensions to spin and finite-size effects (Neutron stars) straightforward
  - Methods used in this talk are also used in other classical approaches

PM expansion

- Natural expansion in the coupling G (post-Minkowskian, PM)

$\rightarrow$ hyperbolic orbits, scattering

- In practice, also expand in $v^2 \sim G/r$ (post-Newtonian, PN, [$\rightarrow$ Talk by Andreas Maier])

$\rightarrow$ circular orbits

- Why care about PM?
  - Scattering and eccentric events in Nature (GW190521?) rapid mergers, multi-body systems
  - Cross-check and complementary PM$\leftrightarrow$PN
  - Understand analytic structure (Using Ansatze to contrain form, exclude certain structures)
  - Reorganization, to be expanded later
  - $v \sim c$ hard for NR

[Damour, Bini, Geralico]
[Blumlein, Maier, Marquard, Schafer]
[Foffa, Sturani] + [Mastrolia, Sturm, Torres Bobadilla]

$$
\begin{array}{l|ccccccccccccccccc|l}
 & \textbf{0PN} & & \textbf{1PN} & & \textbf{2PN} & & \textbf{3PN} & & \textbf{4PN} & & \textbf{5PN} & & \textbf{6PN} & & \textbf{7PN} & & & \\
\textbf{1PM} & (1 & + & v^2 & + & v^4 & + & v^6 & + & v^8 & + & v^{10} & + & v^{12} & + & v^{14} & + & \ldots) & G^1 \\
\textbf{2PM} & & & (1 & + & v^2 & + & v^4 & + & v^6 & + & v^8 & + & v^{10} & + & v^{12} & + & \ldots) & G^2 \\
\textbf{3PM} & & & & & (1 & + & v^2 & + & v^4 & + & v^6 & + & v^8 & + & v^{10} & + & \ldots) & G^3 \\
\textbf{4PM} & & & & & & & (1 & + & v^2 & + & v^4 & + & v^6 & + & v^8 & + & \ldots) & G^4 \\
\textbf{5PM} & & & & & & & & & (1 & + & v^2 & + & v^4 & + & v^6 & + & \ldots) & G^5 \\
\textbf{6PM} & & & & & & & & & & & (1 & + & v^2 & + & v^4 & + & \ldots) & G^6 \\
 & & & & & & & & & & & & \vdots & & & & & &
\end{array}
$$

3PM (cons): [Bern, Cheung, Roiban, Shen, Solon, Zeng '19]

4PM (cons): [Bern, Parra-Martinez, Roiban, MR, Shen, Solon, Zeng '21]

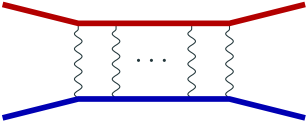
Reproduced by [Dlapa, Kalin, Liu, Porto '21]

3PM rad. energy: [Herrmann, Parra Martinez, MR, Zeng '21]

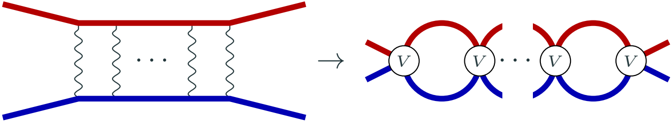
3PM rad. angular momentum: [Manohar, Ridgway, Shen '22]

Classical limit

- Reinstate $\hbar$ in couplings $g \to g/\sqrt{\hbar}$. Loops are classical!
- Large numbers of soft exchanges ($N \sim 1/\hbar$, $k^\mu \sim \hbar$)

$\mathcal{M} =$  $+ \ldots \quad \mathcal{M}^{L-\text{loop}} \sim 1/\hbar^{L+2}$

- Fixed-order+resummation (e.g. EFT, eikonal,...)



Need fixed order $2 \to 2$ amplitudes for soft graviton exchange

Classical limit

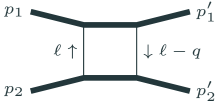
- Hierarchy of scales ($\hbar = 1$): $J^2 \sim 10^{80}$

$$1 \ll J^2 \sim \frac{s}{q^2} \sim \frac{m_i^2}{q^2} \quad \to \quad q^2 \ll m_i^2 \sim s$$

