Renormalization of twist-two operators in QCD and its application to singlet splitting functions

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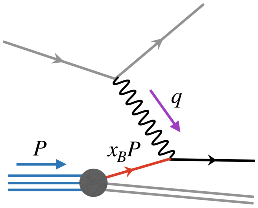
Parton denstities and splitting functions



- Bjorken variable

$$x_B = \frac{-q^2}{2P \cdot q}$$

- Evolution of PDFs

$$\frac{df_{i/N}}{d\ln\mu} = 2\sum_k P_{ik} \otimes f_{k/N}$$

- Quark parton density

$$f_{q/N}(x_B) = \int \frac{dt}{2\pi} e^{-i\,x_B\,t\Delta\cdot p} \langle N(P)|\,[\bar{\psi}W]\,(t\Delta)\frac{\not{\Delta}}{2}\left[W^\dagger\psi\right](0)|N(P)\rangle$$

- Quark collinear Wilson line

$$W^\dagger(x) = \bar{\mathcal{P}}\exp\left(-i\,g_s\int_{-\infty}^0 ds\,\Delta\cdot A(x+s\,\Delta)\right), \Delta^2 = 0$$

Motivations for four-loop splitting functions

- Begin to go into N^3LO precision, but N^3LO PDFs are missing.
- Mismatch between precision of fixed order hard scattering and PDFs
- The scale uncertainties of N^3LO Drell-Yan process using NNLO PDFs don't reduce compared with the uncertainties at NNLO See the talk of Xuan Chen
- The evolution of N^3LO PDFs requires four-loop splitting functions
- Provide data for the study of analytic continuation and reciprocity relation between space-like and time-like singlet splitting functions

Splitting functions & Anomalous dimensions

Mellin transformation

$$\bar{f}_q(n) = -\int_0^1 dz\, z^{n-1} f_q(z)\,, \quad \gamma_{ij}(n) = -\int_0^1 dz\, z^{n-1} P_{i\leftarrow j}(z)$$

DGLAP in moment space: $\frac{d}{d\ln\mu}\bar{f}_q(n,\mu^2) = -2\sum_j \gamma_{qj}(n)\,\bar{f}_j(n,\mu^2)$

Light-cone expansion: $\bar{f}_q(n) \sim \langle N(P)|\,\bar{\psi}_i \not{\Delta} (\Delta\cdot D)_{ij}^{n-1}\psi_j\,|N(P)\rangle$

n-moment of the $P \longleftrightarrow$ anomalous dimension of Twist-2 Spin-n local operator

Twist-two operators

According to the flavor group,

- A single non-singlet operator

$$O_{q,k} = \frac{1}{2}\left[\bar{\psi}_i \not{\Delta} (\Delta \cdot D)^{n-1}_{ij} \frac{\lambda_k}{2} \psi_j\right]$$

where $(D_\mu)_{ij} = \partial_\mu \delta_{ij} - i g_s (T^a)_{ij} A^a_\mu$, λ_k is the flavor generator.

- Two singlet operators

$$O_q = \frac{1}{2}\left[\bar{\psi}_i \not{\Delta} (\Delta \cdot D)^{n-1}_{ij} \psi_j\right],$$
$$O_g = \frac{1}{2}\left[\Delta_{\mu_1} G^{\mu_1}_{a,\mu} (\Delta \cdot D)^{n-2}_{ab} \Delta_{\mu_n} G^{\mu_n \mu}_b\right]$$

Renormalization of twist-two operators

- The non-singlet operator $O_{q,k}$ is distinguished from singlet operators by the quark flavor, and is multiplicatively renormalized, i.e.,

$$O^{\mathsf{R}}_{q,k} = Z^{\mathsf{ns}} O^{\mathsf{B}}_{q,k}\,.$$

- The two singlet operators belong to the same irreducible representation and mix under renormalization,

$$\begin{pmatrix} O_q \\ O_g \end{pmatrix}^{\mathsf{R}} = \begin{pmatrix} Z_{qq} & Z_{qg} \\ Z_{gq} & Z_{gg} \end{pmatrix} \begin{pmatrix} O_q \\ O_g \end{pmatrix}^{\mathsf{B}}\,.$$

- Extract the anomalous dimensions from the renormalization factors,

$$Z_{ij} = \delta_{ij} + \sum_{k=1}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^k \frac{1}{k\,\epsilon} \gamma_{ij}^{(k-1)} + \mathcal{O}\left(\frac{1}{\epsilon^2}\right)$$

