Crewther-Broadhurst-Kataev (CBK) relation: new results

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Loops & Legs 22, Ettal, 29.04.2022

Outline

- intro: CBK-relation, its components, history, current status and why it is of importance

- motivations for computing its components (that is the Adler function $D(\alpha_s)$ and the coef. function C^{Bj}_{pol} /appearing in so-called Bjrken sum rule for polarized DIS/) within a QCD model with artbitrary number of fermion (gauge) representations

- results and technical references

- Perspectives: the CBK relation (that is $D(\alpha_s)$ and C^{Bj}_{pol}) at order α_s^5 in (not very remote) future?

Bjorken Sum Rule (polarized) (reminder)

- **the polarized Bjorken sum rule** ($a_s \equiv \frac{\alpha_s}{\pi}$)

$$\boldsymbol{B}jp(Q^2) = \int_0^1 [g_1^{ep}(x,Q^2) - g_1^{en}(x,Q^2)]dx = \frac{1}{6}|\frac{g_A}{g_V}|C^{Bjp}(a_s)$$

Coefficient function $C^{Bjp}(a_s)$ is fixed by OPE of two EM currents (up to power suppressed corrections)

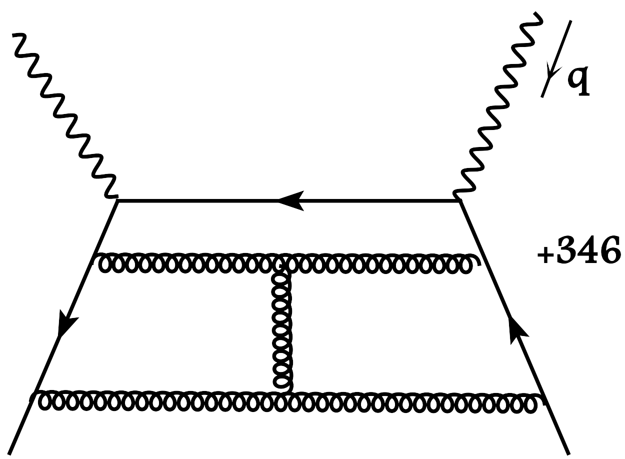
$$i\int TV_\alpha^E(x)V_\beta^E(0)e^{iqx}dx|_{q^2\to\infty} \approx \frac{q^\sigma}{Q^2}\epsilon_{\alpha\beta\rho\sigma}\times$$

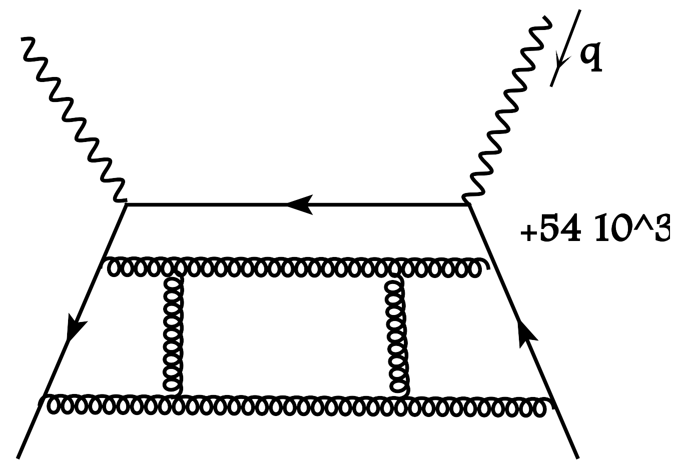
$$\{Tr[E^2 t_c]\ C^{Bjp}(a_s)\ \}\ A_\rho^c(0) + \ldots$$

where $E = diag(Q_i)$, $V_\alpha^E = \overline{\psi}E\gamma_\rho\psi$ **is the EM current,**

$A_\rho^c = \overline{\psi}t^c\gamma_\rho\psi$ **is (non-singlet!) axial current and** $Q^2 = -q^2$

Typical diagrams:





At order α_s^3 the CF was computed in early nineties /Larin, Vermaseren, 91/.