- Relativistic regions:

  hard (h): $\ell \sim m$ $\leftarrow$ short range, UV

  soft (s): $\ell \sim q$ $\leftarrow$ long range



- Soft further splits $v = |\boldsymbol{p}_{\rm COM}|/\sqrt{s}$

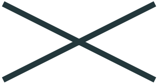
  potential (p): $(\omega, \boldsymbol{\ell}) \sim (|\boldsymbol{q}|v, |\boldsymbol{q}|)$ $\leftarrow$ instantaneous

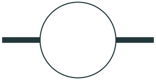
  radiation (r): $(\omega, \boldsymbol{\ell}) \sim (|\boldsymbol{q}|v, |\boldsymbol{q}|v)$

- Classical: soft+threshold expansion keeping (p) and (r) region
- Resummed NRQCD, without "quantum soft" $(\omega, \boldsymbol{\ell}) \sim (|\boldsymbol{q}|, |\boldsymbol{q}|)$ (different terminology!)

Classical limit

- Diagramatics:

 $\overset{(s)}{=} 0\,,$  $\overset{(s)}{=} 0\,,$  $\overset{(s)}{=} 0\,,$

 $\overset{(p)}{=} 0\,,$  $\overset{(p)}{=} 0\,.$

- Conservative dynamics: one matter line per loop
- Many topologies trivial. Important for IBP
- Only one dimensionless variable v

$\rightarrow$ crucial for integration

Conservative dynamics

- In principle can directly compute observables from Amplitudes [Kosower, Maybee, O'Connell (KMOC)]
- However wave-form modeling is based on Hamiltonian ($+$ fluxes)
- To get Hamiltonian, have to separate dissipative/conservative
- Computing the result by pieces is also convenient due to complexity
- Analytic continuation scattering $\rightarrow$ bound is more complicated (if possible) for more complicated in cases when radiation is involved

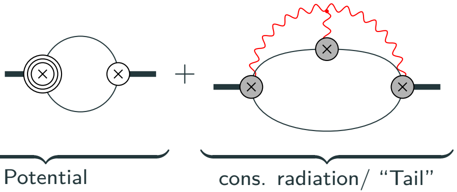
Conservative dynamics

- Up to order G^3 conservative dynamics is synonymous with the potential region $k \sim (vq, q)$

$$\mathcal{M}_3^{(\text{cons})} =$$

- This is an accident, radiation modes $k \sim (vq, vq)$ play an important role in the *conservative* dynamics starting at $\mathcal{O}(G^4)$

$$\mathcal{M}_4^{(\text{cons})} = \cdots +$$



- Since radiation modes can (in principle) go on-shell the boundary conditions are essential

Conservative dynamics

- Time-symmetric propagator [Wheeler–Feynman, Damour '21]

$$G^{\text{sym.}}(x) = \delta(x^2)\,, \quad G^{\text{sym.}}(k) = \mathcal{P}\frac{1}{k^2}$$

- This has several nice properties
  - Well-defined
  - Guarantees elastic unitarity
  - Agrees with EFT intuition [Cheung, Rothstein, Solon] and heuristics no cuts/no real radiation
  - Widely used in the GR literature (self-force)
- Other definitions in [Blumlein et. al. ; Foffa et. al.] based on retarded boundary conditions
- Structure found in [Blumlein et. al. ; Foffa et. al.] impossible to match in amplitude-based approach
- Full result is unambiguous: best way to settle the debate is to do an explicit computation involving dissipative effects

Setup

Computation follows established pattern

1. Integrand construction + soft expansion

    on-shell methods

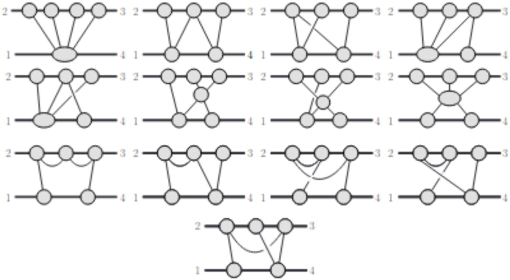
2. IBP reduction
3. Evaluation of master integrals

    Differential equations, boundary conditions

4. Physical observables

    EFT matching, amplitude-action relation, KMOC

- Relevant cuts selected by classical physics: one matter line per loop, no contact interaction, no graviton bubbles



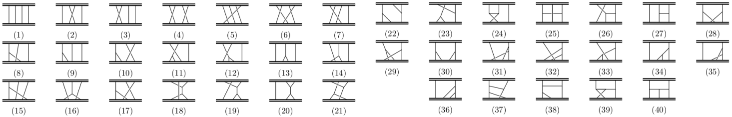
- Tree-level amplitudes computed from double copy

$$\mathcal{M}(1,2,3,4) = s_{12}\mathcal{A}(1,2,3,4)^2$$

- Setup well-established and scales to higher orders

Integrand construction

- Linear dependent propagators: partial fractioning
- 40 families of Feynman integrals



- Classical limit constrains power-counting: max rank 6
- IBP reduction using FIRE6, classical restrictions improve performance, but not crucical (at this order)
- At $\mathcal{O}(G^4)$ only need integrals with odd $\sqrt{-q^2}$-parity.

