Summing Feynman diagrams in the worldline formalism

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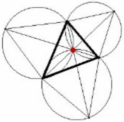

(with N. Ahmadiniaz, V. M. Banda, F. Bastianelli, O. Corradini, J.P. Edwards, C. Lopez-Arcos, M. A. Lopez-Lopez, C. Moctezuma Mata, J. Nicasio)

Loops and Legs in Quantum Field Theory
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Worldline representation of dressed scalar propagator

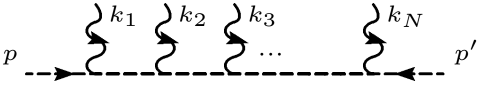
Feynman 1950: Green's function for the operator $-(\partial + ieA)^2 + m^2$

$$
\begin{aligned}
D^{xx'}[A] &= \langle x| \int_0^\infty dT \exp\Big[-T(-(\partial + ieA)^2 + m^2)\Big] |x'\rangle \\
&= \int_0^\infty dT \; \mathrm{e}^{-m^2 T} \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau) \, e^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + ie\dot{x}\cdot A(x(\tau))\right)}
\end{aligned}
$$

Expanding the field in N plane waves,

$$
A^\mu(x(\tau)) = \sum_{i=1}^N \varepsilon_i^\mu \, \mathrm{e}^{ik_i\cdot x(\tau)}
$$

and Fourier transforming the endpoints we get the "photon-dressed propagator",



where summation over all permutations of the photons is understood.

Worldline representation of scalar QED effective action

Similarly for the one-loop effective action:

$$
\begin{aligned}
\Gamma[A] &= -\mathrm{Tr}\ln\Big[-(\partial + ieA)^2 + m^2\Big] \\
&= \int_0^\infty \frac{dT}{T}\,\mathrm{Tr}\exp\Big[-T(-(\partial + ieA)^2 + m^2)\Big] \\
&= \int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2 T}\int_{x(0)=x(T)} \mathcal{D}x(\tau)\, e^{-\int_0^T d\tau\left(\frac{1}{4}\dot{x}^2 + ie\dot{x}\cdot A(x(\tau))\right)}
\end{aligned}
$$

Expanding the field in N plane waves,

$$
A^\mu(x(\tau)) = \sum_{i=1}^{N} \varepsilon_i^\mu\, \mathrm{e}^{ik_i\cdot x(\tau)}
$$

one gets the one-loop N - photon amplitudes, again summed over all permutations.

Worldline representation of spinor QED effective action

Feynman 1951:

Add global factor of $-\frac{1}{2}$ and *spin factor* $\mathrm{Spin}[x, A]$,

$$\mathrm{Spin}[x, A] = \mathrm{tr}_\Gamma \mathcal{P} \exp \left[\frac{i}{4} e \, [\gamma^\mu, \gamma^\nu] \int_0^T d\tau F_{\mu\nu}(x(\tau)) \right]$$

Modern way: Replace spin factor by a Grassmann path integral (E.S. Fradkin 1966)

$$\mathrm{Spin}[x, A] \to \int \mathcal{D}\psi(\tau) \, \exp \left[- \int_0^T d\tau \left(\frac{1}{2} \psi \cdot \dot{\psi} - ie\psi^\mu F_{\mu\nu} \psi^\nu \right) \right]$$

$$\begin{aligned} \psi(\tau_1)\psi(\tau_2) &= -\psi(\tau_2)\psi(\tau_1) \\ \psi(T) &= -\psi(0) \end{aligned}$$

The main point of the Grassmann approach is to replace the path-ordered exponential by an ordinary exponential.