The 4-loop $\mathcal{O}(\alpha_s^4)$ contribution to $C^{Bjp}(a_s)$ for *generic* color group was computed /Baikov, J.Ch. Kühn, 2010/ with two aims: to confront to exp. data (here SU(3) would be enough) and to check the $\mathcal{O}(\alpha_s^4)$ Adler function (technically

significantly more complicated

for evaluation, see later) via CBK relation

The Crewther relation states that in the conformal invariant limit ($\beta \equiv 0$) $C_{Bjp}(a_s)$ is related to the (nonsinglet) Adler function via the following beautiful equality

$$\boldsymbol{C_{Bjp}(a_s) D(a_s))|_{c-i} = 1}$$

its generalization for real QCD (the CBK relation) reads:

$$\boldsymbol{C^{Bjp}(a_s) D(a_s) = 1 + \frac{\beta(a_s)}{a_s} \Big[K = K_1\, a_s + K_2\, a_s^2 + K_3\, a_s^3 + \ldots}$$

Main ingredients of the derivation: the AVV 3-point function and constraints on it from (approximate) conformal invariance + Adler-Bardeen anomaly theorem

Crewther Relation: (short) bibliography

discovered: R.J. Crewther, *Phys. Rev. Lett.* **28**, 1421 (1972).
S.L. Adler, C.G. Callan, D.J. Gross and R. Jackiw, *Phys. Rev.* D **6**, 2982 (1972).

generalized for "real" QCD:

D.J. Broadhurst and A.L. Kataev, *Phys. Lett.* B 315, 179 (1993)

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"proven" (still with some hand-waving):

R.J. Crewther, *Phys. Lett.* B **397**, 137 (1997).
V. M. Braun, G. P. Korchemsky and D. Müller, Prog. Part. Nucl. Phys. **51**, 311 (2003)

Which exactly constraints come from the CBK relation?

$$C^{Bjp}(a_s)D(a_s) = 1 + \frac{\beta(a_s)}{a_s}\Big[K_1\,a_s + K_2\,a_s^2 + K_3\,a_s^3 + \ldots\Big]$$

If it is valid at order a_s^n, then at the next order a_s^{n+1}, we have $(T \equiv T_F\,n_f)$

$$(d_{n+1} - C^{Bjp}_{n+1} + \text{interference terms})\,a_s^{n+1} = \beta_0\,a_s\Big[K_n\,a_s^n\Big]$$

$\alpha_s^1 : (d_1 - C_1) : C_F \Longleftrightarrow K_0 \equiv 0 \leftarrow$ one constraint

$\alpha_s^2 : (d_2 - C_2) : C_F^2, T\,C_F, C_F\,C_A \Longleftrightarrow K_1 : C_F \leftarrow$ two constraints

$$\alpha_s^3 : (d_3 - C_3) : C_F^3\,, C_F^2 C_A\,, C_F C_A^2\,, C_F^2 T\,, C_F C_A T\,, C_F T^2$$

$$\Updownarrow$$

$K_2 : C_F^2, C_F C_A, C_F T \leftarrow$ three constraints

At last, at $\mathcal{O}(\alpha_s^4)$ there exist exactly 12 color strtuctures:

$$C_F^4\,,\ C_F^3 C_A\,,\ C_F^2 C_A^2\,,\ C_F C_A^3\,,\ C_F^3 T_F n_f\,,\ C_F^2 C_A T_F n_f\,,$$
$$C_F C_A^2 T_F n_f\,,\ C_F^2 T_F^2 n_f^2\,,\ C_F C_A T_F^2 n_f^2\,,\ C_F T_F^3 n_f^3\,,\ d_F^{abcd} d_A^{abcd}\,,\ n_f d_F^{abcd} d_F^{abcd}$$