Splitting functions from DIS

- One-loop results $P^{(0)}$[Altarelli and Parisi, 1977]
- Two-loop results $P^{(1)}$[Furmanski and Petronzio, 1980; Curci, Furmanski and Petronzio, 1980]
- Three-loop results $P^{(2)}$
  - Non-singlet[Moch, Vermaseren and Vogt, 2004]
  - Singlet[Vogt, Moch and Vermaseren, 2004]
- Partial four-loop results for singlet splitting functions[S. Moch, B. Ruijl, T. Ueda, J.A. M. Vermaseren, A. Vogt, 2021] see the talk of Sven-Olaf Moch

Splitting functions from on-shell operator matrix elements (OMEs)

Consider the matrix elements with zero-momentum operator insertion,

$$A_{ij} = \langle j(p)|O_i|j(p)\rangle \ , i, j = q \text{ or } g$$

To keep the p^2 onshell, i.e., $p^2 = 0$, one need to introduce a mass scale

- Consider non-zero momentum transfer (like a form factor), one-loop in[B.W. Harris and J. Smith, 1994]
- Introduce an internal mass scale, massive OMEs (by-product), three-loop results in[Ablinger et al. 2010, 2014abc, 2017]
- Introduce an external mass scale, like measuring p_T or the vituality t (by-product), three-loop results in[M.-x. Luo, **TZY**, H. X. Zhu, and Y. J. Zhu, 2019, 2020; M. A. Ebert, B. Mistlberger, and G. Vita, 2020]

Splitting functions from off-shell operator matrix elements

Set $p^2 < 0$,

- Non-singlet, multiplicatively renormalized, no renormalization issues
  - one-loop[D.J. Gross, F. Wilczek, 1973]
  - two-loop[E.G. Floratos, D.A. Ross, C. T. Sachrajda, 1977]
  - three-loop[J. Blümlein, P. Marquard, C. Schneider, K. Schönwald, 2021] See the talk of Johannes Blümlein
  - partial four-loop results from the fix Mellin moments computation[S. Moch, B. Ruijl, T. Ueda, J.A. M. Vermaseren, A. Vogt, 2017] see the talk of Sven-Olaf Moch
- Singlet, there are some renormalization issues, the operators O_q and O_g mix with gauge-variant (alien) operators,
  - one-loop[D.J. Gross, F. Wilczek, 1974] First pointed out the possible mixing with gauge-variant operators
  - two-loop[E.G. Floratos, D.A. Ross, C. T. Sachrajda, 1978] Some flaws due to omitting the gauge-variant operators
  - two-loop[R. Hamberg and W. L. van Neerven, 1992] All flaws resolved using the order g_s gauge-variant operators constructed in[J. A. Dixon, J.C. Taylor, 1974]

Significant efforts in deriving gauge-variant operators

- [D.J. Gross, F. Wilczek, 1974] first pointed out the possible mixing with gauge-variant operators
- [J.A. Dixon and J.C. Taylor, 1974] constructed order g_s gauge-variant operators, not clear how to generalize to higher order
- [Joglekar and Lee, 1975] gave a general theorem about the renormalization of gaug invariant operators. No explicit results are given
- [J. C. Collins and R. J. Scalise, 1994] studied the renormalization of energy-momentum tensor
- [G. Falcioni and F. Herzog, 2022] constructed the gauge-variant operators based on a generalized BRST symmetry See the talk of Giulio Falcioni
  Very promising, however, the operators are constructed for fixed Mellin moments n. More and more number of operators are needed for higher Mellin moments.

Construct all-n gauge-variant operators?

A new framework of deriving gauge-variant operators

- Extending the renormalization of the operator O_g,

$$O_g^{\mathsf{R}} = Z_{gq} O_q^{\mathsf{B}} + Z_{gg} O_g^{\mathsf{B}} + \boxed{\sum_{i=1}^{\infty} \eta_i \left(O_{A_i}^{\mathsf{B}} + O_{B_i}^{\mathsf{B}} + O_{C_i}^{\mathsf{B}} \right)} \, , \eta_i = \mathcal{O}(\alpha_s^i)$$

  where O_{A_i} is composed of gluon fields only, while O_{B_i} and O_{C_i} also involve quark and ghost fields separetaly.
- Infinite gauge-variant operators are needed, only finite number of operators are needed at finite order
- A single gauge-variant operator doesn't satisfy the transverse condition, the sum of them do satisfy

$$p_\mu p_\nu \langle g(p)|O_f|g(p)\rangle^{\mu\nu} \neq 0 \, , O_f = O_{A_i} \, , O_{B_i} \text{ or } O_{C_i} \, ,$$
$$p_\mu p_\nu \langle g(p)|O_{A_i} + O_{B_i} + O_{C_i}|g(p)\rangle^{\mu\nu} = 0 \, .$$