Worldline representation of dressed electron propagator

Feynman 1951; Fradkin 1966; Fradkin and Gitman 1991; M. Reuter, M.G. Schmidt and C.S. 1996,
N. Ahmadiniaz, V.M. Banda Guzmán, F. Bastianelli, O. Corradini, J.P. Edwards and C. S. 2020:
Worldline master formulas for the dressed electron propagator, part 1: Off-shell amplitudes, JHEP 08 (2020) 018
Worldline master formulas for the dressed electron propagator, part 2: On-shell amplitudes, JHEP 01 (2022) 050

Second-order representation of the x-space Dirac propagator $S^{x'x}[A]$ in a Maxwell background:

$$
\begin{aligned}
S^{x'x}[A] &= \left[m + i\not{D}'\right] K^{x'x}[A] \\
K^{x'x}[A] &= \left\langle x' \middle| \left[m^2 - D_\mu D^\mu + \frac{i}{2}\, e\gamma^\mu\gamma^\nu F_{\mu\nu}\right]^{-1} \middle| x \right\rangle \\
&= \int_0^\infty dT\, \mathrm{e}^{-m^2 T}\, \mathrm{e}^{-\frac{1}{4}\frac{(x-x')^2}{T}} \int_{q(0)=0}^{q(T)=0} Dq\, \mathrm{e}^{-\int_0^T d\tau\left(\frac{1}{4}\dot{q}^2 + ie\,\dot{q}\cdot A + ie\frac{x'-x}{T}\cdot A\right)} \\
&\quad \times 2^{-\frac{D}{2}}\, \mathrm{symb}^{-1} \int_{\psi(0)+\psi(T)=0} D\psi\, \mathrm{e}^{-\int_0^T d\tau\left[\frac{1}{2}\psi_\mu\dot{\psi}^\mu - ieF_{\mu\nu}(\psi+\eta)^\mu(\psi+\eta)^\nu\right]}
\end{aligned}
$$

Here η^μ is an external Grassmann Lorentz vector, and the "symbol map" *symb* converts products of ηs into fully antisymmetrised products of Dirac matrices:

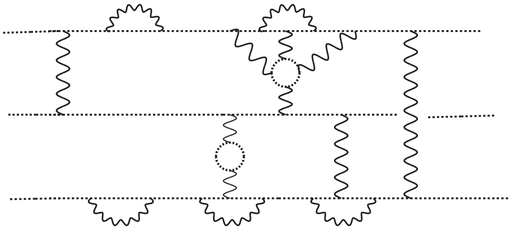
$$
\mathrm{symb}\left(\gamma^{\alpha_1\alpha_2\cdots\alpha_n}\right) \equiv (-i\sqrt{2})^n \eta^{\alpha_1}\eta^{\alpha_2}\ldots\eta^{\alpha_n}
$$

where $\gamma^{\alpha\beta\cdots\rho}$ denotes the totally antisymmetrised product of gamma matrices:

$$
\gamma^{\alpha_1\alpha_2\cdots\alpha_n} \equiv \frac{1}{n!}\sum_{\pi\in S_n} \mathrm{sign}(\pi)\gamma^{\alpha_{\pi(1)}}\gamma^{\alpha_{\pi(2)}}\ldots\gamma^{\alpha_{\pi(n)}}
$$

Higher order QED processes

These are off-shell formulas, therefore arbitrary QED processes can be constructed from these building blocks by sewing:



Equivalent to Feynman diagrams, but does not distinguish between diagrams differing only by the ordering of the photon legs along a line or loop. For some purposes this property becomes useful already at the path-integral level $\longrightarrow$ Landau-Khalatnikov-Fradkin transformations.

2n-point generalization of the LKF transformation

Landau and Khalatnikov 1956, Fradkin 1956:
Nonperturbative behaviour of the electron propagator $S(x;\xi)$ under a change $\xi \to \hat{\xi}$ of the covariant gauge parameter ξ:

$$S(x;\xi) = S(x;\hat{\xi}) \left[\frac{x^2}{x_0^2}\right]^{-\frac{\alpha}{4\pi}(\xi-\hat{\xi})}$$

where x_0 is an IR cutoff.

N. Ahmadiniaz, J.P. Edwards, J. Nicasio and C.S., PRD 104 (2021) 025014:
Generalization from the fermion propagator $\langle\psi(x)\bar{\psi}(x')\rangle$to the 2$n$-point correlator

$$\mathcal{A}(x_1,\dots,x_n;x_1',\dots,x_n'|\xi) \equiv \langle\psi(x_1)\cdots\psi(x_n)\bar{\psi}(x_1')\cdots\bar{\psi}(x_n')\rangle$$