Integration

- Most challenging part of the computation
- Differential equations in pre-canonical form

$$\mathrm{d}\begin{pmatrix} f_1 \\ \vdots \\ f_9 \\ f_{10} \\ \vdots \\ f_{129} \end{pmatrix} = \begin{pmatrix} \epsilon A_{1,9} + B_{1,9} & 0 \\ C(\epsilon) & \epsilon A_{10,129} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_9 \\ f_{10} \\ \vdots \\ f_{129} \end{pmatrix}$$

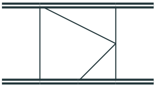
  A, B, C $\mathrm{d}\log$-forms
- 3 elliptic integrals + permutations
- Boundary conditions fixed in static limit $v \to 0$. DE resumms threshold expansion

Integration

- Individual integrals: E and K and iterated integrals
- DE linear: functions dissected by bnd conditions

$$f \overset{v\to 0}{\sim} \sum_i v^{-2n_i\epsilon} c_i g_i(\epsilon) \implies f = \sum_i f_i\,, \quad f_i \overset{v\to 0}{\sim} v^{-2n_i\epsilon} c_i g_i(\epsilon)$$

- Elliptic and polylogarithmic separated (more general?)
- Elliptic integrals only in potential region
- Elliptic part of the amplitude fixed by ansatz



$$= \frac{1}{\epsilon^2}\frac{8}{\sigma+1} K^2\left(\tfrac{\sigma-1}{\sigma+1}\right) + \mathcal{O}(\epsilon^{-1})\,, \quad \sigma = \sqrt{1-v^2}$$

- Individual pieces are divergent in $d = 4 - 2\epsilon$

$$\mathcal{M}^{(\mathrm{p})} = \frac{f^{\mathrm{tail}}}{\epsilon_{\mathrm{IR}}} + \text{finite}\,, \quad \mathcal{M}^{(\mathrm{cr})} = -f^{\mathrm{tail}}\left[\frac{1}{\epsilon_{\mathrm{UV}}} + 4\log(v)\right] + \text{finite}$$

Amplitude

Post-Schwarzschild expansion $\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$

$$\mathcal{M}_4^{(\text{cons})} = G^4 M^7 \nu^2 \pi^2 |\boldsymbol{q}| 2^{2\epsilon} \left(\frac{\boldsymbol{q}^2}{\bar{\mu}^2}\right)^{-3\epsilon} \left\{ f^{\text{pp}} + \nu \left[f^{\text{tail}} \log\left(\frac{\sqrt{\sigma^2-1}}{2}\right) + f^{\text{fin}} \right] \right\}$$
$$+ \underbrace{\int_{\ell} \frac{\tilde{I}_{r,1}^4}{Z_1 Z_2 Z_3} + \int_{\ell} \frac{\tilde{I}_{r,1}^2 \tilde{I}_{r,2}}{Z_1 Z_2} + \int_{\ell} \frac{\tilde{I}_{r,1} \tilde{I}_{r,3}}{Z_1} + \int_{\ell} \frac{\tilde{I}_{r,2}^2}{Z_1}}_{\text{Iterations, } 1/\hbar \text{ and } 1/\epsilon \text{ divergent}}$$

$$f^{\text{pp}} = -\frac{35(1-18\sigma^2+33\sigma^4)}{8(\sigma^2-1)}\,, \qquad f^{\text{tail}} = r_1 + r_2 \log\left(\frac{\sigma+1}{2}\right) + r_3 \frac{\operatorname{arccosh}\sigma}{\sqrt{\sigma^2-1}}$$