Extending to quark operator

- The renormalization can be generalized to O_q directly,

$$O_q^{\mathsf{R}} = Z_{qq} O_q^{\mathsf{B}} + Z_{qg} O_g^{\mathsf{B}} + \sum_{i=1}^{\infty} \kappa_i \left(O_{A_i}^{\mathsf{B}} + O_{B_i}^{\mathsf{B}} + O_{C_i}^{\mathsf{B}} \right), \kappa_i = \mathcal{O}(\alpha_s^{i+1})$$

- Need $k \times k$ mixing matrix for $(k-1)$-loop computations;
  For 3-loop computations,

$$\begin{pmatrix} O_q \\ O_g \\ O_{A_1 B_1 C_1} \\ O_{A_2 B_2 C_2} \end{pmatrix}^{\mathsf{R}} = \begin{pmatrix} Z_{qq} & Z_{qg} & \kappa_1 & \kappa_2 \\ Z_{gq} & Z_{gg} & \eta_1 & \eta_2 \\ 0 & 0 & Z_{A_1 A_1} & Z_{A_1 A_2} \\ 0 & 0 & Z_{A_2 A_1} & Z_{A_2 A_2} \end{pmatrix} \begin{pmatrix} O_q \\ O_g \\ O_{A_1 B_1 C_1} \\ O_{A_2 B_2 C_2} \end{pmatrix}^{\mathsf{B}}$$

with $O_{A_i B_i C_i} = O_{A_i} + O_{B_i} + O_{C_i}$

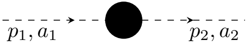
- The renormalizations of gauge-variant operators don't mix with the physical operators, compatible with the theorems given by Joglekar and Lee, 1975

The first type renormalization constants η_1

- Consider the determination of the first type gauge-variant operators

$$O_g^{\mathsf{R}} = Z_{gq} O_q^{\mathsf{B}} + Z_{gg} O_g^{\mathsf{B}} + \eta_1 \left(O_{A_1}^{\mathsf{B}} + O_{B_1}^{\mathsf{B}} + O_{C_1}^{\mathsf{B}} \right) + \mathcal{O}(\alpha_s^2) \, , \eta_1 = \mathcal{O}(\alpha_s)$$

- Idea: we determine the Feynman rules instead of the gauge-variant operators themselves.
- The only input is the Feynman rule of the lowest order ghost operator O_{C_1}, which can be easily determined



$$\rightarrow \quad i\,\delta^{a_1 a_2} \frac{1+(-1)^n}{2} (\Delta \cdot p_1)^n \, .$$

- Consider one loop OME with two ghost states, impose the renormalization conditions

$$0 = \left[\langle c | O_g | c \rangle^{\mathsf{B},(1)} + \eta_1^{(1)} \langle c | O_g | c \rangle^{\mathsf{B},(0)} \right]_{\frac{1}{\epsilon}}$$

$$\rightarrow \eta_1^{(1)} = \frac{1}{\epsilon} \frac{-C_A}{n(n-1)}$$

The first type ghost operators at higher order

- To determine the ghost operator O_{C_1} at order g_s^m, we consider the following correlators with m gluons and two ghosts,

$$\langle c|O_g|c+m\,g\rangle_{\mathrm{1PI}}^{\mu_1\cdots\mu_m,\,\mathrm{R}} = Z_c(\sqrt{Z_A})^m\langle c|Z_{gq}O_q + Z_{gg}O_g + \eta_1\,(O_{A_1}+O_{B_1}+O_{C_1})\,|c+m\,g\rangle_{\mathrm{1PI}}^{\mu_1\cdots\mu_m,\,\mathrm{B}}$$

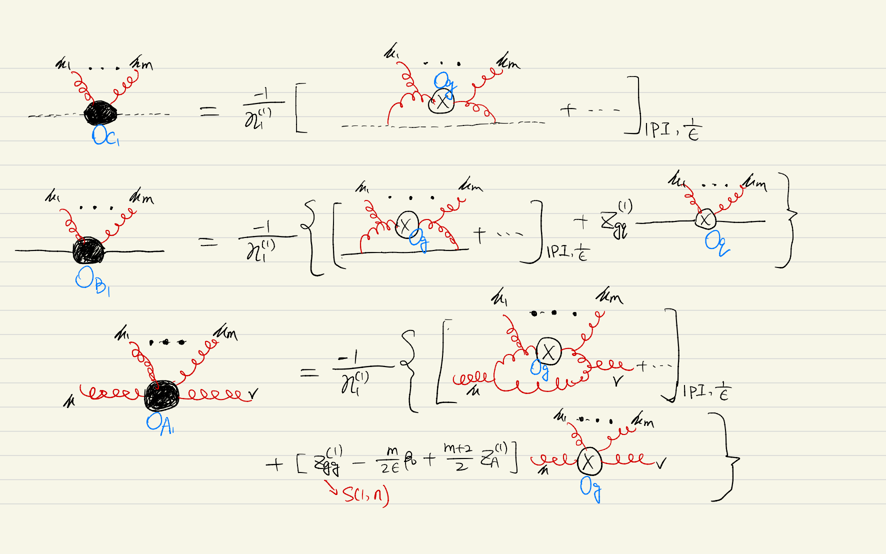
- Expanding the correlators as the following form,