$$\mathcal{A}(x_1,\dots,x_n;x_1',\dots,x_n'|\hat{\xi}) = \prod_{k,l=1}^{n} \mathrm{e}^{(\hat{\xi}-\xi)S^{(k,l)}} \mathcal{A}(x_1,\dots,x_n;x_1',\dots,x_n'|\xi)$$

$$S^{(k,l)} = \frac{e^2}{32\pi^{\frac{D}{2}}}\Gamma\Big(\frac{D}{2}-2\Big)\Big\{\big[\big(x_k-x_l\big)^2\big]^{2-\frac{D}{2}} - \big[\big(x_k-x_l'\big)^2\big]^{2-\frac{D}{2}} - \big[\big(x_k'-x_l\big)^2\big]^{2-\frac{D}{2}} + \big[\big(x_k'-x_l'\big)^2\big]^{2-\frac{D}{2}}\Big\}$$

String-inspired treatment of the worldline path integral

Polyakov 1987, Bern-Kosower 1991, Strassler 1992: Perturbative approach to the evaluation of worldline path integrals using worldline Green's functions $G(\tau_1, \tau_2)$, $G_F(\tau_1, \tau_2)$

$$
\begin{aligned}
\langle x^\mu(\tau_1) x^\nu(\tau_2)\rangle &= -G(\tau_1,\tau_2)\,\delta^{\mu\nu} \\
G(\tau_1,\tau_2) &= |\tau_1-\tau_2| - \frac{1}{T}\Big(\tau_1-\tau_2\Big)^2 \\
\langle \psi^\mu(\tau_1)\psi^\nu(\tau_2)\rangle &= G_F(\tau_1,\tau_2)\,\delta^{\mu\nu} \\
G_F(\tau_1,\tau_2) &= \mathrm{sign}(\tau_1-\tau_2)
\end{aligned}
$$

Leads to compact master formulas for the photon-dressed propagators and the one-loop N-photon amplitudes in scalar and spinor QED.

Master formula for the N-photon amplitudes in scalar QED

Polyakov 1987, Bern-Kosower 1991, Strassler 1992:

$$\Gamma[\{k_i,\varepsilon_i\}] = (-ie)^N \int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}} e^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i$$
$$\times \exp\Big\{ \sum_{i,j=1}^N \big[\frac{1}{2} G_{ij} k_i\cdot k_j + i\dot{G}_{ij} k_i\cdot\varepsilon_j + \frac{1}{2}\ddot{G}_{ij}\varepsilon_i\cdot\varepsilon_j\big]\Big\} \Big|_{\mathrm{lin}(\varepsilon_1,\dots,\varepsilon_N)}$$

$$G(\tau_1,\tau_2) = |\tau_1-\tau_2| - \frac{(\tau_1-\tau_2)^2}{T}$$
$$\dot{G}(\tau_1,\tau_2) = \mathrm{sign}(\tau_1-\tau_2) - 2\frac{(\tau_1-\tau_2)}{T}$$
$$\ddot{G}(\tau_1,\tau_2) = 2\delta(\tau_1-\tau_2) - \frac{2}{T}$$

T is the loop proper-time, τ_i parametrizes the position of photon i along the loop.

Master formula for the N-photon amplitudes in spinor QED

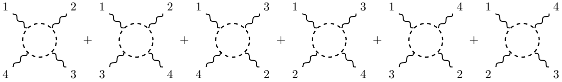
A similar master formula exists for the spinor loop, based on worldline SUSY.
In practice it is usually preferable to use an integration-by-parts,removing the $\ddot{G}_{ij}$, and the Bern-Kosower replacement rule

$$\dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ni_1} \rightarrow \dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ni_1} - G_{Fi_1i_2}G_{Fi_2i_3}\cdots G_{Fi_ni_1} ,$$

with the fermionic worldline Green's function G_F

$$G_F(\tau,\tau') \equiv \mathrm{sign}(\tau-\tau')$$

N=4 (Scalar or Spinor QED)