while the coefficient K_3 is contributed by only **6** color structures:

$$C_F T^2\,, C_F\, C_A^2\,, C_F^2\, T\,, C_F\, C_A\, T\,, C_F^2\, C_A\,, C_F^3$$

Thus, we have 12-6 $=$ **6** constraints on the difference

$$\boldsymbol{d_4 - (C^{Bjp})_4}$$

3 of them are very simple: the above difference cannot contain

$$C_F^4,\ d_F^{abcd} d_A^{abcd} \quad n_f d_F^{abcd} d_F^{abcd}$$

remaining three are a bit more complicated

RESULT: ALL 3 (at $\mathcal{O}(\alpha_s^3)$) and 6 (at $\mathcal{O}(\alpha_s^4)$) constarints are fulfilled identically $\Longrightarrow$ an extremely strong test of $\mathcal{O}(\alpha_s^4)$ results for $D(\alpha_s)$ and C^{Bj}

(and the very CBK relation too!)

The test was then very important due to a controversial situation with an explicit presence of ζ_3 in the 5-loop β-function β^{qQED} in the quenched QED, This was in a conradiction to a widespread belief of experts that β^{qQED} should be rational at 5 (and even more loops) like it does in 3 and four loops

Fine print: **by now the result for the Adler function has beeen confirmed by a completely independnt calculation with the help of FORCER:**

F. Herzog, B. Ruijl, T. Ueda, J. A. M. Vermaseren and A. Vogt, *On Higgs decays to hadrons and the R-ratio at N^4LO*, **JHEP 08 (2017), 113**

	d_4	$(1/C^{Bjp})_4$
C_F^4	$\frac{4157}{2048} + \frac{3}{8}\,\zeta_3$	$\frac{4157}{2048} + \frac{3}{8}\,\zeta_3$
$n_f \frac{d_F^{abcd} d_F^{abcd}}{d_R}$	$-\frac{13}{16} - \,\zeta_3 + \frac{5}{2}\,\zeta_5$	$-\frac{13}{16} - \,\zeta_3 + \frac{5}{2}\,\zeta_5$
$\frac{d_F^{abcd} d_A^{abcd}}{d_R}$	$\frac{3}{16} - \frac{1}{4}\,\zeta_3 - \frac{5}{4}\,\zeta_5$	$\frac{3}{16} - \frac{1}{4}\,\zeta_3 - \frac{5}{4}\,\zeta_5$
$C_F T_f^3$	$-\frac{6131}{972} + \frac{203}{54}\,\zeta_3 + \frac{5}{3}\,\zeta_5$	$-\frac{605}{972}$
$C_F^2 T_f^2$	$\frac{5713}{1728} - \frac{581}{24}\,\zeta_3 + \frac{125}{6}\,\zeta_5 + 3\,\zeta_3^2$	$\frac{869}{576} - \frac{29}{24}\,\zeta_3$
$C_F T_f^2 C_A$	$\frac{340843}{5184} - \frac{10453}{288}\,\zeta_3 - \frac{170}{9}\,\zeta_5 - \frac{1}{2}\,\zeta_3^2$	$\frac{165283}{20736} + \frac{43}{144}\,\zeta_3 - \frac{5}{12}\,\zeta_5 + \frac{1}{6}\,\zeta_3^2$
$C_F^3 T_f$	$\frac{1001}{384} + \frac{99}{32}\,\zeta_3 - \frac{125}{4}\,\zeta_5 + \frac{105}{4}\,\zeta_7$	$-\frac{473}{2304} - \frac{391}{96}\,\zeta_3 + \frac{145}{24}\,\zeta_5$
$C_F^2 T_f C_A$	$\frac{32357}{13824} + \frac{10661}{96}\,\zeta_3 - \frac{5155}{48}\,\zeta_5 - \frac{33}{4}\,\zeta_3^2 - \frac{105}{8}\,\zeta_7$	$-\frac{17309}{13824} + \frac{1127}{144}\,\zeta_3 - \frac{95}{144}\,\zeta_5 - \frac{35}{4}\,\zeta_7$
$C_F T_f C_A^2$	$-\frac{(\cdots)}{(\cdots)} + \frac{8609}{72}\,\zeta_3 + \frac{18805}{288}\,\zeta_5 - \frac{11}{2}\,\zeta_3^2 + \frac{35}{16}\,\zeta_7$	$-\frac{(\cdots)}{(\cdots)} - \frac{59}{64}\,\zeta_3 + \frac{1855}{288}\,\zeta_5 - \frac{11}{12}\,\zeta_3^2 + \frac{35}{16}\,\zeta_7$
$C_F^3 C_A$	$-\frac{253}{32} - \frac{139}{128}\,\zeta_3 + \frac{2255}{32}\,\zeta_5 - \frac{1155}{16}\,\zeta_7$	$-\frac{8701}{4608} + \frac{1103}{96}\,\zeta_3 - \frac{1045}{48}\,\zeta_5$
$C_F^2 C_A^2$	$-\frac{592141}{18432} - \frac{43925}{384}\,\zeta_3 + \frac{6505}{48}\,\zeta_5 + \frac{1155}{32}\,\zeta_7$	$-\frac{435425}{55296} - \frac{1591}{144}\,\zeta_3 + \frac{55}{9}\,\zeta_5 + \frac{385}{16}\,\zeta_7$
$C_F C_A^3$	$\frac{(\cdots)}{(\cdots)} - \frac{(\cdots)}{(\cdots)}\,\zeta_3 - \frac{77995}{1152}\,\zeta_5 + \frac{605}{32}\,\zeta_3^2 - \frac{385}{64}\,\zeta_7$	$\frac{(\cdots)}{(\cdots)} - \frac{(\cdots)}{(\cdots)}\,\zeta_3 - \frac{12545}{1152}\,\zeta_5 + \frac{121}{96}\,\zeta_3^2 - \frac{385}{64}$