$$\langle c|O_g|c+m\,g\rangle^{\mu_1\cdots\mu_m} = \sum_{j=1}^{\infty}\left[\langle c|O_g|c+m\,g\rangle^{\mu_1\cdots\mu_m,\,(j),\,(m)}\right]\left(\frac{\alpha_s}{4\pi}\right)^j g_s^m$$

- To lowest order, imposing the renormalization conditions again

$$\langle c|O_{C_1}|c+m\,g\rangle_{\mathrm{1PI}}^{\mu_1\cdots\mu_m,\,(0),\,(m)} = -\frac{1}{\eta_1^{(1)}}\left[\langle c|O_g|c+m\,g\rangle_{\mathrm{1PI}}^{\mu_1\cdots\mu_m,\,(1),\,(m),\,\mathrm{B}}\right]_{1/\epsilon}$$

The first type gauge-variant operators



The formula to derive other type gauge-variant operators

- Consider the determination of the second type gauge variant operators

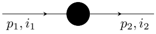
$$O_g^{\mathrm{R}} = Z_{gq} O_q^{\mathrm{B}} + Z_{gg} O_g^{\mathrm{B}} + \eta_1 \left(O_{A_1}^{\mathrm{B}} + O_{B_1}^{\mathrm{B}} + O_{C_1}^{\mathrm{B}} \right) + \eta_2 \left(O_{A_2}^{\mathrm{B}} + O_{B_2}^{\mathrm{B}} + O_{C_2}^{\mathrm{B}} \right) + \mathcal{O}(\alpha_s^3) \,, \eta_2 = \mathcal{O}(\alpha_s^2)$$

- Need to consider two-loop multiple points correlators?
- Look at the renormalization of gauge-variant operators

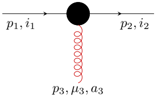
$$O_{A_1 B_1 C_1}^{\mathrm{R}} = Z_{A_1 A_1} \left(O_{A_1}^{\mathrm{B}} + O_{B_1}^{\mathrm{B}} + O_{C_1}^{\mathrm{B}} \right) + Z_{A_1 A_2} \left(O_{A_2}^{\mathrm{B}} + O_{B_2}^{\mathrm{B}} + O_{C_2}^{\mathrm{B}} \right) + \mathcal{O}(\alpha_s^2) \,,$$
$$Z_{A_1 A_1} = \mathcal{O}(1) \,, Z_{A_1 A_2} = \mathcal{O}(\alpha_s)$$

- Only need to consider one-loop one-particle-irreducible OMEs to determine other type of gauge-variant operators
- To determine $\eta_i (i > 2)$, still need to compute multi-loop multi-point correlators, but only in some special kinematics

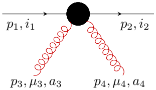
Feynman rules of the first type gauge-variant operator



$$\rightarrow 0$$



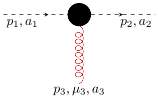
$$-\frac{1}{2}\frac{1+(-1)^n}{2}\, i\, g_s\, \Delta^{\mu_3}\, T^{a_3}_{i_2 i_1}\, \not{\Delta}\left(\Delta\cdot(p_1+p_2)\right)^{n-2}$$



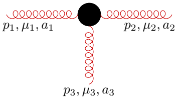
$$-\frac{i}{8}\frac{1+(-1)^n}{2} g_s^2 \Delta^{\mu_3}\Delta^{\mu_4}\left(T^{a_3}T^{a_4}-T^{a_4}T^{a_3}\right)_{i_2 i_1}\not{\Delta}\sum_{j_1=0}^{n-3}\Big(3\left(\Delta\cdot(p_1+p_2)\right)^{-j_1+n-3}\left[(-\Delta\cdot p_3)^{j_1}-(-\Delta\cdot p_4)^{j_1}\right] - (-\Delta\cdot p_4)^{j_1}(\Delta\cdot p_3)^{-j_1+n-3}\Big)$$

New

Feynman rules for O_{B_1} operator with all momentum flowing into the vertex.