Schwarzschild

related to IR divergence

Observables

- Remainder takes a compact form

$$f_4^{\rm fin} = r_4\pi^2 + r_5\, K\left(\tfrac{\sigma-1}{\sigma+1}\right)E\left(\tfrac{\sigma-1}{\sigma+1}\right) + r_6\, K^2\left(\tfrac{\sigma-1}{\sigma+1}\right) + r_7\, E^2\left(\tfrac{\sigma-1}{\sigma+1}\right),$$

$$+ r_8 + \cdots + \frac{r_{17}}{\sqrt{\sigma^2-1}}\left[\mathrm{Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - \mathrm{Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right)\right].$$

$$r_5 = -\frac{1183 + 2929\sigma + 2660\sigma^2 + 1200\sigma^3}{2(\sigma^2-1)}\cdots$$

- Elliptic functions in a classical GR result!
- Radial action: dropping/exponentiating IR divergences

$$I_{r,4} = -\frac{G^4M^7\nu^2\pi\boldsymbol{p}^2}{8EJ^3}\times\left\{f_4^{\rm p} + \nu\left[4f_4^{\rm tail}\log\left(\tfrac{\sqrt{\sigma^2-1}}{2}\right) + f_4^{\rm fin}\right]\right\},$$

- Also computed potential
- Observables

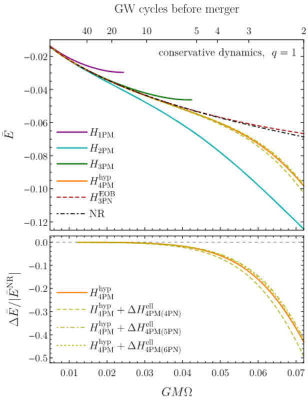
$$\chi = -\frac{\partial I_r}{\partial J}\,, \quad \Delta t = \frac{\partial I_r}{\partial E}$$

Checks

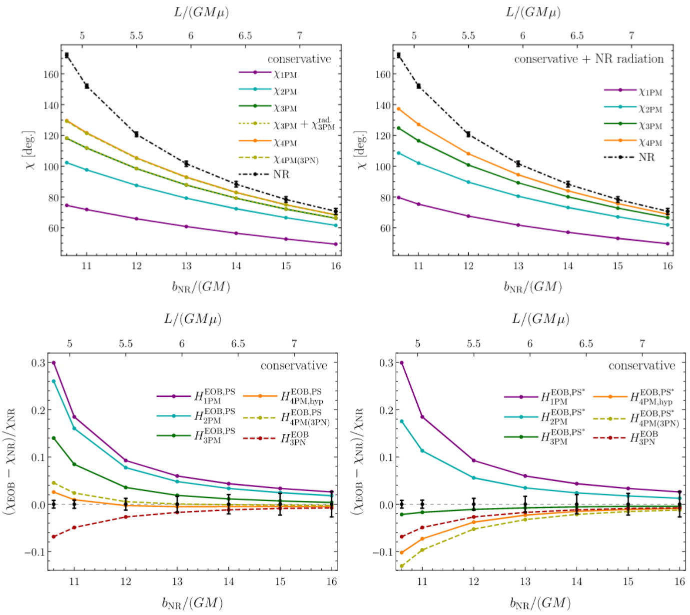
- IR structure agree with lower-order iterations
- Integrals checked against series expansion
- $1/\epsilon$ pole is proportional to $\mathcal{O}(G^3)$ energy loss [Hermann et. al.]
- Potential part matches PN results in the overlap. In particular [Blumlein et. al.] up to $G^4 v^6$.
- Classical computation of radiation-induced piece up to $G^4 v^6$ ([Bini et. al.], caveat: p.v. prescription)
- Independent *classical* computation [Poto et. al.]
- Tension with PN literature related to different boundary conditions

Study by AEI group

- Approximation surprisingly good [Antonelli et. al. '19, Khalil et. al., '22]
- Clear improvements to lower orders
- Tail part is suppressed (power-counting)



Binding energy $E_b = H - M$ vs. orbital frequency Ω [Khalil et. al., 2204.05047]



[Khalil et. al., 2204.05047]

Summary

Collider physics tools have proven immensely for analytic approaches to GR

- Many new results: wave-forms, potentials, radiative losses, ...
- Understanding considerably sharpened in the past ~ 5 years
- Have entered a new phase where main focus is precision (also spin spin)
- Fruitful exchange between communities
- In the future input from collider physics will be even more important