After a large number of integrations by parts:

$$\hat{\Gamma} = \hat{\Gamma}^{(1)} + \hat{\Gamma}^{(2)} + \hat{\Gamma}^{(3)} + \hat{\Gamma}^{(4)} + \hat{\Gamma}^{(5)}$$

$$\begin{aligned}
\hat{\Gamma}^{(1)} &= \hat{\Gamma}^{(1)}_{(1234)} T^{(1)}_{(1234)} + \hat{\Gamma}^{(1)}_{(1243)} T^{(1)}_{(1243)} + \hat{\Gamma}^{(1)}_{(1324)} T^{(1)}_{(1324)} \\
\hat{\Gamma}^{(2)} &= \hat{\Gamma}^{(2)}_{(12)(34)} T^{(2)}_{(12)(34)} + \hat{\Gamma}^{(2)}_{(13)(24)} T^{(2)}_{(13)(24)} + \hat{\Gamma}^{(2)}_{(14)(23)} T^{(2)}_{(14)(23)} \\
\hat{\Gamma}^{(3)} &= \sum_{i=1,2,3} \hat{\Gamma}^{(3)}_{(123)i} T^{(3)r_4}_{(123)i} + \sum_{i=2,3,4} \hat{\Gamma}^{(3)}_{(234)i} T^{(3)r_1}_{(234)i} + \sum_{i=3,4,1} \hat{\Gamma}^{(3)}_{(341)i} T^{(3)r_2}_{(341)i} + \sum_{i=4,1,2} \hat{\Gamma}^{(3)}_{(412)i} T^{(3)r_3}_{(412)i} \\
\hat{\Gamma}^{(4)} &= \sum_{i<j} \hat{\Gamma}^{(4)}_{(ij)ii} T^{(4)}_{(ij)ii} + \sum_{i<j} \hat{\Gamma}^{(4)}_{(ij)jj} T^{(4)}_{(ij)jj} \\
\hat{\Gamma}^{(5)} &= \sum_{i<j} \hat{\Gamma}^{(5)}_{(ij)ij} T^{(5)}_{(ij)ij} + \sum_{i<j} \hat{\Gamma}^{(5)}_{(ij)ji} T^{(5)}_{(ij)ji}
\end{aligned}$$

Tensor basis for the off-shell four-photon amplitudes

The basis of five tensors $T^{(i)}$ is identical with the one found in 1971 by Costantini, De Tollis and Pistoni using the QED Ward identity:

$$
\begin{aligned}
T^{(1)}_{(1234)} &\equiv Z(1234)\,,\\
T^{(2)}_{(12)(34)} &\equiv Z(12)Z(34)\,,\\
T^{(3)r_4}_{(123)i} &\equiv Z(123)\,\frac{r_4\cdot f_4\cdot k_i}{r_4\cdot k_4}\quad (i=1,2,3)\,,\\
T^{(4)}_{(12)11} &\equiv Z(12)\,\frac{k_1\cdot f_3\cdot f_4\cdot k_1}{k_3\cdot k_4}\,,\\
T^{(5)}_{(12)12} &\equiv Z(12)\,\frac{k_1\cdot f_3\cdot f_4\cdot k_2}{k_3\cdot k_4}\,.
\end{aligned}
$$

$$
\begin{aligned}
f_i^{\mu\nu} &\equiv k_i^\mu\varepsilon_i^\nu - \varepsilon_i^\mu k_i^\nu\\
Z(ij) &\equiv \frac{1}{2}\mathrm{tr}\big(f_i f_j\big) = \varepsilon_i\cdot k_j\varepsilon_j\cdot k_i - \varepsilon_i\cdot\varepsilon_j k_i\cdot k_j\\
Z(i_1 i_2\dots i_n) &\equiv \mathrm{tr}\Big(\prod_{j=1}^{n} f_{i_j}\Big)\quad (n\geq 3)
\end{aligned}
$$

Optimized worldline parameter integrals for $N = 4$

$$\hat{\Gamma}^{(k)}_{\dots} = \int_0^\infty \frac{dT}{T} T^{4-\frac{D}{2}} \, \mathrm{e}^{-m^2 T} \int_0^1 \prod_{i=1}^{4} du_i \, \hat{\gamma}^{(k)}_{\dots}(\dot{G}_{ij}) \, \mathrm{e}^{T \sum_{i<j=1}^{4} G_{ij} k_i \cdot k_j}$$