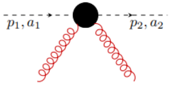
$$\frac{1}{8}\frac{1+(-1)^n}{2}\Delta^{\mu_3} g_s f^{a_1 a_2 a_3}\Big(3\Delta\cdot p_1\,\Delta\cdot p_2\sum_{j_1=0}^{n-3}\Big((-\Delta\cdot p_2)^{j_1}(\Delta\cdot p_1)^{-j_1+n-3}\Big)$$
$$+(\Delta\cdot p_1-\Delta\cdot p_2)(\Delta\cdot(p_1+p_2))^{n-2}-(\Delta\cdot p_1)^{n-1}+(\Delta\cdot p_2)^{n-1}\Big)\,,$$



$$\frac{1}{8}\frac{1+(-1)^n}{2} g_s f^{a_1 a_2 a_3}\Big(-4\Delta^{\mu_3} g^{\mu_1\mu_2}\Delta\cdot p_1(\Delta\cdot(p_1+p_2))^{n-2}$$
$$-3\Delta^{\mu_1}\Delta^{\mu_3}p_2^{\mu_2}\sum_{j_1=0}^{n-2}\Big((-\Delta\cdot p_2)^{j_1}(\Delta\cdot p_1)^{-j_1+n-2}\Big)+2\Delta^{\mu_1}\Delta^{\mu_2}\Big(4p_2^{\mu_3}+p_3^{\mu_3}\Big)(\Delta\cdot p_1)^{n-2}$$
$$-\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}(p_1\cdot p_1-p_1\cdot p_2+p_2\cdot p_2)\sum_{j_1=0}^{n-3}\Big((-\Delta\cdot p_2)^{j_1}(\Delta\cdot p_1)^{-j_1+n-3}\Big)\Big)+\textit{Permutations}\,,$$

Completely agree with [R. Hamberg and W. L. van Neerven, 1992] after correcting some typos there

New, all-n Feynman rules for O_{C_1}



$$\frac{1}{48}\frac{1+(-1)^n}{2} i g_s^2 \Delta^{\mu_3}\Delta^{\mu_4}\Big\{ f^{a_1a_3a}f^{a_2a_4a}\Big(6\,(-\Delta\cdot p_4)^{n-2} + 6\,(\Delta\cdot p_3)^{n-2}$$

$$+6\,(\Delta\cdot(p_1+p_3))^{n-2} + 6\,(\Delta\cdot(p_2+p_3))^{n-2} - \sum_{j_1=0}^{n-2}\Big[$$

$$+[(-\Delta\cdot p_3)^{j_1} + (-\Delta\cdot p_4)^{j_1}][3\,(\Delta\cdot p_1)^{n-j_1-2} + 3\,(\Delta\cdot p_2)^{n-j_1-2} + (\Delta\cdot(p_1+p_2))^{n-j_1-2}]$$

$$+9[(\Delta\cdot p_1)^{n-j_1-2} + (-\Delta\cdot p_2)^{n-j_1-2}][(\Delta\cdot(-p_2-p_3))^{j_1} + (\Delta\cdot(p_1+p_3))^{j_1}]\Big]$$

$$+13\sum_{j_1=0}^{n-2}\sum_{j_2=0}^{j_1}\Big[(-\Delta\cdot p_2)^{j_1-j_2}(\Delta\cdot p_1)^{n-j_1-2}[(\Delta\cdot(-p_2-p_3))^{j_2} + (\Delta\cdot(p_1+p_3))^{j_2}]\Big]\Big)$$

$$+f^{a_1a_2a}f^{a_3a_4a}\Big(-6\,(\Delta\cdot p_3)^{n-2} - 6\,(\Delta\cdot(p_2+p_3))^{n-2} + \sum_{j_1=0}^{n-2}\Big[3\,(-\Delta\cdot p_4)^{j_1}(\Delta\cdot p_1)^{n-j_1-2}$$

$$+3\,(-\Delta\cdot p_3)^{j_1}(\Delta\cdot p_2)^{n-j_1-2} + [5\,(-\Delta\cdot p_3)^{j_1} - 4\,(-\Delta\cdot p_4)^{j_1}](\Delta\cdot(p_1+p_2))^{n-j_1-2}$$