Outlook

- order G^5 and beyond
  - Book-keeping (~ 400 families of integrals/ many thousands of master integrals)
  - Integrand construction
  - IBP-reduction (master topologies with 13 prop/9 ISP, rank 8)
- Radiation at order G^4
- Differential observable $\mathrm{d}E/\mathrm{d}t, \mathrm{d}E/\mathrm{d}\Omega, \ldots, h_{\mu\nu}$
  - Main challange: integration
  - Multivariable elliptic integrals, other functions Bessel-type etc.
- Beyond Schwarzschild-BH: spin, finite size/tidal
- Analytic continuation

Backup

Integration

Mayor simplifications w.r.t e.g. Bhabha scattering

- Only need soft expansion t expansion $\implies$ one dimensionless variable
- Two indipendent DE-systems for even and odd parity in $\sqrt{-t}$
- Three-loop potential is $\frac{1}{r^4} \sim \sqrt{-t}$
- Only sub-regions in momentum space discard $\ell \sim (q, q) \implies$ simpler bnd conditions
- Solve DE independent for contributions of remaining regions

$$f(v) = f^{(\mathrm{ppp})}(v) + v^{-2\epsilon} f^{(\mathrm{ppr})}(v) \tag{1}$$

- split is defined through boundary conditions
- Classical physics imposes cuts, i.e trivial sectors

Integration

Boundary conditions at threshold $v = 0$

- Method of regions in momentum space
- Scaling/regularity fixes most BND conditions
- Resulting integrals are single-scale/numbers
- Individual integrals not regulated in dim-reg: Other regulators

$$I_{\pm} = \int \frac{\mathrm{d}\omega}{\omega \pm \mathrm{i}\varepsilon} = \lim_{\Lambda \to \infty} \int_{-\Lambda}^{\Lambda} \frac{\mathrm{d}\omega}{\omega \pm \mathrm{i}\varepsilon} = -\mathrm{i}\pi \tag{2}$$

- More convenient: keep abstracts, finite combinations in amplitude

$$I_{+} - I_{-} = -2\mathrm{i}\pi \tag{3}$$

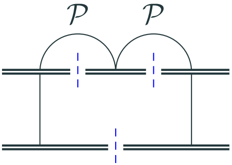
Integration

Boundary conditions at threshold $v \to 0$

- Potential: p.v. prescription implicit

$$\frac{1}{k_0^2 - \vec{k}^2 + \mathrm{i}\varepsilon} = -\frac{1}{\vec{k}^2} \sum_{i=0}^{\infty} \left[\frac{k_0^2}{\vec{k}^2} \right]^i$$

- Radiation: unexpanded propagators, $\mathrm{i}\varepsilon$ matters
- IBP+DE: only depend on $\mathrm{i}\varepsilon$ through BND condition
- For (ppr) one boundary integral to be computed

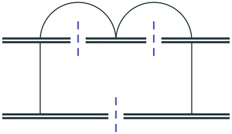
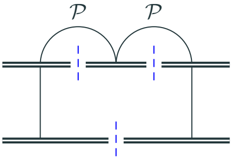


- Poincare-Bertrand

$$\frac{1}{x-a+\mathrm{i}\varepsilon}\,\frac{1}{x-b+\mathrm{i}\varepsilon}$$
$$=\left[\mathcal{P}\left(\frac{1}{x-a}\right)-\mathrm{i}\pi\delta(x-a)\right]\left[\mathcal{P}\left(\frac{1}{x-b}\right)-\mathrm{i}\pi\delta(x-b)\right]+\pi^2\delta(x-a)\delta(x-b)$$

- Applied to the two propagators

$$\mathrm{Re}\left(\frac{1}{\omega_1^2-\boldsymbol{\ell}_1^2+\mathrm{i}\varepsilon}\,\frac{1}{\omega_2^2-\boldsymbol{\ell}_2^2+\mathrm{i}\varepsilon}\right)=\mathcal{P}\left(\frac{1}{\omega_1^2-\boldsymbol{\ell}_1^2}\right)\mathcal{P}\left(\frac{1}{\omega_2^2-\boldsymbol{\ell}_2^2}\right)$$

Re [] = 

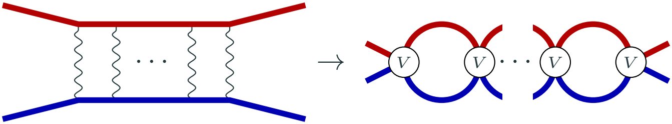
(4)