where, for spinor QED,

$$\begin{aligned}
\hat{\gamma}^{(1)}_{(1234)} &= \dot{G}_{12}\dot{G}_{23}\dot{G}_{34}\dot{G}_{41} - G_{F12}G_{F23}G_{F34}G_{F41} \\
\hat{\gamma}^{(2)}_{(12)(34)} &= \left(\dot{G}_{12}\dot{G}_{21} - G_{F12}G_{F21}\right)\left(\dot{G}_{34}\dot{G}_{43} - G_{F34}G_{F43}\right) \\
\hat{\gamma}^{(3)}_{(123)i} &= \left(\dot{G}_{12}\dot{G}_{23}\dot{G}_{31} - G_{F12}G_{F23}G_{F31}\right)\dot{G}_{4i} \\
\hat{\gamma}^{(4)}_{(12)11} &= \left(\dot{G}_{12}\dot{G}_{21} - G_{F12}G_{F21}\right)\dot{G}_{13}\dot{G}_{41} \\
\hat{\gamma}^{(5)}_{(12)12} &= \left(\dot{G}_{12}\dot{G}_{21} - G_{F12}G_{F21}\right)\dot{G}_{13}\dot{G}_{42}
\end{aligned}$$

(plus permutations thereof). The coefficient functions for scalar QED are obtained from these simply by deleting all the G_{Fij}.

Master formula for the N-photon dressed scalar propagator

Daikouji, Shino and Sumino 1996:

$$\begin{aligned} D^{p'p}(k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N) &= (-ie)^N \int_0^\infty dT\, \mathrm{e}^{-m^2 T} \\ &\times \prod_{i=1}^N \int_0^T d\tau_i\, \mathrm{e}^{-Tb^2 + \sum_{i,j=1}^N [\Delta_{ij} k_i\cdot k_j - 2i\,{}^\bullet\!\Delta_{ij}\varepsilon_i\cdot k_j - {}^\bullet\!\Delta^\bullet_{ij}\varepsilon_i\cdot\varepsilon_j]}\Big|_{\varepsilon_1\varepsilon_2\cdots\varepsilon_N} . \end{aligned}$$

Here we have introduced the vector

$$b \equiv p' + \frac{1}{T}\sum_{i=1}^N (k_i\tau_i - i\varepsilon_i)$$

and a different worldline Green's function $\Delta(\tau,\tau')$ has been used for the q propagator:

$$\begin{aligned} \langle q^\mu(\tau) q^\nu(\tau')\rangle &= -2\Delta(\tau,\tau')\delta^{\mu\nu}, \\ \Delta(\tau,\tau') &= \frac{|\tau-\tau'|}{2} - \frac{\tau+\tau'}{2} + \frac{\tau\tau'}{T} . \end{aligned}$$

Master formula for the photon-dressed electron propagator

N. Ahmadiniaz, V.M. Banda Guzmán, F. Bastianelli, O. Corradini, J.P. Edwards and C. S. 2020:
JHEP 08 (2020) 018, JHEP 01 (2022) 050

$$K_N^{p'p}(k_1,\varepsilon_1;\dots;k_N,\varepsilon_N) = (-ie)^N \mathrm{symb}^{-1}\int_0^\infty dT\, \mathrm{e}^{-m^2T}\int_0^T d\tau_1\cdots\int d\theta_N$$
$$\times\, \mathrm{e}^{-\sqrt{2}\eta\cdot\sum_{i=1}^N(\varepsilon_i+i\theta k_i)+\sum_{i,j=0}^{N+1}\left[\widehat{g}_{ij}k_i\cdot k_j+2iD_i\widehat{g}_{ij}\varepsilon_i\cdot k_j+D_iD_j\widehat{g}_{ij}\varepsilon_i\cdot\varepsilon_j\right]}\Big|_{\varepsilon_1\cdots\varepsilon_N}.$$

$$D = \frac{\partial}{\partial\theta}-\theta\frac{\partial}{\partial\tau}$$
$$\widehat{g}(\tau,\theta;\tau',\theta') = \frac{1}{2}(|\tau-\tau'|+\theta\theta'\mathrm{sign}(\tau-\tau'))$$