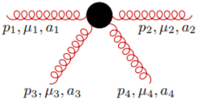
$$+9[(\Delta\cdot p_1)^{n-j_1-2} + (-\Delta\cdot p_2)^{n-j_1-2}](\Delta\cdot(-p_2-p_3))^{j_1}\Big]$$

$$-3\Delta\cdot p_2\sum_{j_1=0}^{n-3}\Big[3[(-\Delta\cdot p_3)^{j_1} - (-\Delta\cdot p_4)^{j_1}](\Delta\cdot(p_1+p_2))^{n-j_1-3} - (-\Delta\cdot p_4)^{j_1}(\Delta\cdot p_3)^{n-j_1-3}\Big]$$

$$+\sum_{j_1=0}^{n-2}\sum_{j_2=0}^{j_1}\Big[(-\Delta\cdot p_2)^{j_1-j_2}(\Delta\cdot p_1)^{n-j_1-2}[(\Delta\cdot(p_1+p_3))^{j_2} - 14\,(\Delta\cdot(p_1+p_4))^{j_2}]\Big]\Big)$$

$$+d_r^{a_1a_2a_3a_4}(\cdots)\Big\}.$$

New, all-n Feynman rules for O_{A_1}



$$
\begin{aligned}
&\frac{1}{96}\frac{1+(-1)^n}{2} i g_s^2 \Bigg\{ 3\Delta^{\mu_1}\Delta^{\mu_2} g^{\mu_3\mu_4} \Bigg(8\left(2 f^{aa_1a_3} f^{aa_2a_4} - f^{aa_1a_2} f^{aa_3a_4}\right)(\Delta\cdot p_1)^{n-2} \\
&- f^{aa_1a_2} f^{aa_3a_4}\left(\Delta\cdot p_1 + \Delta\cdot p_2 + 2\Delta\cdot p_3\right) \\
&\times \sum_{j_1=0}^{n-3}\left[\left[6\left(\Delta\cdot(-p_1-p_2)\right)^{j_1} + (\Delta\cdot p_2)^{j_1}\right](-\Delta\cdot p_1)^{-j_1+n-3}\right]\Bigg) \\
&+ 2\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}\Delta^{\mu_4}\Bigg(\left(p_1\cdot p_1 + p_1\cdot p_2 + p_1\cdot p_3 + p_2\cdot p_2 + p_2\cdot p_3 + p_3\cdot p_3\right) f^{aa_1a_3} f^{aa_2a_4} \\
&+ \left(4p_1\cdot p_1 - 5p_1\cdot p_2 - 5p_1\cdot p_3 - 5p_2\cdot p_2 - 5p_2\cdot p_3 + 4p_3\cdot p_3\right) f^{aa_1a_2} f^{aa_3a_4}\Bigg) \\
&\times \sum_{j_1=0}^{n-4}\sum_{j_2=0}^{j_1}\left((-\Delta\cdot p_3)^{j_2}\left(\Delta\cdot(p_1+p_2)\right)^{j_1-j_2}(\Delta\cdot p_1)^{-j_1+n-4}\right) \\
&- \Delta^{\mu_2}\Delta^{\mu_4}\left(f^{aa_1a_3} f^{aa_2a_4} + 13 f^{aa_1a_2} f^{aa_3a_4}\right)\left(\Delta^{\mu_1} p_3^{\mu_3} - \Delta^{\mu_3} p_1^{\mu_1}\right) \\
&\times \sum_{j_1=0}^{n-3}\sum_{j_2=0}^{j_1}\left(\left(\Delta\cdot(-p_1-p_2)\right)^{j_1-j_2}(\Delta\cdot p_3)^{j_2}(-\Delta\cdot p_1)^{-j_1+n-3}\right) + 3\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_4} \\
&\times \left(4p_1^{\mu_3} + p_3^{\mu_3}\right) f^{aa_1a_3} f^{aa_2a_4} \sum_{j_1=0}^{n-3}\Bigg[\left[4\left(\Delta\cdot(p_1+p_3)\right)^{j_1} + (\Delta\cdot p_4)^{j_1}\right](-\Delta\cdot p_2)^{-j_1+n-3} \\
&+ 2(\Delta\cdot p_4)^{j_1}\left(\Delta\cdot(-p_1-p_3)\right)^{-j_1+n-3}\Bigg] + d_r^{a_1a_2a_3a_4}(\cdots)\Bigg\} + \textit{Permutations}.
\end{aligned}
$$

Feynman rules for other types of gauge-variant operators

- In progress
- Preliminary results

$$
\begin{aligned}
O_{B_2} &= \mathcal{O}(g_s^2)\,, O_{B_3} = \mathcal{O}(g_s^2) \\
O_{A_2} &= \mathcal{O}(g_s)\,, O_{A_3} = \mathcal{O}(g_s) \\
O_{C_2} &= \mathcal{O}(g_s)\,, O_{C_3} = \mathcal{O}(g_s)
\end{aligned}
$$