- At this order we get away with taking matter cuts+real part

Non-relativistic EFT

- Classical (non-relativistic) EFT [Neill, Rothstein; Cheung, Rothstein, Solon]
- Integrate out (potential) off-shell graviton d.o.f.; keep on-shell modes (radiation)
- Integrate out antiparticles

$$
\begin{aligned}
\mathcal{L}[\phi, h^{(\mathrm{r})}_{\mu\nu}] = \sum_{i=1}^{2} \int_{\boldsymbol{k}} \phi_i^\dagger(-\boldsymbol{k}) \left(\mathrm{i}\partial_t - \sqrt{\boldsymbol{k}^2 + m_1^2} \right) \phi_i(\boldsymbol{k}) \\
- \int_{\boldsymbol{k},\boldsymbol{k}'} \phi_1^\dagger(\boldsymbol{k}')\phi_1(\boldsymbol{k}) V(\boldsymbol{k},\boldsymbol{k}') \phi_2^\dagger(-\boldsymbol{k}')\phi_2(-\boldsymbol{k}) + T^{\mu\nu} h^{(\mathrm{r})}_{\mu\nu} + \ldots
\end{aligned}
$$



Classical limit

- Formally $v \ll 1$, resumation to $v \sim 1$.

$$v + \frac{v^3}{3} + \frac{v^5}{5} + \cdots = \operatorname{arctanh}(v) \tag{5}$$

- Systematic through relativistic differential equations

$$\frac{\partial}{\partial \gamma} \vec{f}(\gamma) = A(\epsilon, \gamma) \vec{f}(\gamma) \tag{6}$$

- in $\mathrm{d}\log$ basis BND condition

$$\vec{f}(\gamma) \sim (a_0 + a_1 v + \dots) + v^{-2\epsilon}(a_0 + a_1 v + \dots) + \dots \tag{7}$$

- Only need leading piece from series expansion
- Important simplification: single-variable problem $\gamma = \frac{p_1 \cdot p_2}{m_1 m_2}$

Classical limit (continued)

- Graviton propagators are homogenous in soft, expanded in potential
$$\frac{1}{\ell^2} = \frac{1}{-\boldsymbol{\ell}^2} + \cdots \sim \delta(t) + \ldots$$
- Simpler integrals, matter propagators linearize
$$(\ell - p_1)^2 - m_1^2 = 2m_1 u_1 \cdot \ell + \mathcal{O}(q^2/m_1^2)\,, \quad u_i{\sim}p_i/m_i$$
- Mass scale factors, two variables, q^2, $y = u_1 \cdot u_2$
- Dependence on q^2 fixed by dimensional analysis

Single variable (y) to all orders in the PM expansion!

Classical limit (continued)

- Box integral after expansion:

$$\stackrel{(\mathrm{s})}{=} \frac{(-q^2)^{D/2-3}}{4m_1 m_2} \underbrace{\int \frac{(-q^2)^{3-D/2}\mathrm{d}^D\ell}{\ell^2(\ell-q)^2(u_1\cdot\ell)(-u_2\cdot\ell)}}_{\mathcal{I}(y)} + \ldots$$

- Integrals in the expansions are reduced to a finite set of master integrals
- Evaluate masters $\mathcal{I}(y)$ by differential equations (DE)
- Significant improvements:
  - Reduced number of scales ($3 \to 1$ ratios)
  - Speed-up of integral reduction (days $\to$ minutes)
  - Fewer master integrals ($8 \to 3$ families at 2-loop)
  - Simpler functions (elliptic $\to$ polylogs at 2-loop, ? $\to$ elliptic at 3-loop)

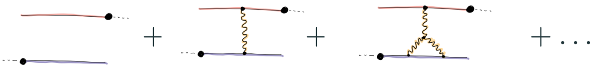
Post-Minkowskian Expansion

- Perturbation theory in G (post-Minkowskian (PM) expansion)
- Iterative solution of Einstein's Eq.

$$T^{\mu\nu} = \sum_{i=1}^{2} \int \mathrm{d}\sigma_i \frac{\delta^{(4)}(x - x_i(\sigma_i))}{\sqrt{-g}} \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma_i} \frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma_i}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \ldots , \quad x_i^\mu = x_{i,0}^\mu + v_i^\mu \sigma + \ldots .$$