In four dimensions, involves only

$$\mathrm{symb}^{-1}(1) = \mathbb{1};$$
$$\mathrm{symb}^{-1}(\eta^{\alpha_1}\eta^{\alpha_2}) = -\frac{1}{4}[\gamma^{\alpha_1},\gamma^{\alpha_2}];$$
$$\mathrm{symb}^{-1}(\eta^{\alpha_1}\eta^{\alpha_2}\eta^{\alpha_3}\eta^{\alpha_4}) = \frac{1}{96}\sum_{\pi\in S_4}\mathrm{sign}(\pi)\gamma^{\alpha_{\pi(1)}}\gamma^{\alpha_{\pi(2)}}\gamma^{\alpha_{\pi(3)}}\gamma^{\alpha_{\pi(4)}} = -\frac{i}{4}\varepsilon^{\alpha_1\alpha_2\alpha_3\alpha_4}\gamma_5\,.$$

Early projection on the Clifford basis $\longrightarrow$ avoidance of long Dirac products.

Properties of the master formula

On-shell:

$$K_N^{p'p} \quad = \quad (-ie)^N \frac{A_N \mathbb{1} + B_N \sigma^{\alpha\beta} - iC_N \gamma_5}{(p^2+m^2)\,(p'^2+m^2)}$$

Spin-averaging can be done without fixing the number or helicity assignments of the photons:

$$\langle |\mathcal{M}_N|^2 \rangle \quad = \quad e^{2N} \Big[|A_N|^2 + 2 B_N^{\alpha\beta} B^*_{N\alpha\beta} - |C_N|^2 \Big]$$

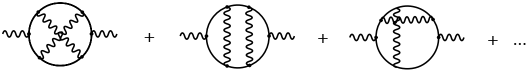
N. Ahmadiniaz, V.M. Banda Guzmán, F. Bastianelli, O. Corradini, J.P. Edwards and C. S. 2020, JHEP 01 (2022) 050: As a check, we have recalculated the Compton scattering amplitude (polarized and unpolarized) and found agreement with Denner and Dittmaier 1999.

On to multiloop

Dealing with the amplitude as a whole becomes important when one uses the one-loop amplitudes to construct higher-loop amplitudes by sewing:

From the four-photon amplitude we can construct the two-loop quenched photon propagator,

From the one-loop six-photon amplitude we get the three-loop quenched propagator



etcetera

This type of sums of diagrams is known to suffer from extensive cancellations...

The fundamental problem of worldline integration

Returning to the one-loop level, using that G_{Fij}s can always be eliminated by

$$G_{Fij}G_{Fjk}G_{Fki} = -(\dot{G}_{ij} + \dot{G}_{jk} + \dot{G}_{ki})$$

the most general integral that one will ever have to compute in the worldline approach to QED (or any abelian theory) is of the form

$$\int_0^1 du_1 du_2 \cdots du_N \, \mathrm{Pol}(\dot{G}_{ij}) \, \mathrm{e}^{\sum_{i<j=1}^N G_{ij} k_i \cdot k_j}$$

with arbitrary N and polynomial $\mathrm{Pol}(\dot{G}_{ij})$, where

$$G_{ij} = |u_i - u_j| - (u_i - u_j)^2, \dot{G}_{ij} = \mathrm{sgn}(u_i - u_j) - 2(u_i - u_j)$$

Without decomposing the integrand into ordered sectors! This leads to a non-standard integration challenge.

Cycle integrals

Cycle integral:

$$b_n \equiv \int_0^1 du_1 du_2 \dots du_n \, \dot{G}_{12}\dot{G}_{23}\cdots\dot{G}_{n1} = \begin{cases} -2^n \frac{B_n}{n!} & n \quad \text{even} \\ 0 & n \quad \text{odd} \end{cases}$$

where B_n denotes the nth Bernoulli number.