- Renormalization of O_g

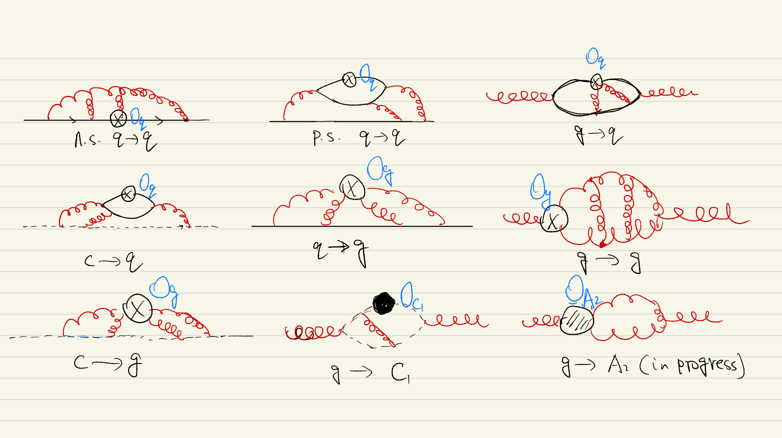
$$
O_g^{\mathsf{R}} = Z_{gq} O_q^{\mathsf{B}} + Z_{gg} O_g^{\mathsf{B}} + \sum_{i=1}^{\infty} \eta_i \left(O_{A_i}^{\mathsf{B}} + O_{B_i}^{\mathsf{B}} + O_{C_i}^{\mathsf{B}} \right), \eta_i = \mathcal{O}(\alpha_s^i)
$$

- O_{A_2} and O_{C_2} also contribute to three-loop singlet splitting functions
- O_{A_3}, O_{B_2} and O_{C_3} only contribute to four-loop singlet splitting functions, O_{B_3} only contributes starting at five-loop order

Three-loop singlet splitting functions from off-shell OMEs

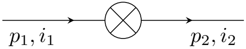
Sample Feynman diagrams

- Once we get the Feynman rules for both gauge-invariant and gauge-variant operators, we only need to compute the two-point off-shell OMEs
- Sample diagrams contribute to three-loop singlet splitting functions



Computational methods

- Non-standard terms appearing in the Feynman rules
- Example: Feynman rules for O_q at lowest order



$$\to \not{\Delta}\,(\Delta \cdot p_1)^{\,n-1}$$

- Sum the non-standard terms into a linear propagtor using a parameter x, first proposed in[J. Ablingera, J. Blumleinb, A. Hasselhuhnb, S. Kleinc, C. Schneidera, and F. Wißbrock, 2012] see also the talk of Johannes Blümlein

$$(\Delta \cdot p)^{n-1} \to \sum_{n=1}^{\infty} x^n (\Delta \cdot p)^{n-1} = \frac{x}{1 - x\Delta \cdot p}$$

- Always work on x-space, take the coefficient of x^n symbolicly in the end using the package `HarmonicSums`[Ablinger 2010–]
- Harmonic polylogrithms[Remiddi and Vermaseren,1999] $\to$ Harmonic sums[Vermaseren 1998,Blumlein and Kurth,1998]

$$H(1,1;x) = \sum_{n=1}^{\infty} x^n \left(-\frac{1}{n^2} + \frac{S(1,n)}{n} \right)$$

Computational procedure up to three loops

- Use QGRAF to generate all relevant Feynman diagrams
- Substitute the x-space Feynman rules in Mathematica
- Use FORM and Color to evaluate dirac matrix and color algebra
- Use a self-written code, Reduze 2 and FeynCalc to classify the topologies
- Use FIRE6, LiteRed, Reduze 2 and Kira to perform IBP reductions
- Derive differential equations with respect to the parameter x, turn them into canonical form proposed by Henn using CANONICA and Libra
- The solutions of master integrals are in terms of HPLs with argument x
- Turn OMEs in terms of HPLs into OMEs in terms of Harmonic sums with the help of the package HarmonicSums.
- In the end, we obtained $\langle j(p)|O_{q/g}|j(p)\rangle$ to three loops, $\langle j(p)|O_{A_1B_1C_1}|j(p)\rangle$ to two loops, in general ξ dependence, where $j = q\,, g\,,$ or ghosts.
- Also need $\langle j(p)|O_{A_2C_2}|j(p)\rangle$ to one loop, in progress.