Motivations for computing the CBK components in an extended QCD model with multiple fermion representations

1. There is no a real all-order proof of the CBK relation (the real one should, imho, provide some prescription for a direct evaluation of the K-factor + clarify in which schemes the CBK-relation holds!)

2. In fact, a change of renormalization scheme might lead to a violation of the CBK relation:

A. Garkusha and A. Kataev, *The absence of QCD β-function factorization property of the generalized Crewther relation in the 't Hooft $\bar{M}S$-based scheme*, **Phys. Lett. B 705 (2011), 400-404**

3. Results within extended QCDe model provide valuable input data for so-called sequential BLM approach (aka β-expansion) /S. Mikhailov ('07), S. Mikhailov, A. Kataev ('12) + $\ldots$/

The approach is trying to isolate the β-independet contributions to pQCD results for solution of the notorious scheme-dependence problem. It is a (rather different) version of the approach based on the Principle of Maximal Conformality /S. Brodsky et al ('12) +$\ldots$/

The Lagrangian of a (massless) QCD-like model extended to include several fermion representations of the gauge group (to be referred as QCDe) is given by

$$\boldsymbol{\mathcal{L}}_{QCD} = \ldots + \sum_{r=1}^{N_{\textbf{rep}}} \sum_{q=1}^{n_{f,r}} \left\{ \frac{i}{2} \bar{\psi}_{q,r} \overleftrightarrow{\hat{\partial}} \psi_{q,r} - m_{q,r} \bar{\psi}_{q,r} \psi_{q,r} + g_s \bar{\psi}_{q,r} \hat{A}^a T^{a,r} \psi_{q,r} \right\},$$

$$\mathrm{R} = (q - \textbf{flavors}, r - \textbf{Representation})$$

Lie Algebra: $\left[T^{a,r}, T^{b,r}\right] = i f^{abc} T^{c,r};\ T^{a,r}_{ik} T^{a,r}_{kj} = \delta_{ij} C_{F,r};\ T_{F,r} \delta^{ab} = \textbf{Tr} \left(T^{a,r} T^{b,r}\right);$