Super cycle integral:

$$b_n - f_n \equiv \int_0^1 du_1 du_2 \dots du_n \left(\dot{G}_{B12}\dot{G}_{B23}\cdots\dot{G}_{Bn1} - G_{F12}G_{F23}\cdots G_{Fn1}\right) = (2-2^n)\, b_n$$

This is sufficient to calculate the one-loop N-photon amplitudes in the low-energy (LE) approximation:

$$\Gamma^{(LE)}_{\rm spin}[\varepsilon_1^+;\varepsilon_2^+;\dots;\varepsilon_N^+] = -\frac{\alpha\pi m^4}{8\pi^2}\left(\frac{2e}{m^2}\right)^N c_{{\rm spin},n}\chi_N^+$$

$$\Gamma^{(LE)}_{\rm scal}[\varepsilon_1^+;\varepsilon_2^+;\dots;\varepsilon_N^+] = \frac{\alpha\pi m^4}{16\pi^2}\left(\frac{2e}{m^2}\right)^N c_{{\rm scal},n}\chi_N^+$$

$$c_{{\rm spin},n} = \frac{(-1)^{n+1}B_{2n}}{2n(2n-2)} = c_{{\rm scal},n}$$

$$\chi_N^+ = \frac{(\frac{N}{2})!}{2^{\frac{N}{2}}}\Big\{[k_1k_2]^2[k_3k_4]^2\cdots[k_{N-1}k_N]^2 + \text{ all distinct permutations}\Big\}. \tag{1}$$

General polynomial integrals

Example:

$$\int_0^1 du\, \dot{G}(u,u_1)\dot{G}(u,u_2)\dot{G}(u,u_3) = -\frac{1}{6}(\dot{G}_{12}-\dot{G}_{23})(\dot{G}_{23}-\dot{G}_{31})(\dot{G}_{31}-\dot{G}_{12})$$

General n-point integral of a polynomial in $\dot{G}$:

$$\int_0^1 du\, \dot{G}(u,u_1)^{k_1}\,\dot{G}(u,u_2)^{k_2}\cdots\dot{G}(u,u_n)^{k_n} = \frac{1}{2n}\sum_{i=1}^n \prod_{j\neq i}\sum_{l_j=0}^{k_j}\binom{k_j}{l_j}\dot{G}_{ij}^{k_j-l_j}\sum_{l_i=0}^{k_i}\binom{k_i}{l_i}$$

$$\times\frac{(-1)^{\sum_{j=1}^n l_j}}{(1+\sum_{j=1}^n l_j)n^{\sum_{j=1}^n l_j}}\left\{\Big(\sum_{j\neq i}\dot{G}_{ij}+1\Big)^{1+\sum_{j=1}^n l_j}-(-1)^{k_i-l_i}\Big(\sum_{j\neq i}\dot{G}_{ij}-1\Big)^{1+\sum_{j=1}^n l_j}\right\}$$

This formula settles all polynomial integrals by recursion

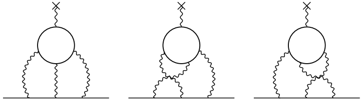
.

Integrating out a low-energy photon

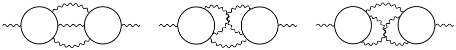
N. Ahmadiniaz, C. Lopez-Arcos, M.A. Lopez-Lopez, C.S., in preparation: Calculation of the four-photon amplitudes off-shell with one leg taken at low energy (linear in momentum) and integrated out.

Possible use as building block in higher-loop calculation:

3-loop g-2 contributions:



4-loop β-function contributions:



Inverse derivative expansion

How to treat the factors $\mathrm{e}^{G_{ij}k_i \cdot k_j}$? Expand them in inverse derivatives:

$$\mathrm{e}^{G_{ij}k_i \cdot k_j} = 1 + 2\sum_{n=1}^{\infty}(k_i \cdot k_j)^{n-1/2} H_{2n-1}\left(\frac{\sqrt{k_i \cdot k_j}}{2}\right)\left(\langle u_i|\partial^{-2n}|u_j\rangle - \langle u_i|\partial^{-2n}|u_i\rangle\right)$$

Here the $H_n(x)$ are Hermite polynomials.