- Organized in terms of Feynman-type diagrams



- For circular systems, double expansion in v^2, G (post-Newtonian expansion)

$$v^2 \sim \frac{GM}{r} \ll 1 \implies v^2 = \mathcal{O}(G)$$

Classical limit (continued)

- Structure of soft-expanded amplitudes:

$$\mathcal{M}^{\text{tree}} = \frac{c_0^{\text{CL}}}{q^2} + \dots$$

$$\mathcal{M}^{1-\text{loop}} = \frac{c_1^{\text{SCL}}}{q^2} + \frac{c_1^{\text{CL}}}{\sqrt{-q^2}} + c_1^{\text{Q}} \log(-q^2) + \dots$$

$$\mathcal{M}^{2-\text{loop}} = \frac{c_2^{\text{SSCL}}}{q^2} + \frac{c_2^{\text{SCL}}}{\sqrt{-q^2}} + c_2^{\text{CL}} \log(-q^2) + \dots$$

- Classical pieces at *any* loop order (loops do not count $\hbar$!)
- "Super-classical" pieces have to be subtracted/canceled
- Quantum pieces can be neglected
- Fourier transform maps non-analytic part to long-range effects

$$\frac{c_0^{\text{CL}}}{q^2} \xrightarrow{\text{FT}} \frac{c_0^{\text{CL}}}{r} = -\frac{G m_1 m_2}{r} + \dots$$

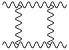
Classical limit (continued)

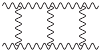
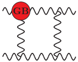
- Graviton propagators are homogenous in soft, expanded in potential
$$\frac{1}{\ell^2} = \frac{1}{-\boldsymbol{\ell}^2} + \cdots \sim \delta(t) + \ldots$$
- Simpler integrals, matter propagators linearize
$$(\ell - p_1)^2 - m_1^2 = 2m_1 u_1 \cdot \ell + \mathcal{O}(q^2/m_1^2)\,, \quad u_i{\sim}p_i/m_i$$
- Mass scale factors, two variables, q^2, $y = u_1 \cdot u_2$
- Dependence on q^2 fixed by dimensional analysis

$\boxed{\text{Single variable } (y) \text{ to all orders in the PM expansion!}}$

Result 3: Four-Graviton Amplitudes [PRL 124, 211601 + PRL 125, 031601 + CPC 267 108069]

- Implemented in `Caravel`: Berends-Giele (BG) recursion, on-shell states, power-counting, graph library
- EFT computation including counter terms

$\mathcal{M}^{(0)} \sim$  $+ \ldots, \quad \mathcal{M}^{(1)} \sim$  $+ \ldots,$

$\mathcal{M}^{(2)} \sim$  $+$ 

$+$ 

$+$ 

$+ \ldots.$

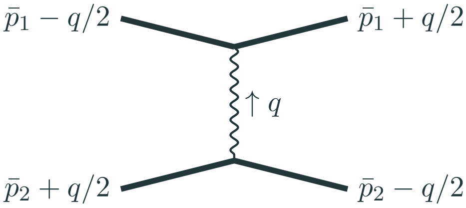
- Challenging computation (linear systems $\sim 30\mathrm{k} \times 30\mathrm{k}$, BG)
- Computation on NEMO ($\sim 10\mathrm{h}/40$ cores/ps-point)
- Reconstruct from ~ 100 (finite field) numerical ps-points.

Special variables

- "soft" variables manifest q-scaling [Sudakov]

$$u_i = \bar{p}_i / |\bar{p}_i| \,, \quad u_1 \cdot u_2 = y \,, \quad u_i \cdot q = 0 \,, \quad u_i^2 = 1 \,.$$

$$y = \frac{p_1 \cdot p_1}{m_1 m_2} + \mathcal{O}(q^2) \equiv \sigma + \mathcal{O}(q^2)$$



Analytic continuation

$$\chi + \pi = 2b \int_{r_{\min}}^{\infty} \frac{\mathrm{d}r}{r\sqrt{r^2 p^2/p_\infty^2 - b^2}} \tag{8}$$

$$\Delta\Phi + 2\pi = 2J \int_{r_-}^{r_+} \frac{\mathrm{d}r}{r\sqrt{r^2 p^2 - J}} \tag{9}$$

$J = p_\infty b$ $r_{+,-,\min}$ zeroes of $p_r^2 = p^2 - J^2/r^2$

$$\Delta\Phi + 2\pi = \chi(J) + \chi(-J) \tag{10}$$