+ purely gauge things f^{abc} and C_A as in the normal QCD

The standard QCD corresponds to the case of $N_{\textbf{rep}} = 1$. If $N_{\textbf{rep}} > 1$ we will consider the first fermion repesentation as a special one in what follows with ($d_{F,r}$ is the dimension of the fermion representation r)

$$C_{F,1} \equiv C_F, T_{F,1} \equiv T_F, n_{f,1}, d_{F,1} \equiv d_F, n_{f,1}, \equiv n_f \ \textbf{ and } \ T^{a,1} \equiv T^a.$$

The following quadratic Casimir operators also appear in our results:

$$\tilde{d}_{FA} = \frac{d_F^{abcd} d_A^{abcd}}{d_F}, \quad \tilde{d}_{FF,r} = \frac{d_F^{abcd} d_{F,r}^{abcd}}{d_F}, \tag{1}$$

with $d_F^{abcd} \equiv d_{F,1}^{abcd}$. But they are not relevant for us as the correspondig coefficients are the same as in normal QCD ...

Which new color structures appear in QCDe? Only in diagrams with internal fermion loop(s). They new structures look as:

$$C_f T_F n_f \to C_f \sum_i T_{F,i} n_{f,i} \equiv C_F\, \boldsymbol{nT}$$

$$C_f^2 T_F n_f \to C_f^2 \sum_i T_{F,i} n_{f,i}, \quad C_f \sum_i T_{F,i} n_{f,i}\, C_{F,i} \equiv C_F\, \boldsymbol{nTC1}$$

$$C_f^3 T_F n_f \to C_f^3 \sum_i T_{F,i} n_{f,i}, \quad C_f \sum_i T_{F,i} n_{f,i} C_{F,i}^2 \equiv C_F\, \boldsymbol{nTC2},$$

$$C_f^2 \sum_i T_{F,i} n_{f,i}\, C_{F,i} \equiv C_F^2\, \boldsymbol{nTC1} \quad \textbf{and so on}$$

As a result, before (for the QCD case) have 12-6 = 6 constraints on the difference

$$\boldsymbol{d_4 - (C^{Bjp})_4}$$

and now (for the QCDe case) we have now 16-7 = 9 constraints and they all are exactly saturated by our resuls!

Our result for, say, d_4 looks as:

$$
\begin{aligned}
d_4 \quad = \quad & C_F^4\left(96\zeta_3+\frac{4157}{8}\right)+ \\
& C_F^3\Big[C_A(-278\zeta_3+18040\zeta_5-18480\zeta_7-2024)-\mathbf{nT}(56\zeta_3+6560\zeta_5-6720\zeta_7-298)\Big]+ \\
& C_F^2\Big[C_A^2\left(-\frac{87850}{3}\zeta_3+\frac{104080}{3}\zeta_5+9240\zeta_7-\frac{592141}{72}\right)+ \\
& C_A(\mathbf{nT})\left(\frac{61912}{3}\zeta_3-\frac{83680}{3}\zeta_5-3360\zeta_7+\frac{67925}{9}\right)+ \\
& (\mathbf{nT})^2\left(-\frac{10240}{3}\zeta_3+\frac{16000}{3}\zeta_5-\frac{13466}{9}\right)+\mathbf{nTC1}(576\zeta_3-960\zeta_5+251)\Big]+ \\
& C_F\Big[C_A^3\left(4840\zeta_3^2-\frac{912446}{27}\zeta_3-\frac{155990}{9}\zeta_5-1540\zeta_7+\frac{52207039}{972}\right)+ \\
& C_A^2(\mathbf{nT})\left(-1408\zeta_3^2+\frac{275488}{9}\zeta_3+\frac{150440}{9}\zeta_5+560\zeta_7-\frac{4379861}{81}\right)+ \\
& C_A(\mathbf{nT})^2\left(-128\zeta_3^2-\frac{83624}{9}\zeta_3-\frac{43520}{9}\zeta_5+\frac{1363372}{81}\right)- \\
& C_A(\mathbf{nTC1})\left(2112\zeta_3^2-7792\zeta_3-400\zeta_5+\frac{375193}{54}\right)\Big]+ \\
& (\mathbf{nT})^3\left(\frac{25984}{27}\zeta_3+\frac{1280}{3}\zeta_5-\frac{392384}{243}\right)+ \\
& (\mathbf{nT})(\mathbf{nTC1})\left(768\zeta_3^2-2784\zeta_3+\frac{63250}{27}\right)+\mathbf{nTC2}\left(272\zeta_3-480\zeta_5+\frac{355}{3}\right)- \\
& 16\left[\sum_r n_{f,r}\,\tilde{d}_{\mathsf{FF},r}\cdot(13+16\zeta_3+40\zeta_5)+\tilde{d}_{\mathsf{FA}}\cdot(-3+4\zeta_3+20\zeta_5)\right]
\end{aligned}
$$