Worldline integration naturally relates to the theory of Bernoulli numbers and polynomials. The path integral is performed in the Hilbert space H'_P of periodic functions orthogonal to the constant functions (because the zero mode must be fixed). In this space the ordinary nth derivative ∂_P is invertible, and the integral kernel of the inverse is given essentially by the nth Bernoulli polynomial $B_n(x)$:

$$\begin{aligned} < u_1 \mid \partial_P^{-n} \mid u_{n+1} > &= -\frac{1}{n!}B_n(|u_1 - u_{n+1}|)\mathrm{sgn}^n(u_1 - u_{n+1}) \quad (n \geq 1) \\ \langle u_i|\partial^0|u_j\rangle &= \delta(u_i - u_j) - 1 \end{aligned}$$

N-point function in ϕ^3 theory

One-loop N-point amplitude for scalar ϕ^3-theory:

$$\hat{I}_N(p_1, \ldots, p_N) = \int_0^\infty \frac{dT}{T} T^{N-D/2} e^{-m^2 T} \int_0^1 du_1 \ldots du_N \exp\left[T \sum_{i<j=1}^{N} G_{ij} k_{ij}\right]$$

$G_{ij} = |u_i - u_j| - (u_i - u_j)^2,\ k_{ij} \equiv k_i \cdot k_j.$

$N = 3$

In the three-point case one can use this to write

$$
\begin{aligned}
e^{k_{12}G_{12}+k_{13}G_{13}+k_{23}G_{23}} &= \left\{1+2\sum_{i=1}^{\infty} k_{12}^{i-\frac{1}{2}} H_{2i-1}\left(\frac{\sqrt{k_{12}}}{2}\right)\left[\langle u_1|\partial^{-2i}|u_2\rangle + \hat{B}_{2i}\right]\right\} \\
&\times \left\{1+2\sum_{j=1}^{\infty} k_{13}^{j-\frac{1}{2}} H_{2j-1}\left(\frac{\sqrt{k_{13}}}{2}\right)\left[\langle u_1|\partial^{-2j}|u_3\rangle + \hat{B}_{2j}\right]\right\} \\
&\times \left\{1+2\sum_{k=1}^{\infty} k_{23}^{k-\frac{1}{2}} H_{2k-1}\left(\frac{\sqrt{k_{23}}}{2}\right)\left[\langle u_2|\partial^{-2k}|u_3\rangle + \hat{B}_{2k}\right]\right\}
\end{aligned}
$$

Since $\int_0^1 du_{i,j}\langle u_i|\partial^{-2i}|u_j\rangle = 0$, the three $\langle u_i|\partial^{-2n}|u_j\rangle$ must go together, and then by $\int_0^1 du|u\rangle\langle u| = \mathbb{1}$,

$$
\int_{123}\langle u_1|\partial^{-2i}|u_2\rangle\langle u_2|\partial^{-2k}|u_3\rangle\langle u_3|\partial^{-2j}|u_1\rangle = \mathrm{Tr}(\partial^{-2(i+j+k)}) = -\hat{B}_{2(i+j+k)}
$$

$N = 3$ coefficients

In this way we get a closed form-expression for the $N = 3$ momentum expansion coefficients,

$$I_3(a,b,c) \equiv \int_{123} G_{12}^a G_{13}^b G_{23}^c = a!b!c! \sum_{i=\lfloor 1+a/2 \rfloor}^{a} \sum_{j=\lfloor 1+b/2 \rfloor}^{b} \sum_{k=\lfloor 1+c/2 \rfloor}^{c} h_i^a h_j^b h_k^c \left(\hat{B}_{2i} \hat{B}_{2j} \hat{B}_{2k} - \hat{B}_{2(i+j+k)} \right)$$

Here we have assumed that a, b, c are all different from zero, and the coefficients h_i^a are (from the explicit formula for the Hermite polynomials)

$$h_i^a = (-1)^{a+1} \frac{2(2i-1)!}{(2i-a-1)!(2a-2i+1)!}$$

At $N = 4$, we encounter the "cubic worldline vertex"

$$V_3^{ijk} \quad \equiv \quad \int_0^1 du \langle u | \partial^{-i} | u_1 \rangle \langle u | \partial^{-j} | u_2 \rangle \langle u | \partial^{-k} | u_3 \rangle$$

but it can be converted to chain integrals by integration by parts.

Summary and Outlook

1. In the worldline formalism, we can integrate out photons in the low-energy limit, or to any finite order in the external momentum, without fixing the ordering of the remaining legs.
2. At full momentum, we can use the inverse derivative expansion, and try to resum. Needed: formulas relating the Bernoulli numbers to hypergeometric functions.
3. In x-space, this provides also a new approach to the calculation of the ϕ^3 and QED heat-kernel expansions.