Renormalization and results

- The correct renormalization to three loops should be

$$\begin{pmatrix} O_q \\ O_g \\ O_{A_1 B_1 C_1} \\ O_{A_2 B_2 C_2} \end{pmatrix}^{\mathsf{R}} = \begin{pmatrix} Z_{qq} & Z_{qg} & \kappa_1 & \kappa_2 \\ Z_{gq} & Z_{gg} & \eta_1 & \eta_2 \\ 0 & 0 & Z_{A_1 A_1} & Z_{A_1 A_2} \\ 0 & 0 & Z_{A_2 A_1} & Z_{A_2 A_2} \end{pmatrix} \begin{pmatrix} O_q \\ O_g \\ O_{A_1 B_1 C_1} \\ O_{A_2 B_2 C_2} \end{pmatrix}^{\mathsf{B}}$$

- $O_{A_2 B_2 C_2}$ is still missing, try the 3 by 3 mixing matrix,

$$\begin{pmatrix} O_q \\ O_g \\ O_{A_1 B_1 C_1} \end{pmatrix}^{\mathsf{R}} = \begin{pmatrix} Z_{qq} & Z_{qg} & \kappa_1 \\ Z_{gq} & Z_{gg} & \eta_1 \\ 0 & 0 & Z_{A_1 A_1} \end{pmatrix} \begin{pmatrix} O_q \\ O_g \\ O_{A_1 B_1 C_1} \end{pmatrix}^{\mathsf{B}}$$

- Results: confirm MVV in Feynman gauge $\xi = 1$

$$\gamma_{qq}^{(2)} - \gamma_{qq}^{(2)}[\mathsf{MVV}] = 0\,, \gamma_{qg}^{(2)} - \gamma_{qg}^{(2)}[\mathsf{MVV}] = 0\,,$$
$$\gamma_{gq}^{(2)} - \gamma_{gq}^{(2)}[\mathsf{MVV}] = 0\,, \gamma_{gg}^{(2)} - \gamma_{gg}^{(2)}[\mathsf{MVV}] = (1-\xi)\,[\cdots]$$

Summary

- The off-shell OMEs method is one of the most efficient methods to compute the splitting functions
- For off-shell OMEs, the physical singlet operators mix with unknown gauge-variant operators
- We developed a new framework based on renormalization conditions to find the all-n Feynman rules of the gauge-variant operators
- As a first application, we apply it to the 3-loop singlet splitting functions and recovered the well-known results by MVV
- The method of finding the Feynman rules of gauge-variant operators presented here can be generalized to four loops directly

Thank you very much for your attention!

Extra material

Lorentz structures of one-loop multi-point correlators

- Pure gluons correlators, for exmample, 4−gluon correlators

$$\left[\langle g|O_g|ggg\rangle_{\text{1PI}}^{\mu_1\mu_2\mu_3\mu_4,\,(1),\,(2),\,\text{B}}\right]_{1/\epsilon} = b_1\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}\Delta^{\mu_4} + b_2\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}p_1^{\mu_4}$$
$$+ b_3\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_4}p_1^{\mu_3} + b_4\Delta^{\mu_1}\Delta^{\mu_3}\Delta^{\mu_4}p_1^{\mu_2} + b_5\Delta^{\mu_2}\Delta^{\mu_3}\Delta^{\mu_4}p_1^{\mu_1}$$
$$+ b_6\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}p_2^{\mu_4} + b_7\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_4}p_2^{\mu_3} + b_8\Delta^{\mu_1}\Delta^{\mu_3}\Delta^{\mu_4}p_2^{\mu_2} + b_9\Delta^{\mu_2}\Delta^{\mu_3}\Delta^{\mu_4}p_2^{\mu_1}$$
$$+ b_{10}\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_3}p_3^{\mu_4} + b_{11}\Delta^{\mu_1}\Delta^{\mu_2}\Delta^{\mu_4}p_3^{\mu_3} + b_{12}\Delta^{\mu_1}\Delta^{\mu_3}\Delta^{\mu_4}p_3^{\mu_2} + b_{13}\Delta^{\mu_2}\Delta^{\mu_3}\Delta^{\mu_4}p_3^{\mu_1}$$
$$+ b_{14}\Delta^{\mu_1}\Delta^{\mu_2}g^{\mu_3\mu_4} + b_{15}\Delta^{\mu_1}\Delta^{\mu_3}g^{\mu_2\mu_4} + b_{16}\Delta^{\mu_1}\Delta^{\mu_4}g^{\mu_2\mu_3} + b_{17}\Delta^{\mu_2}\Delta^{\mu_3}g^{\mu_1\mu_4}$$
$$+ b_{18}\Delta^{\mu_2}\Delta^{\mu_4}g^{\mu_1\mu_3} + b_{19}\Delta^{\mu_3}\Delta^{\mu_4}g^{\mu_1\mu_2}$$

- An example, why not $b_{20}\Delta^{\mu_1}\Delta^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$?
- Property of twist-two operators (spin-n and mass $n+2$)
- Property of Feynman rules for a vertex (no mass scale can be generated in the denominator)
- counting the mass dimension of $b_{20}\Delta^{\mu_1}\Delta^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$,

$$[b_{20}] + 2 + 4 = n - 2 + 2 + 4 = n + 4\,,$$

where 4 is mass dimension of the external 4 gluons. Twist-4 operators