Some technical details and references

We have computed both functions $D(\alpha_s)$ and $C^{Bj}(\alpha_s)$ at order α_s^4 in QCDe using essentially the same methods which were used in /P. Baikov, K. Ch., J. Kühn (2010)/:

- **reduction of FI's with 1/D expansion [P.Baikov 2000. . .]**
- **tFORM [J. Vermaseren 1990, J. Vermaseren, M.Tentyukov and J.Vollinga (2009), J. Vermaseren, T. Kaneko, J. Kuipers, B.'Ruijl, M. Tentyukov, T. Ueda and J. Vollinga (2018)**
- **new tool: an extension of FORM package COLOR for color structures [T. van Ritbergen, A.Schellekens, J. Vermaseren 1999 . . .] made for QCDe by M.Zoller (2016)]**

What about an $\mathcal{O}(\alpha_s^5)$ calculation in some (not too remote) future?

In fact, it is not too crasy idea, especially for $C^{Bj}(\alpha_s)$. **ALL** what we need is an ability to reduce 5-loop p-integrals (that is "massless" propagators) (At 4-loop level it is now a routine task due to FORCER /B. Ruijl, T. Ueda and J.A.M. Vermaseren, ('17)/

Main reason for hope:

Continuous development of "Laporta algorith" approach

AIR, /Anastasiou,Lazopoulos ('04)/

Reduze [Studerus'09; Manteuffel,Studerus /'12+/

FIRE /Smirnov et al.'08+/

Kira [Maierhöfer,Usovitsch,Uwer'17; Klappert,Lange,Maierhöfer,Usovitsch'20]

Crusher /Marquard,Seidel ('00 +) / ...

as well as various aprovements in parametric solution of IBP eqs (see the talk by T. Huber)

In fact, recently master 5-loop p-integrals have been all analytically computed in

A. Georgoudis, V. Gonçalves, E. Panzer, R. Pereira, A. Smirnov and V. Smirnov, "Glue-and-cut at five loops,", JHEP 09 (2021)

The work have used the the glueing method /K. Ch,. P. Baikov (2010)/; the method requires essentially an ability to do reduction to masters (the reduction was successfully done with FIRE)

FI's to be reduced were still relatively simple, MUCH simpler than those required even for $C^{Bj}(\alpha_s)$, not speaking on the Adler function.

Still, $\mathcal{O}(\alpha_s^3)$ calculation of the Adler function was done at 1990-1991 (Kataev, Larin, Gorishny), $\mathcal{O}(\alpha_s^4)$ at 2010. Thus, one might expect $\mathcal{O}(\alpha_s^5)$ calculation by, say, 2030.